Well-Ordering Principles, Omega & Beta Models

Michael Rathjen

Department of Pure Mathematics, University of Leeds
West Yorkshire

Model Theory and Proof Theory of Arithmetic

A Memorial Conference in Honour of
Henryk Kotlarski and Zygmunt Ratajczyk

Będlewo, 23 July 2012

Aims

Aims

To present a general proof-theoretic machinery for investigating statements about well-orderings from a reverse mathematics point of view.

Aims

To present a general proof-theoretic machinery for investigating statements about well-orderings from a reverse mathematics point of view.

These statements are of the form **WOP**(f)

"if** X **is well ordered then** $f(X)$ **is well ordered"

where f is a standard proof theoretic function from ordinals to ordinals.

Aims

To present a general proof-theoretic machinery for investigating statements about well-orderings from a reverse mathematics point of view.

These statements are of the form **WOP**(f)

"if X is well ordered then $f(X)$ is well ordered"

where f is a standard proof theoretic function from ordinals to ordinals.

There are by now several examples of functions f where the statement **WOP**(f) has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$).

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic Comprehension

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic Comprehension
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic Comprehension
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA_0}$ the following are equivalent:

1. Arithmetic Comprehension
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

Slogan: **one quantifier = one exponential**

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic Comprehension
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

Slogan: **one quantifier = one exponential**

$2^{\mathfrak{X}}$ is effectively computable from $\mathfrak{X}$.

$2^{\mathfrak{X}}$ *and Arithmetic Comprehension*

$$2^{\mathfrak{X}} \quad := \quad (|2^X|, <_{2^X})$$

Theorem: (Girard 1987) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic Comprehension
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

Slogan: **one quantifier = one exponential**

$2^{\mathfrak{X}}$ is effectively computable from $\mathfrak{X}$.

Abstract property WO of real object $2^{\mathfrak{X}}$ versus existence of abstract sets **ACA**.

The theory ACA_0^+

The theory $\mathbf{ACA}_0^+$

$\mathbf{ACA}_0^+$ is $\mathbf{ACA}_0$ plus the axiom

$$\forall X\,\exists Y\,[(Y)_0 = X \wedge \forall n\,(Y)_{n+1} = \mathrm{jump}((Y)_n)].$$

The theory $\mathbf{ACA}_0^+$

$\mathbf{ACA}_0^+$ is $\mathbf{ACA}_0$ plus the axiom

$$\forall X \, \exists Y \, [(Y)_0 = X \wedge \forall n \, (Y)_{n+1} = \mathrm{jump}((Y)_n)].$$

The theory $\mathbf{ACA}_0^+$

$\mathbf{ACA}_0^+$ is $\mathbf{ACA}_0$ plus the axiom

$$\forall X \, \exists Y \, [(Y)_0 = X \wedge \forall n \, (Y)_{n+1} = \text{jump}((Y)_n)].$$

- Hindman's Theorem and the Auslander/Ellis theorem are provable in $\mathbf{ACA}_0^+$.

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

$\varepsilon_{\mathfrak{X}}$ and $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ACA}_0^+$

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ACA}_0^+$
2. $\forall \mathfrak{X}\ [\mathsf{WO}(\mathfrak{X}) \rightarrow \mathsf{WO}(\varepsilon_{\mathfrak{X}})]$.

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ACA}_0^+$
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varepsilon_{\mathfrak{X}})]$.

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ACA}_0^+$
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varepsilon_{\mathfrak{X}})]$.

- A. Marcone, A. Montalbán: *The epsilon function for computability theorists*, draft, 2007.

$\varepsilon_{\mathfrak{X}}$ *and* $\mathbf{ACA}_0^+$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ACA}_0^+$
2. $\forall \mathfrak{X}\, [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varepsilon_{\mathfrak{X}})]$.

- A. Marcone, A. Montalbán: *The epsilon function for computability theorists*, draft, 2007.
- B. Afshari, M. Rathjen: *Reverse Mathematics and Well-ordering Principles: A pilot study*, APAL 160 (2009) 231-237.

