Nonstandard Analysis: a new way to compute

Sam Sanders[1]

Model Theory and Proof Theory of Arithmetic

A Memorial Conference in Honour of H. Kotlarski and Z. Ratajczyk

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UNIVERSITEIT GENT

TOHOKU UNIVERSITY

[1]This research is generously supported by the John Templeton Foundation.

Take-home message

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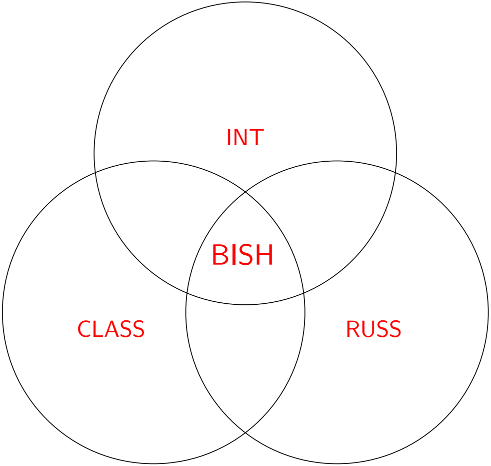
Most results from CRM (= RM based on BISH) translate to NSA under a natural translation $\mathbb{B}$.

Son of a…



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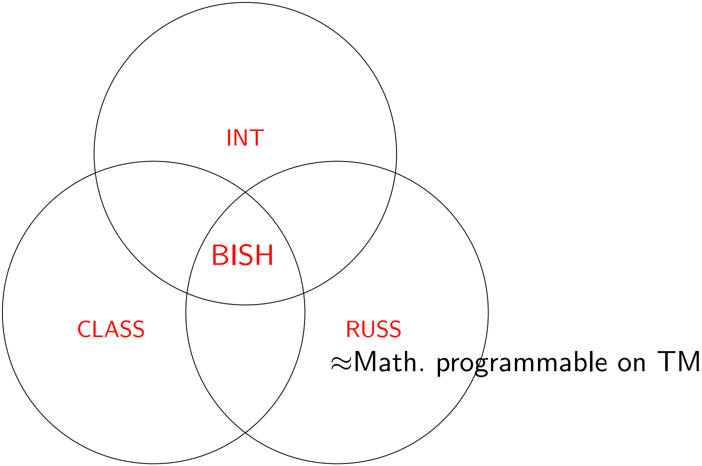

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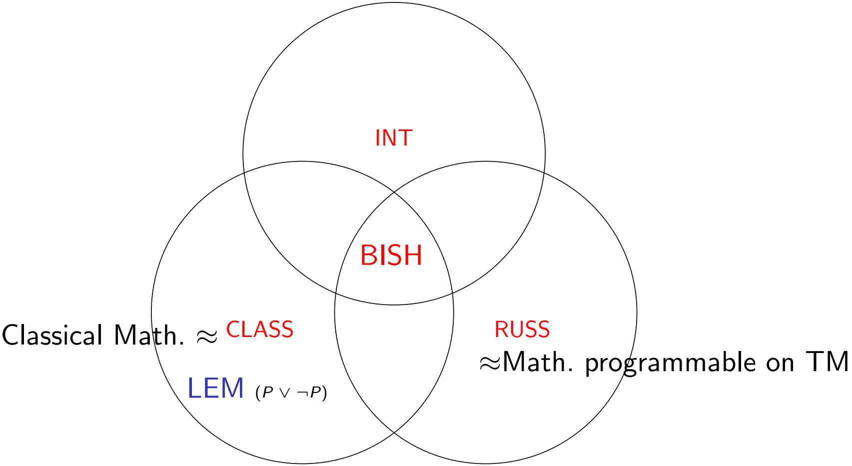
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RUSS
≈Math. programmable on TM
LEM (P ∨ ¬P)

