The strength of Ramsey theorem for coloring ω–large sets

Konrad Zdanowski
(join work with Lorenzo Carlucci, La Sapienza)

University of Cardinal Stefan Wyszyński, Warsaw

Kotlarski–Ratajczyk conference 2012

Outline

Outline

1. Basic notions
   - Second order arithmetics
   - Ordinals and large sets
   - Recursion theory
2. Ramsey principle for coloring n–tuples
   - $\mathrm{RT}(n)$
   - $\forall n \mathrm{RT}(n)$
3. Ramsey principle for coloring α–large sets
   - Farmaki theorem
   - Coloring ω–large sets
   - Arithmetics with truth predicates
4. Open problems

Outline

1. Basic notions
   - Second order arithmetics
   - Ordinals and large sets
   - Recursion theory
2. Ramsey principle for coloring n–tuples
3. Ramsey principle for coloring α–large sets
4. Open problems

- We consider theories of second order arithmetic.
- First order formulas in the usual hierarchy Σ^0_n, Π^0_n may contain second order parameters.
- Basic axioms:
  - $n+1 \neq 0$,
  - $n+1 = m+1 \rightarrow n = m$,
  - $m+0 = m$,
  - $m+(n+1) = (m+n)+1$,
  - $m \cdot 0 = 0$,
  - $m \cdot (n+1) = (m \cdot n)+m$,
  - $\neg m < 0$,
  - $m < n+1 \rightarrow (m < n \vee m = n)$.

- For a set of formulas $\mathcal{F}$, by $\mathcal{F}$ comprehension scheme we define the set of formulas

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

for $\varphi \in \mathcal{F}$.

- By Δ^0_1 comprehension scheme we define

$$\forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

for $\varphi \in \Sigma^0_1$, $\psi \in \Pi^0_1$.

Definition 1

By RCA_0 *we denote arithmetic containing basic axioms,* Σ^0_1 *induction and* Δ^0_1 *comprehension.*

Definition 2

By ACA_0 *we denote* RCA_0 *extended by first order comprehension.*

Definition 3

By ATR_0 *we denote* RCA_0 *extended by definitions of sets by transfinite recursion.*

Outline

Definition 4

For each ordinal $\alpha < \omega_1$ let us fixed a sequence $\{\alpha\}(x)$, for $x \in \omega$ such that

- $\{\beta + 1\}(x) = \beta$,
- $\{\alpha\}(x) \leq \{\alpha\}(y)$, *for* $x \leq y$,
- $\lim_{x\in\omega} \{alpha\}(x) = \alpha$.

Definition 5

Let λ be limit and let $\lambda_0 = \{\lambda\}(a)$, $\lambda_{i+1} = \{\lambda_i\}(a)$. By $\{\lambda\}^*(a)$ we denote the first successor ordinal in the sequence $\lambda_0, \lambda_1, \ldots$

Example

- $\{\omega\}(a) = a$,
- $\{\alpha + \beta\}(a) = \alpha + \{\beta\}(a)$,
- $\{\omega^{\alpha+1}\}(a) = \omega^{\alpha} a$,
- $\{\omega^{\lambda}\}(a) = \omega^{\{\lambda\}(a)}$.

Let h be a, possibly finite, function from $\mathbb{N}$ to $\mathbb{N}$. We define the Hardy hierarchy of functions:

- $h_0(x) = x$,
- $h_{\alpha+1}(x) = h_\alpha(h(x))$,
- $h_\lambda(x) = h_{\{\lambda\}(x)}(x) = h_{\{\lambda\}^*(x)-1}(h(x))$.

Example

Let $h(x) = x + 1$.

- $h_n(x) = h^n(x)$,
- $h_\omega(x) = h_x(x) = 2x$,
- $h_{\omega 2}(x) = h_{\omega + x}(x) = h_\omega(2x) = 2^2 x$,
- $h_{\omega^2}(x) = h_{\omega x}(x) = 2^x x$,
- $h_{\omega^\omega}(x)$ is ackermanian.

Let $X = \{x_0, \dots, x_k\}$. Let h be a successor in the sense of X:

$$h(x_i) = x_{i+1}.$$

Thus, $h(\max X)$ is undefined.