The ordering $<_{\varepsilon_{\mathfrak{X}}}$

Let $\mathfrak{X} = \langle X, <_X \rangle$ be an ordering where $X \subseteq \mathbb{N}$.
$<_{\varepsilon_{\mathfrak{X}}}$ and its field $|\varepsilon_{\mathfrak{X}}|$ are inductively defined as follows:

1. $0 \in |\varepsilon_{\mathfrak{X}}|$.
2. $\varepsilon_u \in |\varepsilon_{\mathfrak{X}}|$ for every $u \in X$, where $\varepsilon_u := \langle 0, u \rangle$.
3. If $\alpha_1, \ldots, \alpha_n \in |\varepsilon_{\mathfrak{X}}|$, $n > 1$ and $\alpha_n \leq_{\varepsilon_{\mathfrak{X}}} \ldots \leq_{\varepsilon_{\mathfrak{X}}} \alpha_1$, then

   $$\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$$

   where $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} := \langle 1, \langle \alpha_1, \ldots, \alpha_n \rangle \rangle$.
4. If $\alpha \in |\varepsilon_{\mathfrak{X}}|$ and α is not of the form ε_u, then $\omega^{\alpha} \in |\varepsilon_{\mathfrak{X}}|$, where $\omega^{\alpha} := \langle 2, \alpha \rangle$.

1. $0 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ for all $u \in X$.
2. $0 <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ for all $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$.
3. $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \varepsilon_v$ if $u, v \in X$ and $u <_X v$.
4. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$ and $\alpha_1 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ then $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$.
5. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$, and $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \alpha_1$ or $\varepsilon_u = \alpha_1$, then $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$.
6. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ and $\omega^{\beta_1} + \ldots + \omega^{\beta_m} \in |\varepsilon_{\mathfrak{X}}|$, then

$$\begin{aligned} &\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \omega^{\beta_1} + \ldots + \omega^{\beta_m} \text{ iff} \\ &n < m \wedge \forall i \leq n\ \alpha_i = \beta_i \quad \text{or} \\ &\exists i \leq \min(n, m)[\alpha_i <_{\varepsilon_{\mathfrak{X}}} \beta_i \wedge \forall j < i\ \alpha_j = \beta_j]. \end{aligned}$$

Let $\varepsilon_{\mathfrak{X}} = \langle |\varepsilon_{\mathfrak{X}}|, <_{\varepsilon_{\mathfrak{X}}} \rangle$.

O. Veblen, 1908

Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:

O. Veblen, 1908

Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**

O. Veblen, 1908

Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**
  - **Transfinite Iteration**

O. Veblen, 1908

Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**
  - **Transfinite Iteration**
- Let **ON** be the class of ordinals. A (class) function $f : \mathbf{ON} \rightarrow \mathbf{ON}$ is said to be **increasing** if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and **continuous** (in the order topology on **ON**) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$.

Derivations

- f is called **normal** if it is increasing and continuous.

Derivations

- f is called **normal** if it is increasing and continuous.
- The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at ω since $\lim_{\xi<\omega}(\xi + \omega) = \omega$ but $(\lim_{\xi<\omega} \xi) + \omega = \omega + \omega$.

Derivations

- f is called **normal** if it is increasing and continuous.
- The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at ω since $\lim_{\xi<\omega}(\xi + \omega) = \omega$ but $(\lim_{\xi<\omega} \xi) + \omega = \omega + \omega$.
- The **derivative** f' of a function $f : \mathbf{ON} \to \mathbf{ON}$ is the function which enumerates in increasing order the solutions of the equation

$$f(\alpha) = \alpha,$$

also called the **fixed points** of f.

Derivations

- f is called **normal** if it is increasing and continuous.
- The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at ω since $\lim_{\xi<\omega}(\xi + \omega) = \omega$ but $(\lim_{\xi<\omega} \xi) + \omega = \omega + \omega$.
- The **derivative** f' of a function $f : \mathbf{ON} \to \mathbf{ON}$ is the function which enumerates in increasing order the solutions of the equation

$$f(\alpha) = \alpha,$$

  also called the **fixed points** of f.
- If f is a normal function,

$$\{\alpha : f(\alpha) = \alpha\}$$

  is a proper class and f' will be a normal function, too.