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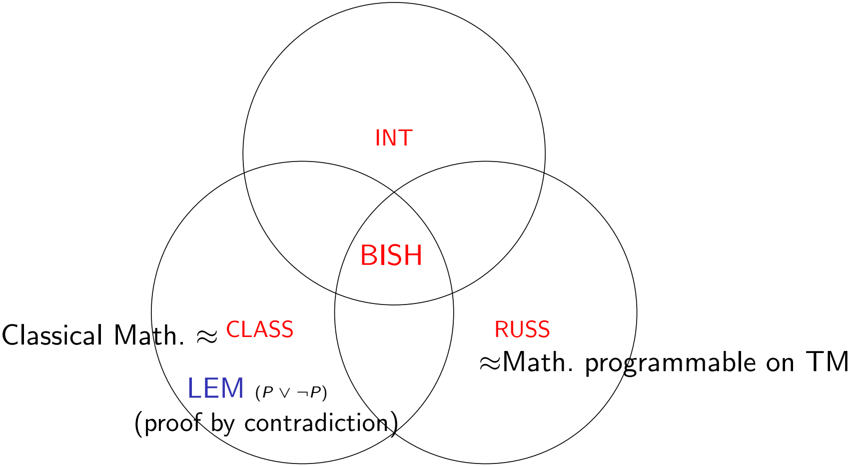
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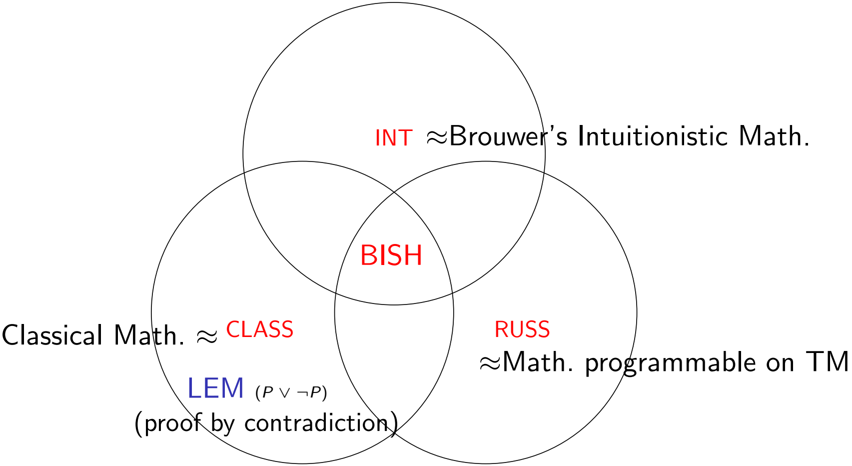
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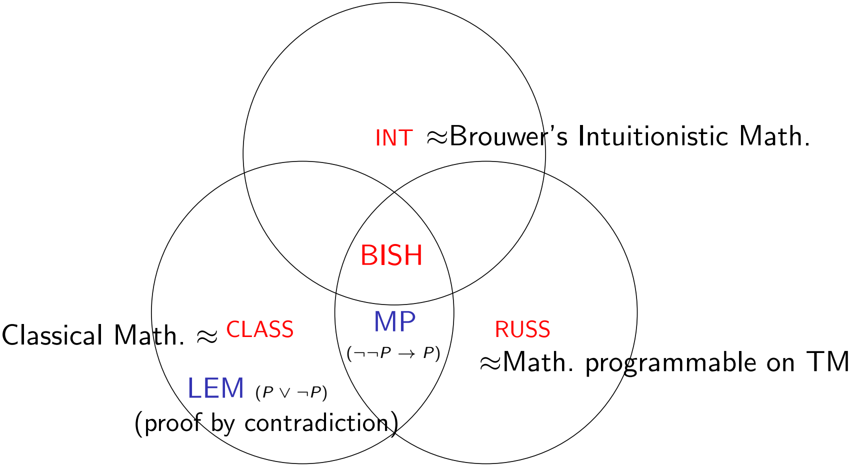
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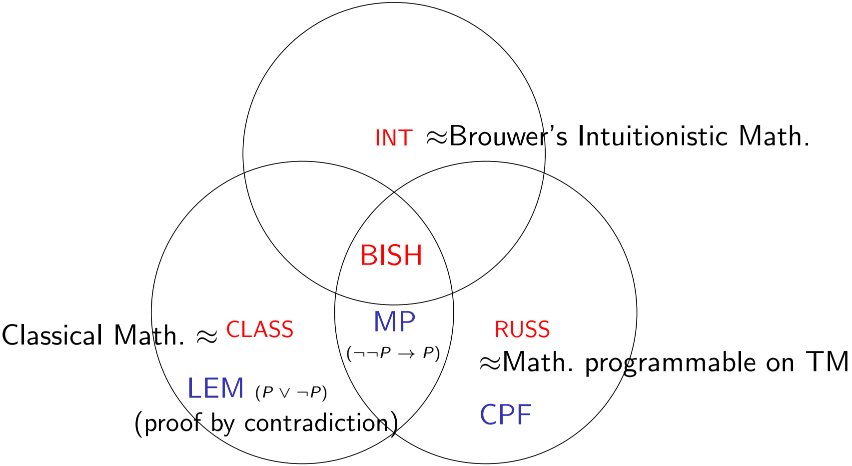