Definition 6

We say that X is α–large if $h_\alpha(\min X)$ is defined.

We say that X is exactly α–large if $h_\alpha(\min X) = \max(X)$.

Definition 7

For a given X, by $[X]^{!\alpha}$ we denote its exactly α–large subsets.

Example

Let X be a finite set.

- X is 0–large if $h_0(\min X)\downarrow$, X is nonempty,
- X is n–large if $h_n(\min X)\downarrow$, X has $n+1$ elements,
- X is ω–large if $h_\omega(\min X) = h_x(\min X)\downarrow$, X has $\min(X)+1$ elements.

Outline

1. Basic notions
   - Second order arithmetics
   - Ordinals and large sets
   - Recursion theory
2. Ramsey principle for coloring n–tuples
3. Ramsey principle for coloring α–large sets
4. Open problems

Definition 8

For a Turing machine e, by $\{e\}(x)\downarrow$ we denote the fact that e stops on the input x.

By $\{e\}_z(x)\downarrow$ we denote the fact that e stops on the input x with a computation less than z.

Definition 9

The jump of the set X is defined as

$$X' = \{e\colon \{e\}^X(0)\downarrow\}.$$

The $(n+1)$-th jump of X is defined as $X^{(n+1)} = (X^{(n)})'$.
The ω–jump of X is defined as

$$X^\omega = \{(i,j)\colon j \in X^{(i)}\}.$$

The above notions can be easily generalized to higher ordinals α's provided (recursive) fundamental sequences up to α are fixed.

Theorem 10

ACA_0 *can be characterized as* $\mathrm{RCA}_0 + \forall X\, X'$ *exists..*

Definition 11

ACA_0^+ *is* $\mathrm{RCA}_0 + \forall X\, X^\omega$ *exists..*

Outline

Outline

Theorem 12 (RT(n))

For each coloring $C\colon [\mathbb{N}]^n \to \{0,1\}$ there exists an infinite set $X \subseteq \mathbb{N}$ such that all tuples from $[X]^n$ have the same color under C.

Such an X will be called C–homogeneous.

Theorem 13 (Jockush'72)

For each $n \geq 2$ there exists a recursive coloring $C\colon [\mathbb{N}]^n \to \{0, 1\}$ such that there each C–homogeneous set computes $0^{(n)}$.

Theorem 14

The following are equivalent over RCA_0*:*

- RT(3),
- RT(n), *for any* $n \geq 3$,
- *for each X there exists jump of X,*
- ACA_0.

Theorem 15 (Cholak, Jockush and Slaman'01)

$I\Sigma^0_2 + \mathrm{RT}(2)$ *is* Π^1_1 *conservative over* $I\Sigma^0_2$.

Theorem 16 (Hirst'87)

$\mathrm{RCA}_0 + \mathrm{RT}(2)$ *proves* $B\Sigma^0_2$, *hence is not* Σ^0_3 *conservative over* RCA_0.

Theorem 17 (Liu' 12)

$\mathrm{RCA}_0 + \mathrm{RT}(2)$ *does not proves* $\mathrm{RCA}_0 + WKL_0$.

It is a long standing open problem whether $\mathrm{RCA}_0 + \mathrm{RT}(2)$ is Π^0_2 conservative over RCA_0.

Outline

1. Basic notions
2. Ramsey principle for coloring n–tuples
   - RT(n)
   - $\forall n$RT(n)
3. Ramsey principle for coloring α–large sets
4. Open problems

Let TJ(α, X) be the α-th Turing jump of X.

Theorem 18 (McAloon'85)

The following are equivalent:

- $\mathrm{RCA}_0 + \forall n \mathrm{RT}(n)$,
- $\mathrm{RCA}_0 + \forall n \forall X$ *TJ*(n, X) *exists.*

Theorem 19 (McAloon'85)

The ordinal of $\mathrm{RCA}_0 + \forall n \mathrm{RT}(n)$ *is* ε_ω.

See also PhD's by Afshari (2009) and De Smet (2011).

