PEANO DOWNSTAIRS

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Reduction Relations

Two Groups of Theories

The Theory PA^{-}

Cut-Interpretability

The $\Sigma_{1,n}$-Hierarchy

Peano Downstairs and Peano Cellar

Universiteit Utrecht

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Reduction Relations

- $V \rhd U$ iff, there is a K with $K : V \rhd U$.
  This relation is *interpretability*.
- $V \rhd_{\mathrm{mod}} U$ iff, for all models $\mathcal{M}$ of V, there is an translation τ such that $\widetilde{\tau}(\mathcal{M})$ is a model of U.
  This relation is *model interpretability*.
- $V \rhd_{\mathrm{loc}} U$ iff, for all finitely axiomatized subtheories U_0 of U, $V \rhd U_0$.
  This relation is *local interpretability*.

Fact: Suppose A is finitely axiomatized. We have:

$$U \rhd A \quad \Leftrightarrow \quad U \rhd_{\mathrm{mod}} A.$$

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Finitely Axiomatized Sequential Theories

S^1_2, EA, $I\Sigma_1$, ACA_0, GB.

Let A be a consistent, finitely axiomatized, sequential theory and let $N : S^1_2 \lhd A$.

- There is a Σ_1-sentence S such that $A \rhd (A + S^N)$ and $A \rhd (A + \neg S^N)$.
- Suppose $A \vdash \mathrm{Supexp}^N$. Then the interpretability logic of A w.r.t. N is ILP.
  $\vdash \phi \rhd \psi \to \Box(\phi \rhd \psi)$.
- There is a Σ_1-sound $M : S^1_2 \lhd A$.

Essentially Reflexive Sequential Theories

PA, ZF and their extensions in the same language.

U is *essentially reflexive* (w.r.t. $N : \mathrm{PA}^- \lhd U$) iff U proves the full uniform reflection principle for predicate logic in the signature of U. This implies full induction w.r.t. N. If U is sequential, full induction w.r.t. N implies full uniform reflection.

Let U be consistent, sequential and essentially reflexive w.r.t. N.

- There is a Δ_2-sentence B such that $U \rhd (A + B^N)$ and $A \rhd (A + \neg B^N)$, but no Σ_1-sentence has this property.
- The interpretability logic of A w.r.t. N is ILM.
  $\vdash \phi \rhd \psi \to (\phi \wedge \Box\chi) \rhd (\psi \wedge \Box\chi)$.
- $U + \mathrm{incon}^N(U)$ is consistent and no $M : \mathrm{S}^1_2 \lhd (U + \mathrm{incon}^N(U))$ is Σ_1-sound.
- U is not locally mutually interpretable with a finitely axiomatized theory.

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Peano Downstairs en Peano Cellar

The theories Peano Downstairs (or $\mathrm{PA}^{\downarrow}$) and Peano Cellar (or $\mathrm{PA}^{\downarrow\downarrow}$) are in many respects like PA:

- They satisfy an induction principle that is in some respects more like full induction than Σ_n-induction.
- They are sententially essentially reflexive (w.r.t. restricted provability).
- They have no consistent finitely axiomatized extension in the same language. So e.g. $\mathrm{PA}^{\downarrow}$ is not a subtheory of $\mathrm{I}\Sigma_n$. It is a subtheory of PA.

On the other hand they are locally weak, i.e. they are locally interpretable (and even cut-interpretable) in PA^{-}.

I predict that almost all results of Per Lindstöm's book *Aspects of Incompleteness* transfer to extensions of $\mathrm{PA}^{\downarrow\downarrow}$ / $\mathrm{PA}^{\downarrow}$. But what about model theoretic results? This is far less clear.

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The Theory PA^-, 1

The theory PA^- is the theory of discretely ordered commutative semirings with a least element.

The theory is mutually interpretable with Robinson's Arithmetic Q. However, PA^- has a more mathematical flavor. Moreover, it has the additional good property that it is sequential. This was shown recently by Emil Jeřábek.