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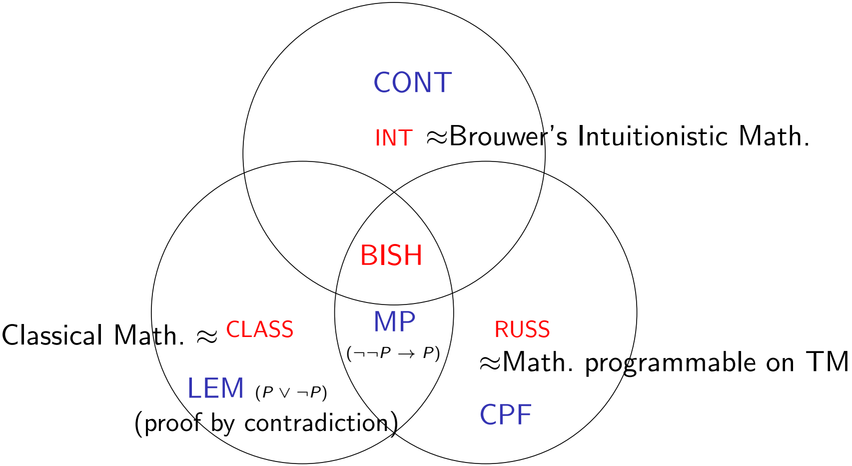
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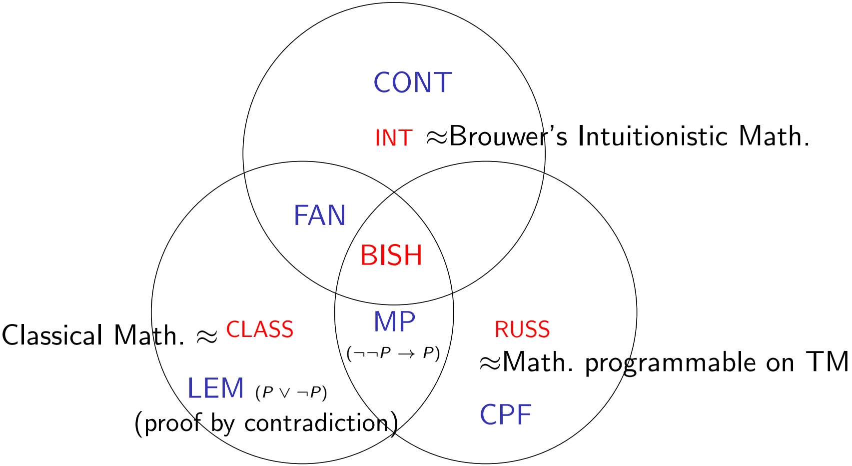