Outline

1. Basic notions
   - Second order arithmetics
   - Ordinals and large sets
   - Recursion theory
2. Ramsey principle for coloring n–tuples
   - RT(n)
   - $\forall n$RT(n)
3. Ramsey principle for coloring α–large sets
   - Farmaki theorem
   - Coloring ω–large sets
   - Arithmetics with truth predicates
4. Open problems

Outline

Definition 20

Let $\alpha < \omega_1$. By $\mathrm{RT}(\alpha)$ we denote the following statement:

for each infinite set $X \subseteq \mathbb{N}$, for each coloring $C\colon [X]^{!\alpha} \to \{0,1\}$

there exists an infinite set $Y \subseteq X$ such that
Y is C–homogeneous.

Theorem 21 (Pudlak and Rodl'82, see also Farmaki'98)

For each $\alpha < \omega_1$, $\mathrm{RT}(\alpha)$.

Let $\mathrm{RT}(\alpha)$ be the statement of the theorem. Assume $\mathrm{RT}(\beta)$, for $\beta < \alpha$ and let $C\colon [X]^{!\alpha} \to \{0,1\}$.

For $a \in X$ let $C_a \colon [X]^{!(\{\alpha\}^*(a)-1)} \to \{0,1\}$ defined as

$$C_a(a_1,\dots,a_k) = C(a,a_1,\dots,a_k).$$

We construct a sequence $\{(a_i, Y_i)\}_{i\in\omega}$ such that

- $Y_0 = X$,
- $a_i = \min Y_i$,
- $Y_{i+1} \subseteq Y_i$ is infinite, C_{a_i}–homogeneous, $a_i \notin Y_{i+1}$.

The sequence $\{a_i\}_{i\in\omega}$ is infinite, C–homogeneous.

The amount of induction in the above proof is extravagant – Σ^1_1.

Another proof

Let a function $f\colon \mathbb{N} \to \mathbb{N}$ code a set $\{f(i)\colon i \in \mathbb{N}\}$.

Definition 22

By Σ^0_1–RT *we denote the following scheme: for* φ *is* Σ^0_1,

$$\exists g(\forall f \varphi(g \cdot f) \vee \forall f \neg\varphi(g \cdot f)).$$

Another proof

For $f\colon \mathbb{N} \to \mathbb{N}$ let f^α be the set $\{f(0), \ldots, f(k)\}$ which is exactly α–large.
Let $C\colon [\mathbb{N}]^{!\alpha} \to \{0,1\}$. Let g be such that

$$\forall f\, C((g \cdot f)^\alpha) = 0 \vee \forall f\, C((g \cdot f)^\alpha) = 1.$$

Then, $\{g(i)\colon i \in \mathbb{N}\}$ is C–homogeneous.

The above proof can be done in ATR_0.

Theorem 23

The following are equivalent over RCA_0*:*

- ATR_0,
- Σ^0_1–RT.

While doing it in ATR_0 we should restrict ourselves to ordinals below Γ_0.

Outline

Theorem 24

The following are equivalent over RCA_0*:*

- $\mathrm{RT}(\omega)$,
- *for each X there exists $TJ(\omega, X)$.*

Assume $\mathrm{RT}(\omega)$ and let A be an arbitrary set.
For brevity we use “computable” to mean “computable in A”.

We define family of computable colorings $C_n : [\mathbf{N}]^{n+1} \to \{0,1\}$, for $n \in \mathbf{N}$ and $n \geq 2$, and Turing machines $M_n(x,y)$ such that for any $n \geq 2$,

1. All infinite homogeneous sets for C_n have color 1.
2. If X is an infinite homogeneous set for C_n then for any for any $a_1 < \cdots < a_{n+1} \in X$ it holds that if a is a code for a sequence $(a_1, \ldots, a_{n+1})$ then $M_n(x,a)$ decides $0^{(n-1)}$ for machines with indices less than or equal to a_1.
3. Machines M_n are total. If their inputs are not from an infinite homogeneous set for C_n then we have no guarantee on the correctness of their output.

We define C_2 as

$$C_2(k, y, z) = \begin{cases} 1 & \text{if } \forall e \leq k(\{e\}_y^A(0){\downarrow} \Leftrightarrow \{e\}_z^A(0){\downarrow}) \\ 0 & \text{otherwise.} \end{cases}$$

Now, $M_2(e, (k, b, b'))$ searches for a computation of e below b, provided that $e \leq k$.