The theory PA^- is given by the following axioms.

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The Theory PA$^-$, 2

PA$^-$1	$\vdash x + 0 = x$
PA$^-$2	$\vdash x + y = y + x$
PA$^-$3	$\vdash (x + y) + z = x + (y + z)$
PA$^-$4	$\vdash x \cdot 1 = x$
PA$^-$5	$\vdash x \cdot y = y \cdot x$
PA$^-$6	$\vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)$
PA$^-$7	$\vdash x \cdot (y + z) = x \cdot y + x \cdot z$
PA$^-$8	$\vdash x \leq y \vee y \leq x$
PA$^-$9	$\vdash (x \leq y \wedge y \leq z) \rightarrow x \leq z$
PA$^-$10	$\vdash x + 1 \not\leq x$
PA$^-$11	$\vdash x \leq y \rightarrow (x = y \vee x + 1 \leq y)$
PA$^-$12	$\vdash x \leq y \rightarrow x + z \leq y + z$
PA$^-$13	$\vdash x \leq y \rightarrow x \cdot z \leq y \cdot z$

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The Theory PA^-, 3

The subtraction axiom is:

sbt $\quad \vdash x \leq y \to \exists z\; x + z = y$

In many presentations the subtraction axiom is part of the axioms of PA^-. We call $PA^-_{sbt} := PA^- + sbt$.

sbt is interpretable in PA^- on a cut.

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What is a Cut?

We are mostly speaking about *definable* cuts. A definable cut is a virtual class that is downwards closed w.r.t. $\leq$ and closed under successor.

If a cut is closed under addition it is an a-cut. If a cut is closed under addition and multiplication it is an am-cut. Etc.

Solovay's method of shortening cuts:
a definable cut can always be shortened to a definable am-cut. And similarly for closure under the any element of the ω_n-hierarchy.

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Cut-interpretability in PA^-

A central result: $\mathsf{PA}^- \rhd_{\mathsf{cut}} (\mathrm{I}\Delta_0 + \Omega_1)$.

Given that exponentiation is undefined for some n, there is a unique element $\mathfrak{s}$, *Solovay's number*, such that $\mathsf{supexp}(\mathfrak{s})$ is defined and $\mathsf{supexp}(\mathfrak{s}+1)$ is undefined. The following theories are interpretable on a cut:

- For $k < n$: $\mathrm{I}\Delta_0 + (\mathsf{Exp} \vee \mathfrak{s} \equiv k \, (\mathsf{mod}\ n))$.
- $\mathrm{I}\Delta_0 + (\Omega_1 \to \mathsf{Exp})$.

There are $2^{\aleph_0}$ theories locally cut-interpretable in PA^-. To each $\alpha : \omega \to \{0, 1\}$, we assign an extension of $\mathrm{I}\Delta_0$ that says: either Exp or the binary expansion of $\mathfrak{s}$ ends with $\ldots\alpha_2\alpha_1\alpha_0$. These theories are pairwise incompatible in the sense that their union implies Exp.

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The Hierarchy Defined

- The class $\Sigma_{1,0}$ consists of formulas of the form $\exists \vec{x}\, S_0(\vec{x}, \vec{y})$, where S_0 is Δ_0.
- The class $\Sigma_{1,n+1}$ consists of formulas of the form $\exists \vec{x}\, \forall \vec{y} \leq \vec{t}\, S_0(\vec{x}, \vec{y})$, where S_0 is $\Sigma_{1,n}$.
- The class $\Sigma_{1,\infty}$ is the union of the $\Sigma_{1,n}$.

In a similar way we define the formula classes $\Pi_{1,n}$ and $\Pi_{1,\infty}$.

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Collection

The scheme $\mathrm{B}\Sigma_1$ is given as follows:

- $\forall a, \vec{z}\, (\forall x \leq a \exists y\, A(x, y, \vec{z}\,) \to \exists b \forall x \leq a \exists y \leq b\, A(x, y, \vec{z}\,))$, where A is Δ_0.