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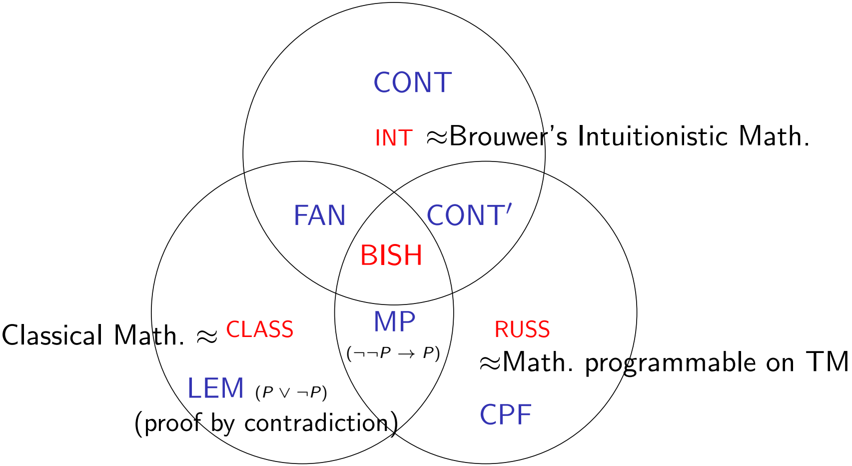

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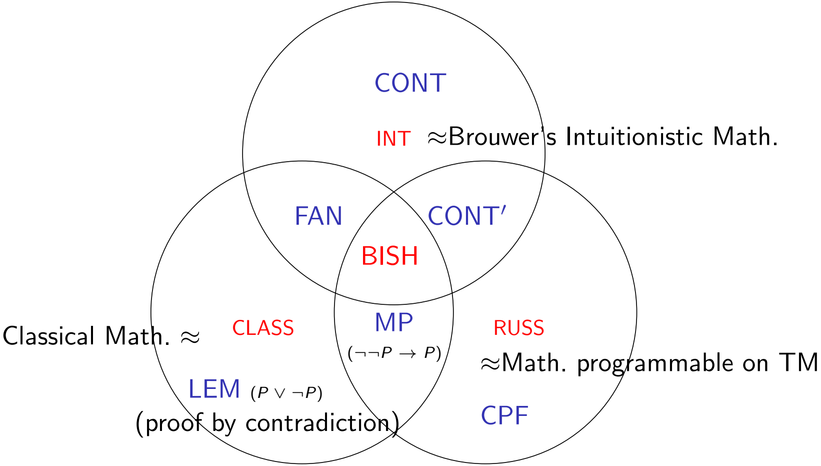

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6. $(\forall x \in A) P(x)$: for all x, $x \in A \to P(x)$.

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3. Levels of infinity (Stratified NSA).

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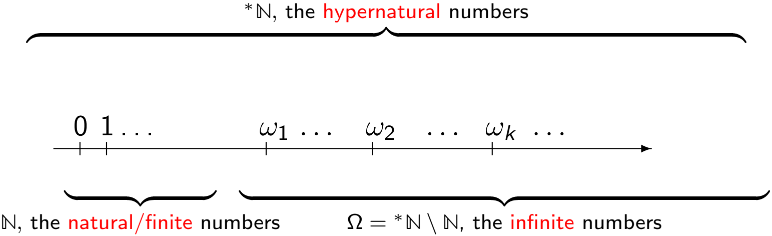

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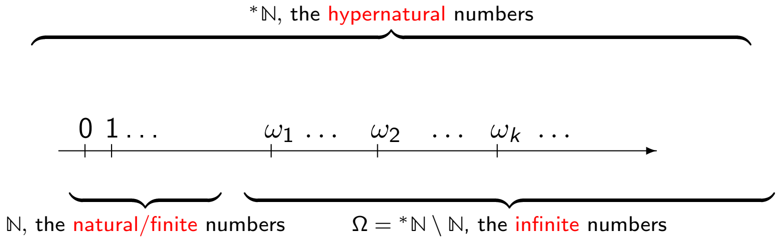

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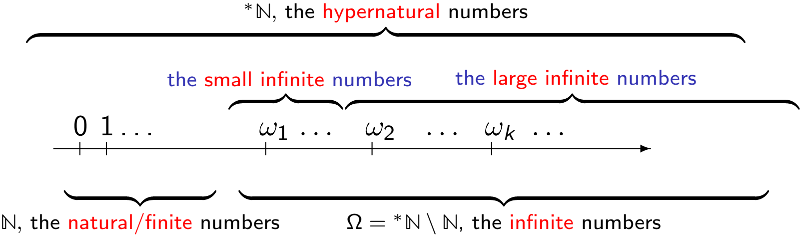