We define $C_{n+1}(a_1, \ldots, a_{n+2})$ as

$$C_{n+1}(\ldots) = \begin{cases} 1 & \text{if } \{a_1, \ldots, a_{n+2}\} \text{ is } C_n\text{–homogeneous and} \\ & \text{and } \forall e \leq a_1(\{e\}^Y_{a_2}(0)\downarrow \Leftrightarrow \{e\}^Y_{a_3}(0)\downarrow), \text{where} \\ & Y = \{i \leq a_2 : M_n(i, (a_2, \ldots, a_{n+2})) \text{ accepts,}\} \\ 0 & \text{otherwise.} \end{cases}$$

We would like to replace the condition in the second line of the above definition by

$$\forall e \leq a_1(\{e\}_{a_2}^{A^{(n-1)}}(0)\downarrow \Leftrightarrow \{e\}_{a_3}^{A^{(n-1)}}(0)\downarrow.$$

We use approximations of these sets computed by machines M_n.

For each $a_1 < a_2$ from an infinite C_{n+1}–homogeneous set and for all $e < a_1$ we have

$$\{e\}_{a_1}^{A^{(n-1)}}(0)\downarrow \Leftrightarrow \{e\}_{a_2}^{A^{(n-1)}}(0)\downarrow$$

and consequently, by infinity of a given C_{n+1}–homogeneous set,

$$\{e\}_{a_1}^{A^{(n-1)}}(0)\downarrow \Leftrightarrow \{e\}^{A^{(n-1)}}(0)\downarrow.$$

$M_{n+1}(e, (a_1, \ldots, a_{n+2}))$ computes firstly the set

$$Y = \{i \leq a_2 : M_n(i, (a_2, \ldots, a_{n+1})) \text{ accepts}\}.$$

Then, it checks whether $\{e\}^{Y}_{a_2}(0){\downarrow}$ and if this holds, M_{n+1} accepts.

Finally, we define C_ω as follows.

$$C_\omega(a_1,\dots,a_k) = C_{a_1}(a_1,\dots,a_k).$$

For a sequence $a = (a_1,\dots,a_k)$, we define

$$M_\omega(e,a) = M_{a_1}(e,a).$$

If a comes from an infinite C_ω–homogeneous set, then $M_\omega(x,a)$ decides $\mathrm{TJ}(a_1, A)$ for machines up to a_1.

For the direction from the existence of $\mathrm{TJ}(\omega, X)$ to $\mathrm{RT}(\omega)$ one needs a lemma.

Lemma 25

Let $a \geq 1$. Let $C\colon [U]^a \to 2$. One can find effectively a machine f_a with oracle $(C \otimes U)^{(2a)}$ such that f_a computes a C–homogeneous set.

With some uniformity of a inductive construction one may replace all oracles by one $\mathrm{TJ}(\omega, C_\omega)$, for a given $C_\omega\colon [\mathbb{N}]^{!\omega} \to \{0, 1\}$.

Regressive colorings

Definition 26

A coloring C is regressive if for every $S \subseteq \mathbf{N}$ of the appropriate type, $C(S) < \min(S)$, whenever $\min(S) > 0$.

Definition 27

By $\mathrm{KM}(n)$ we denote the statement that for every regressive coloring of n–tuples $\mathbb{N}$ there exists a infinite homogeneous subset.

By $\mathrm{KM}(!\omega)$ we denote the statement that for every regressive coloring of ω–large subsets of $\mathbb{N}$ there exists a infinite homogeneous subset.

Proposition 28

Over RCA_0, $\mathrm{KM}(!\omega)$ *and* $\mathrm{RT}(!\omega)$ *are equivalent.*

It is easy to reduce $\mathrm{KM}(d)$ to $\mathrm{RT}(d+1)$. Having ω–large sets we have a lot of finite tuples at our disposal.

Outline

For an ordinal α, let L_α be a language of PA extended by predicates $\mathrm{Tr}_\beta(x)$, for $\beta < \alpha$.

