Axiom schema for a model of arithmetic with a cut

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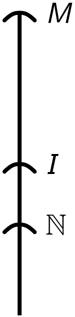
Cuts

Definition

- PA stands for first order Peano arithmetic.
- A *cut* of a model of PA is a proper nonempty initial segment that is closed under successor, $+$ and $\times$.



Example

$\mathbb{N}$ is a cut of every nonstandard model of PA.

Why cuts?

- Model theory
- Second order arithmetic
- Nonstandard analysis
- Independence results

Language for cuts

Observation

In a model of PA, no cut is definable.

Definition

$\mathscr{L}_{\mathsf{cut}} = \{0, 1, +, \times, <, \mathbb{I}\}$, where $\mathbb{I}$ is a unary predicate symbol.

Definition

$\mathsf{PA}^{\mathsf{cut}} = \mathsf{PA} +$ "$\mathbb{I}$ is a cut".

Convention

We write models of $\mathsf{PA}^{\mathsf{cut}}$ as pairs (M, I) where $M \models \mathsf{PA}$ and I is a cut of M.

Overview

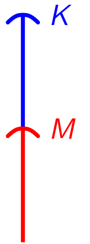
Aim

Understand cuts as models of $\mathrm{PA}^{\mathrm{cut}}$.

Plan

1. Elementary extensions of models of PA
2. Elementary extensions of models of $\mathrm{PA}^{\mathrm{cut}}$
3. Second order strength of $\mathscr{L}_{\mathrm{cut}}$ theories
4. Further topics

End extensions and cofinal extensions



Definition

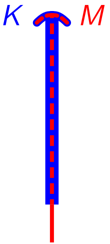
Let M, K be ordered sets and $K \supsetneq M$.

- K is an *end extension* of M, denoted $K \supset_e M$, if

$$\forall x \in K \setminus M \;\; \forall y \in M \;\; y < x.$$

- K is a *cofinal extension* of M, denoted $K \supset_{cf} M$, if

$$\forall x \in K \setminus M \;\; \exists y \in M \;\; x \leqslant y.$$



The Splitting Theorem

Definition

A structure $\mathfrak{K}$ is an *elementary extension* of another structure $\mathfrak{M}$, denoted $\mathfrak{K} \succ \mathfrak{M}$, if $\mathfrak{K} \supset \mathfrak{M}$ and

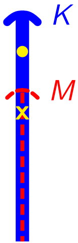
$$\mathfrak{M} \models \theta(\bar{c}) \quad \Leftrightarrow \quad \mathfrak{K} \models \theta(\bar{c})$$

for all formulas $\theta(\bar{x})$ and all $\bar{c} \in \mathfrak{M}$.

Splitting Theorem

If $K \succ M \models \mathrm{PA}$, then there is $\overline{M}$ such that $K \succeq_{\mathrm{e}} \overline{M} \succeq_{\mathrm{cf}} M$.

End extensions



Theorem (Mac Dowell–Specker 1961, Keisler 1966, Paris–Kirby 1978)

For a countable $M \models \mathrm{I}\Delta_0$, the following are equivalent.

(a) $M \models \mathrm{PA}$.

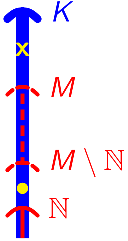
(b) There is $K \succ_{\mathrm{e}} M$.

(c) There is $K \succ_{\mathrm{e}} M$ that is ω_1-*like*,
i.e., every proper cut of K is countable, but K is uncountable.

Remark

The regularity scheme plays an important role here.

Cofinal extensions



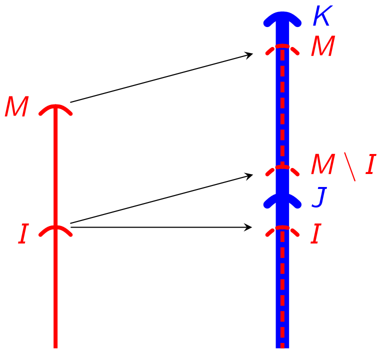
Theorem (Rabin 1962)

For every nonstandard $M \models \mathrm{PA}$, there is $K \succ_{\mathrm{cf}} M$.