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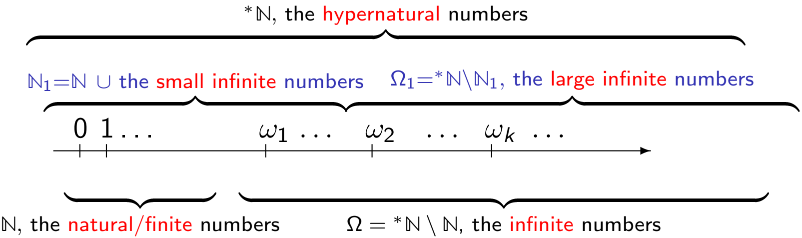

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$\mathbb{NSA}$ has Ω-CA instead of Δ_1-CA.

Principle (Ω-CA)

For all Ω-invariant $\psi(n, \omega)$, we have

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \psi(n, \omega)).$$

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$$\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$$

$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$

$\sim A$: $A \Rrightarrow (0 = 1)$

$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x}, \omega)$ s.t. $\psi(\vec{x}, \omega) \rightarrow [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x}, \omega) \rightarrow [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \rightarrow B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \rightarrow \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \rightarrow (0 = 1)$	$\sim A$: $A \Rrightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"
	$\sim[(\forall n \in \mathbb{N})A(n)]$

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbin{\mathbb{V}} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \to (0=1)$	$\sim A$: $A \Rrightarrow (0=1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

$\sim[(\forall n \in \mathbb{N})A(n)] \equiv (\exists n \in \mathbb{N}_1)\sim A(n)$

WEAKER than $(\exists n \in \mathbb{N})\sim A(n)$.

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)

Central: algorithm and proof

$A \vee B$: an algo yields a proof of A or of B

$A \to B$: an algo converts a proof of A to a proof of B

$\neg A$: $A \to (0 = 1)$

$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$

$\neg[(\forall n \in \mathbb{N})A(n)]$ is WEAKER than $(\exists n \in \mathbb{N})\neg A(n)$.

$\mathbb{NSA}$ (based on CL)

Central: Ω-invariance and Transfer ($\mathbb{T}$)

$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x}, \omega)$ s.t.

$$\psi(\vec{x}, \omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$$
$$\wedge$$
$$\neg\psi(\vec{x}, \omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$$

$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$

$\sim A$: $A \Rrightarrow (0 = 1)$

$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

$\sim[(\forall n \in \mathbb{N})A(n)] \equiv (\exists n \in \mathbb{N}_1)\sim A(n)$

WEAKER than $(\exists n \in \mathbb{N})\sim A(n)$.

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbin{\mathbb{V}} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \to (0 = 1)$	$\sim A$: $A \Rrightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

We know: If BISH $\vdash X$ then $X \not\to$ LPO, LLPO, MP, ... (princ. rejected in BISH)

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \to (0 = 1)$	$\sim A$: $A \Rrightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

We know: If BISH $\vdash X$ then $X \not\to$ LPO, LLPO, MP, ... (princ. rejected in BISH)

We show: If $\mathbb{NSA} \vdash Y$ then $Y \not\Rrightarrow \mathbb{LPO}, \mathbb{LLPO}, \mathbb{MP}, \dots$

The translation $\mathbb{B}$ from BISH to $\mathbb{NSA}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x}, \omega)$ s.t. $\psi(\vec{x}, \omega) \rightarrow [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x}, \omega) \rightarrow [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \rightarrow B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \rightarrow \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \rightarrow (0 = 1)$	$\sim A$: $A \Rrightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

We know: If BISH $\vdash X$ then $X \not\rightarrow$ LPO, LLPO, MP, ... (princ. rejected in BISH)

We show: If $\mathbb{NSA} \vdash Y$ then $Y \not\Rrightarrow \mathbb{LPO}, \mathbb{LLPO}, \mathbb{MP}, \ldots$ (e.g. $\mathbb{LPO}$ is $\mathbb{B}$(LPO), unprovable in $\mathbb{NSA}$