The scheme $\mathrm{B}\Sigma_1^{\mathrm{j}}$ is given as follows:

- $\forall a, \vec{z}\, \exists u \leq a \forall b\, (A(u, b, \vec{z}\,) \to \forall x \leq a \exists y \leq b\, A(x, y, \vec{z}\,))$, where A is Δ_0.

Over $\mathrm{I}\Delta_0$ these schemes coincide (Jeřábek). Note that $\mathrm{B}\Sigma_1^{\mathrm{j}}$ is $\Pi_{1,1}$.

Over $\mathsf{PA}^- + \mathrm{B}\Sigma_1$ the $\Sigma_{1,n}$-hierarchy collapses to $\Sigma_{1,0}$. Over $\mathrm{I}\Delta_0 + \neg\,\mathrm{B}\Sigma_1$ the hierarchy explodes to the full language.

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Marker's Theorem

Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable models of PA^- that are jointly recursively saturated. Suppose further that, for all sentences S of $\Sigma_{1,\infty}$, we have: if $\mathcal{M} \models S$, then $\mathcal{N} \models S$. Then there is an initial embedding of $\mathcal{M}$ in $\mathcal{N}$.

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Definition of the Theories

- $\mathrm{I}\Sigma_\infty[\Sigma_{1,n}]$ is PA^- plus $\vdash S \to S^I$, where S is a $\Sigma_{1,n}$-sentence and where I is a PA^--cut.
- Peano Cellar is $\mathrm{PA}^{\downarrow\downarrow}$ is $\mathrm{I}\Sigma_\infty[\Sigma_{1,0}]$.
- Peano Downstairs is $\mathrm{PA}^{\downarrow}$ is $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$.

The theories have many equivalent formulations. E.g., Peano Cellar is equivalent to $\mathrm{I}\Delta_0$ plus: "all Σ_1-definable elements are in each inductive virtual class."

Each of these theories says that inductive classes / cuts are large. Thus we are looking at a variant of the induction principle.

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Basic Facts

$\Box_m$ is provability in predicate logic where all formulas in the proof have complexity $\leq m$.

We have:

1. $\mathsf{PA}^{\downarrow\downarrow} \vdash \mathrm{I}\Pi_1^-$.
2. $\mathsf{PA}^{\downarrow\downarrow} \nvdash \mathrm{B}\Sigma_1$.
3. for $n \geq 1$, $\mathsf{PA}^{\downarrow} = \mathrm{I}\Sigma_\infty[\Sigma_{1,n}] = \mathsf{PA}^{\downarrow\downarrow} + \mathrm{B}\Sigma_1$.
4. $\mathsf{PA}^{\downarrow\downarrow} \vdash \Box_m A \rightarrow A$,
   for all sentences A in the language of arithmetic.

We say that $\mathsf{PA}^{\downarrow\downarrow}$ is sententially essentially reflexive.

In fact $\mathsf{PA}^{\downarrow\downarrow}$ is equivalent to the restricted sentential reflection scheme $\Box_m A \rightarrow A$ over CFL, which is $\mathrm{I}\Delta_0$ plus “exponentiation is defined for all Σ_1-definable numbers”.

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Basic Facts 2

The theories $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ do not have a finitely axiomatized extension. So they are subtheories of any of the $\mathrm{I}\Sigma_n$. Also they are not mutually interpretable with a finitely axiomatized theory but they are locally mutually interpretable with a finitely axiomatized theory.

$\mathsf{PA}^{\downarrow\downarrow} + \mathsf{Exp}$ is not locally mutually interpretable with a finitely axiomatized theory.
So, in this respect $\mathsf{PA}^{\downarrow\downarrow} + \mathsf{Exp}$ is more like PA than $\mathsf{PA}^{\downarrow\downarrow}$ is.

