Automorphisms and cofinal extensions

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Three papers with Henryk

- *Results on automorphisms of recursively saturated models of arithmetic*, Fundamenta Mathematicae, vol 129, pp. 9-15, 1988.
- *On extending automorphisms of models of Peano Arithmetic*, Fundamenta Mathematicae, vol. 149, pp. 245-263, 1996.
- *More on extending automorphisms of recursively saturated models of PA,* Fundamenta Mathematicae 200, pp. 133-143, 2008.

All models in this talk are countable recursively saturated models of PA

For $X \subseteq M$, let $\mathfrak{A}(X) = |\{f(X) : f \in \mathrm{Aut}(M)\}|$.

Let $M \prec N$. Then $\mathrm{Cod}(N/M) = \{a \cap M : a \in N\}$.

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Counting automorphic images

- (Krajewski) Let $S \subseteq M$ be a partial inductive satisfaction class. Then $\mathfrak{A}(S) = 2^{\aleph_0}$
- (RK, Kotlarski) If $M \prec_{\text{end}} N$ and $X \in \text{Cod}(N/M) \setminus \text{Def}(M)$, then $\mathfrak{A}(X) = 2^{\aleph_0}$.
- (Schmerl) If $X \in \text{Class}(M) \setminus \text{Def}(M)$, then $\mathfrak{A}(X) = 2^{\aleph_0}$.

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The question

If $M \prec N$ and $f \in \text{Aut}(M)$, does f extend to N, i.e. is there a $g \in \text{Aut}(N)$ such that $f \subseteq g$?

Extending to a given end extension

- (RK, Kotlarski) If $M \prec_{\text{end}} N$, then $2^{\aleph_0}$ automorphisms of M do not extend to N.

- (RK) For every M there is an N such that $M \prec_{\text{end}} N$ and identity is the only automorphism of M that extends to N.

- (Schmerl) Let $\mathfrak{A}$ be a countable linearly ordered structure. For every M there is an N such that $M \prec_{\text{end}} N$ and

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Extendibility

- Let $f \in \mathrm{Aut}(M)$ be given. Is there an N such that $M \prec_{\mathrm{end}} N$ and f extends to N? Could there be an f that is not extendible to any elementary end extension?
- If there is a partial inductive satisfaction class S such that $f \in \mathrm{Aut}(M, S)$, then there is an N such that $M \prec_{\mathrm{end}} N$ and f extends to N.
- If M is arithmetically saturated then there are $f \in \mathrm{Aut}(M)$ such that $f \notin \mathrm{Aut}(M, S)$ for all partial inductive satisfaction classes S.

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A condition for extendibility

- (RK, Kotlarski) If $M \prec_{\text{end}} N$ and there are no $a \in N$ coding decreasing infinite sequences such that $M = \inf\{(a)_i : i < \omega\}$ then every $f \in \text{Aut}(M, \text{Cod}(N/M))$ extends to N.

- (RK) If $M \prec_{\text{end}} N$ and M is strong in N, then for every $f \in \text{Aut}(M, \text{Cod}(N/M))$, there is a $g \in \text{Aut}(N)$ such that $f \subseteq g$ and $\text{fix}(f) = \text{fix}(g)$.

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Description Property I

Definition

The extension $M \prec_{\text{cof}} N$ has the description property if for every $a \in N \setminus M$ there is a coded in N nested sequence $\langle A_i : i < \omega \rangle$ of M-finite sets such that

1. $N \models a \in A_i$ for all $i < \omega$;
2. For each M-finite B such that $a \in B$, there is an $i < \omega$ such that $A_i \subseteq B$.

Theorem

(RK, Kotlarski) For each M, there are K and, N such that $K \prec_{\text{cof}} M \prec_{\text{cof}} N$ and both extensions have the description property.

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Description Property II

Question

For a given M is there an N such that $M \prec_{\mathrm{cof}} N$ has the description property and $\mathrm{SSy}(M) = \mathrm{SSy}(N)$?