Constructive Reverse Mathematics under $\mathbb{B}$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK) | $\mathbb{NSA}$ (based on CL)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK) | $\mathbb{NSA}$ (based on CL)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	
$\updownarrow$	
CIT: Cantor intersection thm	

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim}P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} {\sim}(x > 0))$

$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

$\Longleftrightarrow$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$	$\Longleftrightarrow$
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)		$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic		non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$		$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbin{\mathbb{V}} \sim P$
$\updownarrow$		$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$		$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbin{\mathbb{V}} \sim(x > 0))$
$\updownarrow$		$\Longleftrightarrow$
MCT: monotone convergence thm		$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm		$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
(point in intersection computed by algo)	

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo) $\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

(point in intersection computed by algo)

(point in intersection computed by Ω-inv. proc.)

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$ (limit computed by algo)

CIT: Cantor intersection thm

$\Longleftrightarrow$

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$

$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

$\Longleftrightarrow$ (limit computed by Ω-inv. proc.)

$\mathbb{CIT}$: Cantor intersection thm

$\Longleftrightarrow$

Universal Transfer: For all $\varphi \in \Delta_0$

$(\forall n \in \mathbb{N})\varphi(n) \rightarrow (\forall n \in {}^*\mathbb{N})\varphi(n)$

Constructive Reverse Mathematics under $\mathbb{B}$

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim (x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo) $\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
	$\Longleftrightarrow$
	Universal Transfer: For all $\varphi \in \Delta_0$
	$(\forall n \in \mathbb{N})\varphi(n) \rightarrow (\forall n \in {}^*\mathbb{N})\varphi(n)$

$\mathbb{NSA}$ does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \Rrightarrow (x > 0) \mathbb{V} (x < \delta)\big]$.

BISH does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \rightarrow (x > 0) \vee (x < \delta)\big]$.

Constructive Reverse Mathematics under $\mathbb{B}$ II

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \Vvee \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \Vvee x \leq 0)$

$\Longleftrightarrow$

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$

$\Longleftrightarrow$

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem
(int. value computed by algo)

$\mathbb{NSA}$ (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \veebar \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \veebar x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \veebar y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK) non-constructive/non-algorithmic	$\mathbb{NSA}$ (based on CL) non-Ω-invariant
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\Longleftrightarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\Longleftrightarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem (int. value computed by algo)	$\mathbb{IVT}$: Intermediate value theorem (int. value computed by Ω-inv. proc.)

Constructive Reverse Mathematics under $\mathbb{B}$ II

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NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \,\mathbb{V}\, y = 0)$
$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem	$\mathbb{IVT}$: Intermediate value theorem
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WKL	$\Longleftrightarrow$ $\mathbb{WKL}$

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WKL	$\Longleftrightarrow$ $\mathbb{WKL}$ $\Longleftrightarrow$ $\mathbb{V}$-Transfer

Constructive Reverse Mathematics under $\mathbb{B}$ II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

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WKL

Axioms of $\mathbb{R}$: $\neg(x > 0 \wedge x < 0)$

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non-Ω-invariant

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$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem (int. value computed by algo)	$\mathbb{IVT}$: Intermediate value theorem (int. value computed by Ω-inv. proc.)
$\updownarrow$ WKL	$\Longleftrightarrow$ $\mathbb{WKL}$ $\Longleftrightarrow$ $\vee$-Transfer
Axioms of $\mathbb{R}$: $\neg(x > 0 \wedge x < 0)$	Axioms of $\mathbb{R}$: $\sim(x > 0 \wedge x < 0)$

Constructive Reverse Mathematics under $\mathbb{B}$ III

Constructive Reverse Mathematics under $\mathbb{B}$ III

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics under $\mathbb{B}$ III

BISH (based on BHK)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ III

BISH (based on BHK)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics under $\mathbb{B}$ III

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non-constructive/non-algorithmic MP: For $P \in \Sigma_1$, $\neg\neg P \to P$ $\updownarrow$ MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$ $\updownarrow$ EXT: the extensionality theorem	

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MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rightarrow P$
$\updownarrow$	$\Longleftrightarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rightarrow x > 0)$
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WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$	
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(Four Remarks)

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A formula ψ is Δ_1 if $\psi \iff (\exists n \in \mathbb{N})\varphi_1(n) \iff (\forall m \in \mathbb{N})\varphi_2(m)$.