Let Tarski_β be a theory stating that $\mathrm{Tr}_\beta(x)$ is a truth predicate for L_β:

- $\mathrm{Tr}_\beta(\varphi) \leftrightarrow \varphi$, for each atomic $\varphi \in L_\beta$,
- $\forall \ulcorner \varphi \urcorner \ (\mathrm{Tr}_\beta(\ulcorner \neg\varphi \urcorner) \leftrightarrow \neg \mathrm{Tr}_\beta(\ulcorner \varphi \urcorner))$,
- $\forall \ulcorner \varphi \urcorner \forall \ulcorner \psi \urcorner \ (\mathrm{Tr}_\beta(\ulcorner \varphi \wedge \psi \urcorner) \leftrightarrow \mathrm{Tr}_\beta(\ulcorner \varphi \urcorner) \wedge \mathrm{Tr}_\beta(\ulcorner \psi \urcorner))$,
- $\forall \ulcorner x \urcorner \forall \ulcorner \varphi \urcorner (\mathrm{Tr}_\beta(\ulcorner \exists x \varphi \urcorner) \leftrightarrow \exists a \mathrm{Tr}_\beta(\ulcorner \varphi(a) \urcorner))$.

Let $\mathrm{PA}(L_\alpha)$ be PA with full induction in L_α and Tarski_β, for $\beta < \alpha$.

Theorem 29 (Kotlarski–Ratajczyk)

The set of arithmetical consequences of $PA(L_1)$ is axiomatized by the scheme of transfinite induction up to $\varepsilon_{\varepsilon_0}$.
The ordinal of $PA(L_1)$ is $\varepsilon_{\varepsilon_0}$.

One of the last Kotlarski's article on Pudlak's principle up to Γ_0 was intended, among other things, as a preparatory work for generalizing the above theorem to arithmetic with Γ_0 truth predicates.

Theorem 30

The following theories are equivalent over the language of PA,

- $\mathrm{RCA}_0 + \forall X$ *there exists* $TJ(\omega, X)$,
- $PA(L_\omega)$.

Let $M \models \mathrm{PA}(L_\omega)$ and let $\mathcal{F}_i$ be the family of sets Δ^0_1 definable in the language L_i.

$$(M, \bigcup_{i\in\omega} \mathcal{F}_i) \models \mathrm{RCA}_0 + \forall X \text{ there exists } \mathrm{TJ}(\omega, X).$$

If $\varphi_1, \ldots, \varphi_n$ is a proof in $\mathrm{PA}(L_{i+1})$ then we can replace the use of $\mathrm{Tr}_0, \ldots, \mathrm{Tr}_i$ by $\emptyset^\omega, \emptyset^{\omega 2}, \ldots, \emptyset^{\omega(i-1)}$.

The proof theoretic ordinal of $\mathrm{ACA}_0 + \forall X$ there exists $\mathrm{TJ}(\omega, X)$ is the limit of the sequence

$$\varepsilon_0, \varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots$$

Outline

A very plausible conjecture

Conjecture 31

For each ordinal $\alpha < \varepsilon_0$ $(< \Gamma_0, \dots)$*, the following are equivalent*

- $\mathsf{RCA}_0 + \mathsf{RT}(\alpha)$,
- $\mathsf{RCA}_0 + \forall X$ *there exists* $TJ(\alpha, X)$.

Let us note, that we do not have a correspondence with $\mathsf{PA}(L_\alpha)$ since $\mathsf{RT}(\omega + 1)$ is equivalent to $\mathsf{RT}(\omega)$ while $\mathsf{PA}(L_{\omega+1})$ is stronger than $\mathsf{PA}(L_\omega)$.

Operations on well orderings

Various second order arithmetics may be characterized in terms of well ordering preserving operations, e.g.,

- $\forall X(\mathrm{WO}(X) \implies \mathrm{WO}(\omega^X))$ (equivalent to ACA_0),
- $\forall X(\mathrm{WO}(X) \implies \mathrm{WO}(\omega^X))$ (equivalent to ACA_0^+).

It would be good to prove these principles from corresponding Ramsey principles.

Thank you.