Theorem

(RK, Kotlarski) If $M \prec_{\mathrm{cof}} N$, $f \in \mathrm{Aut}(M, \mathrm{Cod}(N/M))$, and the extension $M \prec_{\mathrm{cof}} N$ has the the description property, then f extends to N.

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Smoryński Stavi Theorem

Theorem

If M is recursively saturated M and N are countable and $M \prec_{\mathrm{cof}} N$, then $M \cong N$ iff $\mathrm{SSy}(M) = \mathrm{SSy}(N)$.

Gaps

Definition

For $a \in M$

$$\mathrm{gap}(a) = \bigcap\{K \prec_{\mathrm{end}} M : a \in K\} \setminus \bigcup\{K \prec_{\mathrm{end}} M : a \notin K\}$$

Theorem

(Moving Gaps Lemma) For each $a \in M$ there are (cofinally many) b such that for all $c \in \mathrm{gap}(b)$, $a \in \mathrm{Scl}(c)$.

Corollary

For every proper extension $M \prec_{\mathrm{cof}} N$ there (cofinally many) are new gaps, i.e. there are cofinally many $c \in N$ such that $\mathrm{gap}(c) \cap M = \emptyset$.

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Isolated gaps

Definition

If $M \prec_{\mathrm{cof}} N$ *and* $b \in N \setminus M$, *then* $\mathrm{gap}^N(b)$ *is non-isolated if there are* $d < \mathrm{gap}(b) < e \in N$ *such that* $[d, e] \cap M = \emptyset$

Theorem

(RK, Kotlarski) Every cofinal extension has non-isolated gaps.

Theorem

(RK, Kotlarski) No extension with the description property has isolated gaps.

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Are there cofinal extensions with isolated gaps?

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Are there cofinal extensions with isolated gaps?

Conservative cofinal extensions?

Definition

For $M \prec_{\text{cof}} N$ and $b \in N \setminus M$, let $M_b = \sup([0,b] \cap M)$.

Definition

An extension $M \prec_{\text{cof}} N$ is conservative if, for each $b \in N \setminus M$ there is $a \in M$ such that $b \cap M_b = a \cap M_b$.

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Do conservative cofinal extensions exist?

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Diversity in cofinal extensions (RK, Schmerl 2012)

Lemma

There is a $\mathcal{L}_{\mathrm{PA}} \cup \{M\}$-sentence σ such that for all $M \prec_{\mathrm{cof}} N$ iff $(N, M) \models \sigma$.

Dowód.

$\sigma = \exists x \forall y \exists z \in M(y = (x)_z)$. □

Lemma

Suppose that $M \prec_{\mathrm{cof}} N \models \mathrm{PA}$. Then, $\mathrm{Lt}_0(N/M)$ is interpretable in (N, M).

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In (N, M), the relation $R = \{\langle x, y \rangle \in N : M(x) \prec M(y)\}$ is definable by the formula $\forall u \in M \exists v \in M[(u)_x = (v)_y]$. □

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Diversity in cofinal submodels

- There are K_α, $\alpha < 2^{\aleph_0}$ such that $K_\alpha \prec_{\text{cof}} M$ and if $\alpha \neq \beta$, then $\text{SSy}(K_\alpha) \neq \text{SSy}(K_\beta)$.

- (Smoryński) There are K_α, $\alpha < 2^{\aleph_0}$ such that $K_\alpha \prec_{\text{cof}} M$, $K_\alpha \cong M$, and if $\alpha \neq \beta$, then $\text{Th}(\text{GCIS}(M, K_\alpha)) \neq \text{Th}(\text{GCIS}(M, K_\beta))$; hence $\text{Th}(M, K_\alpha) \neq \text{Th}(M, K_\beta)$.

- (RK, Schmerl) For every $J \subseteq_{\text{end}} M$ that is closed under exponentiation, there are K_α, $\alpha < 2^{\aleph_0}$ such that $K_\alpha \prec_{\text{cof}} M$, for all α, $\text{GCIS}(M, K_\alpha) = J$, and if $\alpha \neq \beta$, then $\text{Th}(M, K_\alpha) \neq \text{Th}(M, K_\beta)$.

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