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Fannying about: FAN_Δ vs WKL

FAN_Δ (Every detachable bar is uniform) is accepted in INT.

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$\mathbb{WKL}$ $(\forall n \in \mathbb{N})(\exists \alpha \in 2^{\mathbb{N}})(\overline{\alpha}n \in T) \Rightarrow (\exists \alpha \in 2^{\mathbb{N}})(\forall n \in \mathbb{N})(\overline{\alpha}n \in T)$

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In $\mathbb{NSA}$, we have $\mathbb{WKL} \Rightarrow \mathbb{FAN}_\Delta$.

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Recall that '$A \in \mathbb{T}$' means 'A satisfies Transfer'.

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$\text{LLPO} \rightarrow \text{MP}^\vee$	$\mathbb{LLPO} \Rrightarrow \mathbb{MP}^\vee$
$\text{LPO} \rightarrow \text{BD-N}$	$\mathbb{LPO} \Rrightarrow \mathbb{BD}\text{-}\mathbb{N}$
$\text{LLPO} \rightarrow \text{FAN}_\Delta$	$\mathbb{LLPO} \Rrightarrow \mathbb{FAN}_\Delta$
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Reverse-engineering Reverse Mathematics (Fuchino-sensei)

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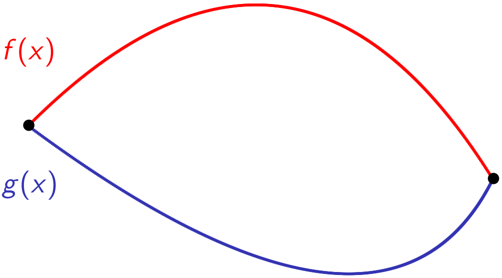
Homotopy:



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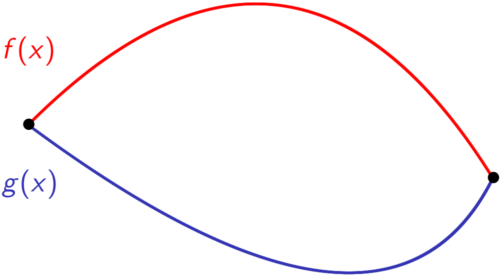

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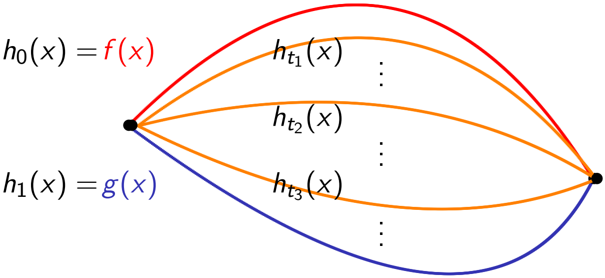

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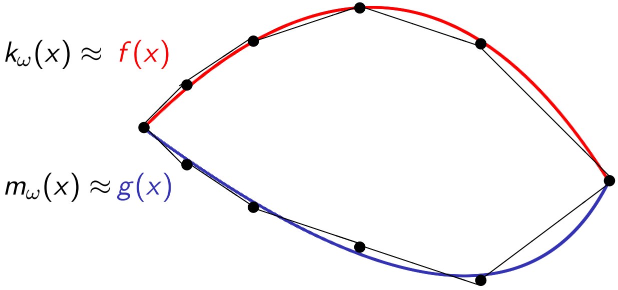

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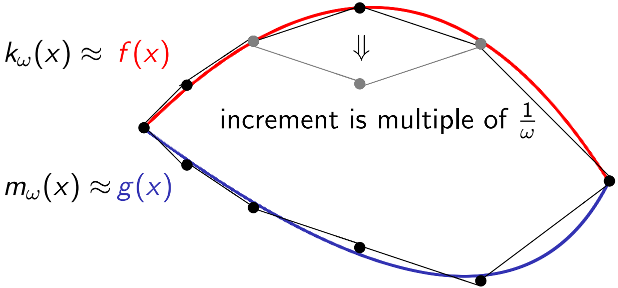