A Hierarchy of Ordinal Functions

- Given a normal function $f : \mathbf{ON} \rightarrow \mathbf{ON}$, define a hierarchy of normal functions as follows:

A Hierarchy of Ordinal Functions

- Given a normal function $f : \mathbf{ON} \rightarrow \mathbf{ON}$, define a hierarchy of normal functions as follows:
- $f_0 = f$

A Hierarchy of Ordinal Functions

- Given a normal function $f : \mathbf{ON} \rightarrow \mathbf{ON}$, define a hierarchy of normal functions as follows:
- $f_0 \;=\; f$
- $f_{\alpha+1} \;=\; {f_\alpha}'$

A Hierarchy of Ordinal Functions

- Given a normal function $f : \mathbf{ON} \rightarrow \mathbf{ON}$, define a hierarchy of normal functions as follows:
- $f_0 \;=\; f$
- $f_{\alpha+1} \;=\; f_\alpha{}'$
- $$f_\lambda(\xi) \;=\; \xi^{th} \text{ element of } \bigcap_{\alpha<\lambda} \{\text{Fixed points of } f_\alpha\} \qquad \text{for } \lambda \text{ limit.}$$

The Feferman-Schütte Ordinal Γ_0

- From the normal function f we get a two-place function,

$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

We are interested in the hierarchy with starting function

$$f = \ell, \qquad \ell(\alpha) = \omega^\alpha.$$

The Feferman-Schütte Ordinal Γ_0

- From the normal function f we get a two-place function,

$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

We are interested in the hierarchy with starting function

$$f = \ell, \qquad \ell(\alpha) = \omega^\alpha.$$

- The least ordinal $\gamma > 0$ closed under φ_ℓ, i.e. the least ordinal > 0 satisfying

$$(\forall \alpha, \beta < \gamma)\ \varphi_\ell(\alpha, \beta) < \gamma$$

is the famous ordinal Γ_0 which Feferman and Schütte determined to be the least ordinal 'unreachable' by **predicative means**.

ATR_0 *and* $\varphi\mathfrak{X}0$

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

- Friedman's proof uses computability theory and also some proof theory. Among other things it uses a result which states that if $P \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that

$\mathbf{ATR_0}$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

- Friedman's proof uses computability theory and also some proof theory. Among other things it uses a result which states that if $P \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that
  - for every n, A_{n+1} is the unique set such that $P(A_n, A_{n+1})$,

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: (Friedman, unpublished) Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall \mathfrak{X}\, [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

- Friedman's proof uses computability theory and also some proof theory. Among other things it uses a result which states that if $P \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that
  - for every n, A_{n+1} is the unique set such that $P(A_n, A_{n+1})$,
  - for every n, $A'_{n+1} \leq_T A_n$.

ATR_0 *and* $\varphi\mathfrak{X}0$

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\,[\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\ [\mathsf{WO}(\mathfrak{X}) \to \mathsf{WO}(\varphi\mathfrak{X}0)]$.

- A. Marcone, A. Montalbán: *The Veblen function for computability theorists*, JSL 76 (2011) 575–602.

$\mathbf{ATR}_0$ *and* $\varphi\mathfrak{X}0$

Theorem: Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\mathbf{ATR}_0$
2. $\forall\mathfrak{X}\ [\mathsf{WO}(\mathfrak{X}) \rightarrow \mathsf{WO}(\varphi\mathfrak{X}0)]$.

- A. Marcone, A. Montalbán: *The Veblen function for computability theorists*, JSL 76 (2011) 575–602.
- M. Rathjen, A. Weiermann, *Reverse mathematics and well-ordering principles*, Computability in Context: Computation and Logic in the Real World (S. B. Cooper and A. Sorbi, eds.) (Imperial College Press, 2011) 351–370.

Countable coded ω-models

Countable coded ω-models

- An ω-model of a theory T in the language of second order arithmetic is one where the first order part is standard.

Countable coded ω-models

- An ω-model of a theory T in the language of second order arithmetic is one where the first order part is standard.
- Such a model is isomorphic to one of the form

$$\mathfrak{M} = (\mathbb{N}, \mathfrak{X}, 0, 1, +, \times, \in)$$

with $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$.