EA is Σ_2-conservative over $\mathsf{PA}^{\downarrow\downarrow}$ (since it is Σ_2-conservative over $\mathrm{I}\Pi_1^-$) and $\mathsf{PA}^{\downarrow}$ is Π_2-conservative over EA.

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Local Cut-interpretability

Consider any finite set $\mathcal{S}$ of $\Sigma_{1,\infty}$-sentences. Consider any model $\mathcal{M}$ of PA^-. The set $\mathcal{S}$ splits into $\mathcal{S}_0$ the set of S in $\mathcal{S}$ that are true in all definable $\mathcal{M}$-am-cuts J, and $\mathcal{S}_1$ the set of S in $\mathcal{S}$ such that for some definable $\mathcal{M}$-am-cut J_S we have $\mathcal{M} \models (\neg S)^{J_S}$. Let J^* be the intersection of the J_S for S in $\mathcal{S}_1$. Then clearly we have $J^*(\mathcal{M}) \models S \to S^J$, for all am-cuts J and for all $S \in \mathcal{S}$.

We may conclude that:
$\mathsf{PA}^- \rhd_{\text{mod,cut}} (\mathsf{PA}^- + \{S \to S^I \mid \text{am-cut}(I) \text{ and } S \in \mathcal{S}\})$.

So, *a fortiori*:
$\mathsf{PA}^- \rhd_{\text{loc,cut}} (\mathsf{PA}^- + \{S \to S^I \mid \text{am-cut}(I) \text{ and } S \in \Sigma_{1,1}\text{-sent}\})$.

Characterization Theorem

Suppose $\mathcal{M}$ is a countable, recursively saturated model of PA^-. Then $\mathcal{M}$ satisfies $\mathrm{PA}^{\downarrow}$ iff there is a, not necessarily definable, initial embedding of $\mathcal{M}$ into the intersection $\mathcal{J}_{\mathcal{M}}$ of all definable am-cuts in $\mathcal{M}$.

Thus $\mathrm{PA}^{\downarrow}$ is the theory of all countable, recursively saturated models $\mathcal{M}$ that have an initial embedding in $\mathcal{J}_{\mathcal{M}}$.

This uses Marker's theorem plus the fact that, by chronic resplendence, we can extend $\mathcal{M}$ with an non-definable am-cut $\mathcal{I} \subseteq \mathcal{J}_{\mathcal{M}}$ such that every Σ_1-definable element is in $\mathcal{I}$, where $\mathcal{M}$ and $\mathcal{I}$ are jointly recursively saturated.

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A Consequence

There is no finitely axiomatizable extension of $\mathsf{PA}^{\downarrow\downarrow}$ in the same language. So $\mathsf{PA}^{\downarrow\downarrow}$ is not a subtheory of $\mathrm{I}\Sigma_n$.

Let U be any sequential theory with p-time decidable axiom set. We consider a sentence Θ such that:

$$\mathsf{PA}^{-} \vdash \Theta \leftrightarrow \forall x\,(\mathsf{con}_x(\mathsf{PA}^{\downarrow\downarrow} + \Theta) \to \mathsf{con}_x(U)).$$

We have:

$$(\mathsf{PA}^{\downarrow\downarrow} + \Theta) \equiv_{\mathsf{loc}} U.$$

Specifically, we have:

$$(\mathsf{PA}^{\downarrow\downarrow} + \Theta) \rhd U \text{ and } U \rhd_{\mathsf{loc}} (\mathsf{PA}^{\downarrow\downarrow} + \Theta).$$

This last result is an immediate adaptation of a result of Per Lindström.

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Another Consequence

Suppose U is an extension of $\mathsf{PA}^{\downarrow}$ and V is sequential. Then: $U \rhd V$ iff, for every countable recursively saturated model $\mathcal{N}$ of U, there is a model $\mathcal{M}$ of V such that, for every internal model $\mathcal{K}$ of PA^{-} in $\mathcal{M}$, there is an initial embedding of $\mathcal{N}$ in $\mathcal{K}$.

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