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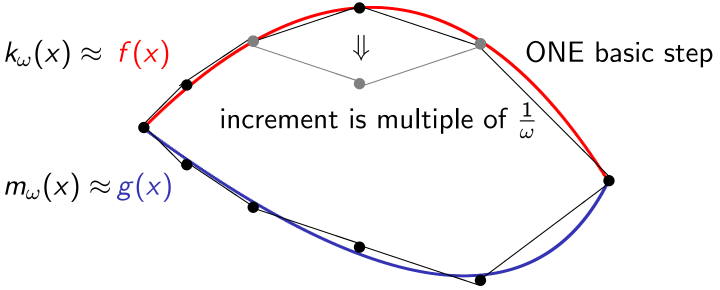

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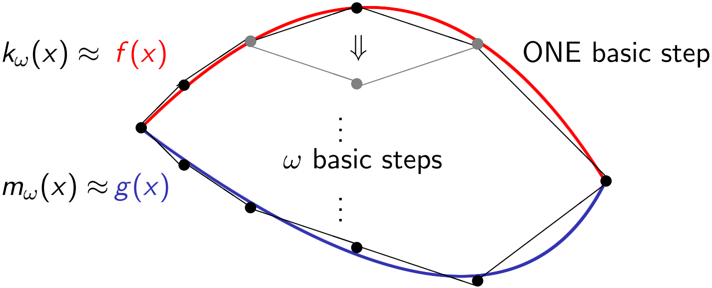

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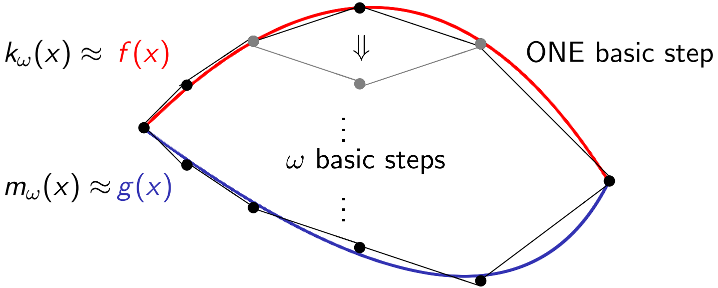

Independent of the choice of ω

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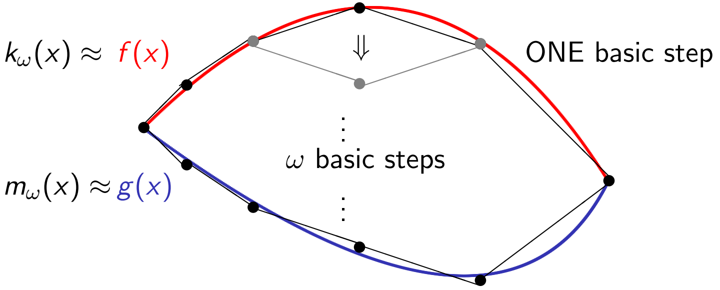
Homotopy: $\approx$ Ω-invariant broken-line transformation $h_{\omega,t}$ of f to g.



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A mathematical result with physical meaning will not depend on the choice of infinite number/infinitesimal used, i.e. it is Ω-invariant. (Alain Connes)

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When life gives you lemons... you make Ω-invariance:

Arithmetic is about a computationally robust variety of structures.

Despite Tennenbaum's Theorem, one can define computability/constructivity via Ω-invariance in each nonstandard model of arithmetic.

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Any questions?

Take-home message

In Nonstandard Analysis, an algorithm is any object whose definition is independent of the choice of infinitesimal (Ω-invariance).

More technically, we define a translation between Constructive Analysis (BISH) and Nonstandard Analysis (NSA):

(Proof and Algorithm) in BISH = (Transfer and Ω-invariance) in NSA

Most results from CRM (= RM based on BISH) translate to NSA via a natural translation $\mathbb{B}$.