Countable coded ω-models

- An ω-model of a theory T in the language of second order arithmetic is one where the first order part is standard.
- Such a model is isomorphic to one of the form

$$\mathfrak{M} = (\mathbb{N}, \mathfrak{X}, 0, 1, +, \times, \in)$$

  with $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$.
- **Definition.** $\mathfrak{M}$ is a countable coded ω-model of T if

$$\mathfrak{X} = \{(C)_n \mid n \in \mathbb{N}\}$$

  for some $C \subseteq \mathbb{N}$ where $(C)_n = \{k \mid 2^n 3^k \in C\}$.

Characterizing theories in terms of countable coded ω-models

Characterizing theories in terms of countable coded ω-models

Theorem ($\mathbf{RCA}_0$)

$\mathbf{ACA}_0^+$ is equivalent to the statement that every set is contained in a countable coded ω-model of **ACA**.

Characterizing theories in terms of countable coded ω-models

Theorem ($\mathbf{RCA}_0$)

$\mathbf{ACA}_0^+$ is equivalent to the statement that every set is contained in a countable coded ω-model of **ACA**.

Theorem ($\mathbf{ACA}_0$)

$\mathbf{ATR}_0$ is equivalent to the statement that every set is contained in a countable coded ω-model of Δ^1_1-CA.

A Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

A Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X} \, [\mathsf{WO}(\mathfrak{X}) \rightarrow \mathsf{WO}(\Gamma_{\mathfrak{X}})]$

A Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X}\ [\mathsf{WO}(\mathfrak{X}) \rightarrow \mathsf{WO}(\Gamma_{\mathfrak{X}})]$
2. Every set is contained in an ω-model of **ATR.**

A Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X}\ [\mathsf{WO}(\mathfrak{X}) \rightarrow \mathsf{WO}(\Gamma_{\mathfrak{X}})]$
2. Every set is contained in an ω-model of **ATR.**

To appear in: *Foundational Adventures*, Proceedings in honor of Harvey Friedman's 60th birthday.

Some famous theories of the 1960s and 1970s

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathsf{WF}(\prec_X) \rightarrow \mathrm{TI}(\prec_X, F)\,]$$

where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathsf{WF}(\prec_X) \rightarrow \mathrm{TI}(\prec_X, F)\,]$$

where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathrm{WF}(\prec_X) \to \mathrm{TI}(\prec_X, F)\,]$$

  where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

  Theorem. The following theories have the same proof-theoretic strength:

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathsf{WF}(\prec_X) \to \mathrm{TI}(\prec_X, F)\,]$$

  where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

  Theorem. The following theories have the same proof-theoretic strength:

  1. The theory of positive arithmetic inductive definitions $\mathbf{ID}_1$.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathsf{WF}(\prec_X) \to \mathrm{TI}(\prec_X, F)\,]$$

  where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

  Theorem. The following theories have the same proof-theoretic strength:

  1. The theory of positive arithmetic inductive definitions $\mathbf{ID}_1$.
  2. Kripke-Platek set theory, **KP**.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\, [\, \mathsf{WF}(\prec_X) \to \mathrm{TI}(\prec_X, F)\,]$$

where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

Theorem. The following theories have the same proof-theoretic strength:

1. The theory of positive arithmetic inductive definitions $\mathbf{ID}_1$.
2. Kripke-Platek set theory, **KP**.
3. **BI**.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X \, [\, \mathsf{WF}(\prec_X) \to \mathrm{TI}(\prec_X, F) \,]$$

  where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

  Theorem. The following theories have the same proof-theoretic strength:

  1. The theory of positive arithmetic inductive definitions $\mathbf{ID}_1$.
  2. Kripke-Platek set theory, **KP**.
  3. **BI**.
  4. $\mathbf{ACA}_0$ + parameter-free $\Pi^1_1 - \mathrm{CA}$.

Some famous theories of the 1960s and 1970s

- Every set $X \subseteq \mathbb{N}$ gives rise to a binary relation $\prec_X$ via $n \prec_X m$ iff $2^n 3^m \in X$.
- Let BI be the schema

$$\forall X\,[\,\mathsf{WF}(\prec_X) \rightarrow \mathrm{TI}(\prec_X, F)\,]$$

  where $F(x)$ is an arbitrary formula of $\mathcal{L}_2$.
- Let **BI** be the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

  Theorem. The following theories have the same proof-theoretic strength:

  1. The theory of positive arithmetic inductive definitions $\mathbf{ID}_1$.
  2. Kripke-Platek set theory, **KP**.
  3. **BI**.
  4. $\mathbf{ACA}_0$ + parameter-free $\Pi^1_1 - \mathrm{CA}$.
- Their proof-theoretic ordinal is the Howard-Bachmann ordinal.

