A NATURAL NONSTANDARD COUNTERPART OF RCA_0

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Abstract. Our open problem is to find a *more natural* nonstandard counterpart of RCA_0, the well-known base theory of Reverse Mathematics ([3]). We discuss why the existing systems ([1, 2, 4]) are unsatisfactory and propose a natural system ourselves.

Nonstandard RCA_0

Keisler and Yokoyama have introduced a number of nonstandard counterparts of RCA_0; See e.g. [1, 2, 4]. These systems fall into two[1] categories:

(1) The correspondence[2] between standard and nonstandard numbers is *weak* and the comprehension principle[3] is a *natural nonstandard principle*.
(2) The correspondence between standard and nonstandard numbers is *strong* and the comprehension principle is not a *natural nonstandard principle*.

An example of the first category is Keisler's system ${}^*\mathrm{RCA}_0'$, which has the same comprehension principle STP as ${}^*\mathrm{WKL}_0$ ([1, 2]), stating that every nonstandard number codes a set of standard numbers. However, ${}^*\mathrm{RCA}_0'$ cannot even prove bounded induction on the nonstandard segment, whereas the standard segment satisfies $I\Sigma_1$. Moreover, ${}^*\mathrm{RCA}_0'$ cannot even prove very elementary overspill. An example of the second category is Keisler's ${}^*\mathrm{RCA}_0$ ([1]), which does prove e.g. overspill, but the comprehension principle, called Δ^0_1-STP, is clearly similar to that of RCA_0, and not a natural principle of nonstandard arithmetic.

Ω-invariance

We are interested in a system which combines an intermediate correspondence between the standard and the nonstandard numbers and an natural nonstandard comprehension principle. For the latter, we propose Ω-INV-CA, i.e. comprehension[4] for every Ω-invariant formula. The notion of Ω-invariance is defined as follows.

1. **Definition** (Ω-invariance)**.** *Let $\psi(n,m)$ be standard[5] and bounded, and fix $\omega \in \Omega := {}^*\mathbb{N} \setminus \mathbb{N}$. Then ${}^*\psi(n,\omega)$ is Ω-invariant if*

$$(\forall n \in \mathbb{N})(\forall \omega' \in \Omega)\big[{}^*\psi(n,\omega) \leftrightarrow {}^*\psi(n,\omega')\big]. \tag{1}$$

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[1] Note that the terms 'weak', 'natural' and 'strong' are used informally here, and relative to the context of RCA_0.

[2] By 'correspondence', we mean principles connecting the standard and nonstandard universe, such as overspill and transfer.

[3] In nonstandard arithmetic, the comprehension principle may be given in terms of which infinite numbers code sets of natural numbers.

[4] Here, Γ-CA is the statement that for every formula $\psi \in \Gamma$, we have $(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \psi(n))$. The collection '$\Omega$-INV' consists of all Ω-invariant formulas.

[5] We denote the set of standard numbers by $\mathbb{N}$ and the nonstandard extension of $\mathbb{N}$ by ${}^*\mathbb{N}$.

Note that $\psi(n, \omega)$ from (1) is nonstandard, but independent of the choice of the infinite $\omega \in \Omega$. This notion is inspired by the practice of Nonstandard Analysis in Physics and Applied Mathematics, where calculations are independent of the *choice* of the infinitesimal used. This is a natural property, as end results of calculations with physical meaning should be independent of the choice of calculus tool, in casu the infinitesimal used in the calculation.

We do not know exactly which correspondence there should be between the standard and nonstandard numbers, but [2, Proposition 6.11] signals caution. We believe overspill limited to Ω-invariant formulas to be a good candidate.

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References

[1] H. Jerome Keisler, *Nonstandard arithmetic and reverse mathematics*, Bull. Symbolic Logic **12** (2006), no. 1, 100–125.

[2] ———, *Nonstandard arithmetic and recursive comprehension*, Ann. Pure Appl. Logic **161** (2010), no. 8, 1047–1062.

[3] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

[4] Keita Yokoyama, *Non-standard analysis within second-order arithmetic*, Lecture presented at the Conference on Computability,Reverse Mathematics, and Combinatorics in Banff (2008).