The Big Veblen Number

- Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

$$\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta),$$

only a finite number of the arguments

$$\alpha_\nu$$

may be non-zero.

The Big Veblen Number

- Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

$$\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta),$$

only a finite number of the arguments

$$\alpha_\nu$$

may be non-zero.

- Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

$$\Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$$

with $\eta < \delta$, and each $\alpha_\gamma < \delta$.

The Big Leap: H. Bachmann 1950

- Bachmann's novel idea: Use **uncountable ordinals** to keep track of the functions defined by diagonalization.

The Big Leap: H. Bachmann 1950

- Bachmann's novel idea: Use **uncountable ordinals** to keep track of the functions defined by diagonalization.
- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$.

The Big Leap: H. Bachmann 1950

- Bachmann's novel idea: Use **uncountable ordinals** to keep track of the functions defined by diagonalization.
- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi<\tau_\lambda} \lambda[\xi] = \lambda$.
- Let Ω be the first uncountable ordinal. A hierarchy of functions $(\varphi^{\mathfrak{B}}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

$$\varphi^{\mathfrak{B}}_0(\beta) = 1 + \beta \qquad \varphi^{\mathfrak{B}}_{\alpha+1} = \left(\varphi^{\mathfrak{B}}_\alpha\right)'$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \bigcap_{\xi<\tau_\lambda} (\text{Range of } \varphi^{\mathfrak{B}}_{\lambda[\xi]}) \qquad \lambda \text{ limit}, \tau_\lambda < \Omega$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \{\beta < \Omega : \varphi^{\mathfrak{B}}_{\lambda[\beta]}(0) = \beta\} \qquad \lambda \text{ limit}, \tau_\lambda = \Omega.$$

The Howard-Bachmann ordinal

The Howard-Bachmann ordinal

Let Ω be a "big" ordinal. By recursion on α we define sets $C_\Omega(\alpha)$ and the ordinal $\psi_\Omega(\alpha)$ as follows:

$$C_\Omega(\alpha) \quad = \quad \begin{cases} \textit{closure of } \{0, \Omega\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega(\xi))_{\xi<\alpha} \end{cases} \tag{1}$$

$$\psi_\Omega(\alpha) \quad \simeq \quad \min\{\rho < \Omega : \ \rho \notin C_\Omega(\alpha)\}. \tag{2}$$

The Howard-Bachmann ordinal

Let Ω be a "big" ordinal. By recursion on α we define sets $C_\Omega(\alpha)$ and the ordinal $\psi_\Omega(\alpha)$ as follows:

$$C_\Omega(\alpha) \quad = \quad \begin{cases} \textit{closure of } \{0, \Omega\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega(\xi))_{\xi<\alpha} \end{cases} \tag{1}$$

$$\psi_\Omega(\alpha) \quad \simeq \quad \min\{\rho < \Omega : \ \rho \notin C_\Omega(\alpha)\}. \tag{2}$$

The **Howard-Bachmann ordinal** is $\psi_\Omega(\varepsilon_{\Omega+1})$, where $\varepsilon_{\Omega+1}$ is the next ε-number after Ω.

How to relativize the Howard-Bachmann ordinal?

How to relativize the Howard-Bachmann ordinal?

- Let $\mathfrak{X}$ be a well-ordering.

How to relativize the Howard-Bachmann ordinal?

- Let $\mathfrak{X}$ be a well-ordering.
- **Idea 1**: Define $C_\Omega^{\mathfrak{X}}(\alpha)$ by adding ε-numbers $\mathfrak{E}_u$ **BELOW** Ω for every $u \in |\mathfrak{X}|$:

$$C_\Omega^{\mathfrak{X}}(\alpha) \quad = \quad \begin{cases} \textit{closure of } \{0, \Omega\} \cup \{\mathfrak{E}_u \mid u \in |\mathfrak{X}|\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega^{\mathfrak{X}}(\xi))_{\xi<\alpha} \end{cases} \tag{3}$$

$$\psi_\Omega^{\mathfrak{X}}(\alpha) \quad \simeq \quad \min\{\rho < \Omega : \ \rho \notin C_\Omega^{\mathfrak{X}}(\alpha)\}. \tag{4}$$

How to relativize the Howard-Bachmann ordinal?

How to relativize the Howard-Bachmann ordinal?

- **Idea 2**: Define $C^{\mathfrak{X}}_{\Omega}(\alpha)$ by adding ε-numbers $\mathfrak{E}_u$ **ABOVE** Ω for every $u \in |\mathfrak{X}|$:

$$C^{\mathfrak{X}}_{\Omega}(\alpha) = \begin{cases} \textit{closure of } \{0, \Omega\} \cup \{\mathfrak{E}_u \mid u \in |\mathfrak{X}|\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \longmapsto \psi^{\mathfrak{X}}_{\Omega}(\xi))_{\xi<\alpha} \end{cases} \tag{5}$$

$$\psi^{\mathfrak{X}}_{\Omega}(\alpha) \simeq \min\{\rho < \Omega : \rho \notin C^{\mathfrak{X}}_{\Omega}(\alpha)\}. \tag{6}$$

How to relativize the Howard-Bachmann ordinal?

- **Idea 2**: Define $C_\Omega^{\mathfrak{X}}(\alpha)$ by adding ε-numbers $\mathfrak{E}_u$ **ABOVE** Ω for every $u \in |\mathfrak{X}|$:

$$C_\Omega^{\mathfrak{X}}(\alpha) \quad = \quad \begin{cases} \textit{closure of } \{0, \Omega\} \cup \{\mathfrak{E}_u \mid u \in |\mathfrak{X}|\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega^{\mathfrak{X}}(\xi))_{\xi<\alpha} \end{cases} \tag{5}$$

$$\psi_\Omega^{\mathfrak{X}}(\alpha) \quad \simeq \quad \min\{\rho < \Omega : \ \rho \notin C_\Omega^{\mathfrak{X}}(\alpha)\}. \tag{6}$$

- Let $\psi_\Omega^{\mathfrak{X}}$ be $\psi_\Omega^{\mathfrak{X}}(*)$, where $* = \sup\{\mathfrak{E}_u \mid u \in |\mathfrak{X}|\}$.

Another Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

Another Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\psi_\Omega^{\mathfrak{X}})]$.

Another Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\psi_{\Omega}^{\mathfrak{X}})]$.
2. *Every set is contained in a countable coded ω-model of* **BI**.

Another Theorem

Over $\mathbf{RCA}_0$ the following are equivalent:

1. $\forall \mathfrak{X}$ [WO($\mathfrak{X}$) $\rightarrow$ WO($\psi_{\Omega}^{\mathfrak{X}}$)].
2. *Every set is contained in a countable coded ω-model of* **BI**.

Joint work with Pedro Francisco Valencia Vizcaino.

History of proving completeness via search trees

History of proving completeness via search trees

An extremely elegant and efficient proof procedure for first order logic consists in producing the search or decomposition tree (in German “Stammbaum") of a given formula. It proceeds by decomposing the formula according to its logical structure and amounts to applying logical rules backwards. This decomposition method has been employed by Schütte (1956) to prove the completeness theorem. It is closely related to the method of “semantic tableaux" of Beth (1959) and methods of Hintikka (1955). Ultimately, the whole idea derives from Gentzen (1935).

The decomposition tree method can also be extended to prove the ω-completeness theorem due to Henkin (1954) and Orey (1956). Schütte (1951) used it to prove ω-completeness in the arithmetical case.

Prospectus

Prospectus

A statement of the form **WOP**(f) is Π^1_2 and therefore cannot be equivalent to a theory whose axioms have a higher complexity, like for instance Π^1_1-comprehension.

Prospectus

A statement of the form **WOP**(f) is Π^1_2 and therefore cannot be equivalent to a theory whose axioms have a higher complexity, like for instance Π^1_1-comprehension.

After ω-models come β-models.

Prospectus

A statement of the form **WOP**(f) is Π^1_2 and therefore cannot be equivalent to a theory whose axioms have a higher complexity, like for instance Π^1_1-comprehension.

After ω-models come β-models.

The question arises whether the methodology of this paper can be extended to more complex axiom systems, in particular to those characterizable via β-models?

First of all, to get equivalences one has to climb up in the type structure. Given a functor

$$F : (\mathbb{LO} \to \mathbb{LO}) \to (\mathbb{LO} \to \mathbb{LO}),$$

where $\mathbb{LO}$ is the class of linear orderings, we consider the statement:

$$\textbf{WOPP}(F): \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO})\ [\textbf{WOP}(f) \to \textbf{WOP}(F(f))].$$

First of all, to get equivalences one has to climb up in the type structure. Given a functor

$$F : (\mathbb{LO} \to \mathbb{LO}) \to (\mathbb{LO} \to \mathbb{LO}),$$

where $\mathbb{LO}$ is the class of linear orderings, we consider the statement:

$$\mathbf{WOPP}(F) : \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO}) \, [\mathbf{WOP}(f) \to \mathbf{WOP}(F(f))].$$

There is also a variant of **WOPP**(F) which should basically encapsulate the same “power”. Given a functor

$$G : (\mathbb{LO} \to \mathbb{LO}) \to \mathbb{LO}$$

consider the statement:

$$\mathbf{WOPP}_1(G) : \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO}) \, [\mathbf{WOP}(f) \to \mathrm{WO}(G(f))].$$

Conjecture

Conjecture

Statements of the form

$$\textbf{WOPP}(F)$$

(or $\textbf{WOPP}_1(F)$), where F comes from some ordinal ordinal representation system used for an ordinal analysis of a theory T_F, are equivalent to statements of the form

"every set belongs to a countable coded β-model of T_F".

Conjecture

Statements of the form

$$\mathbf{WOPP}(F)$$

(or $\mathbf{WOPP}_1(F)$), where F comes from some ordinal ordinal representation system used for an ordinal analysis of a theory T_F, are equivalent to statements of the form

"every set belongs to a countable coded β-model of T_F".

The conjecture may be a bit vague, but it has been corroborated in some cases (around Π^1_1-**CA**), and, what is perhaps more important, the proof technology exhibited in this paper seems to be sufficiently malleable as to be applicable to the extended scenario of β-models, too.

β-models I

β-models I

Every ω-model $\mathfrak{M}$ of a theory T in the language of second order arithmetic is isomorphic to a structure

$$\mathfrak{A} = \langle \omega; \mathfrak{X}; 0, +, \times, \ldots \rangle$$

where $\mathfrak{X} \subseteq \mathcal{P}(\omega)$.

β-models I

Every ω-model $\mathfrak{M}$ of a theory T in the language of second order arithmetic is isomorphic to a structure

$$\mathfrak{A} = \langle \omega; \mathfrak{X}; 0, +, \times, \ldots \rangle$$

where $\mathfrak{X} \subseteq \mathcal{P}(\omega)$.

Definition:

$\mathfrak{A}$ is a **β-model** *if the concept of well ordering is absolute with respect to $\mathfrak{A}$, i.e. for all $X \in \mathfrak{X}$,*

$$\mathfrak{A} \models WO(<_X) \quad \text{iff} \quad <_X \text{ is a well ordering.}$$

β-models I

Every ω-model $\mathfrak{M}$ of a theory T in the language of second order arithmetic is isomorphic to a structure

$$\mathfrak{A} = \langle \omega; \mathfrak{X}; 0, +, \times, \ldots \rangle$$

where $\mathfrak{X} \subseteq \mathcal{P}(\omega)$.

Definition:

$\mathfrak{A}$ is a **β-model** *if the concept of well ordering is absolute with respect to $\mathfrak{A}$, i.e. for all $X \in \mathfrak{X}$,*

$$\mathfrak{A} \models WO(<_X) \quad \text{iff} \quad <_X \text{ is a well ordering.}$$

- $n <_X m :\Rightarrow 2^n 3^m \in X$.

Mostowski's question

Mostowski's question

Is there a "syntactical" rule which characterizes validity in all β-models?

Mostowski's question

Is there a "syntactical" rule which characterizes validity in all β-models?

$T \models_\beta F$ iff F holds in all β-models of T.

The End