Theorem (Kaye 1991)

The existence of elementary cofinal extensions does not imply PA.

Elementary extensions of models of $\mathrm{PA}^{\mathrm{cut}}$



- $K \supset_{\mathrm{e}} M$
- $K \supset_{\mathrm{cf}} M$

$K \setminus J$ is an end segment of $M \setminus I$.

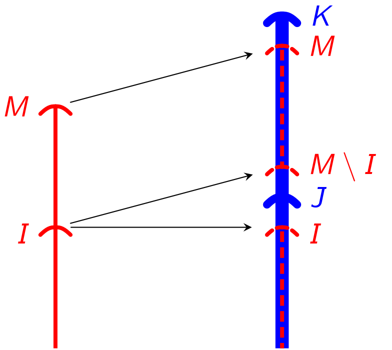
- $\mathrm{rev}(K \setminus J) \supset_{\mathrm{e}} \mathrm{rev}(M \setminus I)$
- $\mathrm{rev}(K \setminus J) \supset_{\mathrm{cf}} \mathrm{rev}(M \setminus I)$
- $K \setminus J = M \setminus I$

$M \setminus I$ is downward cofinal in $K \setminus J$.

- $J \supset_{\mathrm{e}} I$
- $J \supset_{\mathrm{cf}} I$
- $J = I$

$\mathrm{rev}(X)$ denotes the reverse of the ordered set X.

End segments



- $K \supset_e M$
- $K \supset_{cf} M$

- ~~$\mathrm{rev}(K \setminus J) \supset_e \mathrm{rev}(M \setminus I)$~~
- $\mathrm{rev}(K \setminus J) \supset_{cf} \mathrm{rev}(M \setminus I)$
- ~~$K \setminus J = M \setminus I$~~

- $J \supset_e I$
- $J \supset_{cf} I$
- $J = I$

Theorem (Smoryński 1984)

If two models of PA share some end segment, then they are equal.

End extensions

Theorem (Smith 1989)

If $I \subset_{\mathrm{e}} M \prec_{\mathrm{e}} K \models \mathrm{PA}$, then $(M, I) \prec (K, I)$.

- $K \supset_{\mathrm{e}} M$ ✓
- $K \supset_{\mathrm{cf}} M$ ✓

Proof

Back-and-forth in the style of Kotlarski, Smoryński, and Vencovská. □

- $\cancel{\mathrm{rev}(K \setminus J) \supset_{\mathrm{e}} \mathrm{rev}(M \setminus I)}$
- $\mathrm{rev}(K \setminus J) \supset_{\mathrm{cf}} \mathrm{rev}(M \setminus I)$
- $\cancel{K \setminus J = M \setminus I}$

Corollary

For every $(K, J) \succ (M, I)$, there exists $\overline{M}$ such that $(K, J) \succeq_{\mathrm{e}} (\overline{M}, J) \succeq_{\mathrm{cf}} (M, I)$.

- $J \supset_{\mathrm{e}} I$
- $J \supset_{\mathrm{cf}} I$
- $J = I$ and $K \neq M$

End extending the cut

Theorem

For any countable $(M, I) \models \mathrm{PA}^{\mathrm{cut}}$, the following are equivalent.

(a) (M, I) satisfies the regularity scheme.

(b) There is $(K, J) \succ (M, I)$ such that $J \supset_{\mathrm{e}} I$.

(c) There is $(K, J) \succ (M, I)$ such that $J \supset_{\mathrm{e}} I$ and J is ω_1-like.

Regularity scheme

For each formula $\theta(x, y)$ in $\mathscr{L}_{\mathrm{cut}}$,

$$\forall a{\in}\mathbb{I}\ \big(\mathrm{Q}x{\in}\mathbb{I}\ \ \exists y{<}a\ \ \theta(x,y) \rightarrow \exists y{<}a\ \ \mathrm{Q}x{\in}\mathbb{I}\ \ \theta(x,y)\big),$$

where $\mathrm{Q}x{\in}\mathbb{I}$ means "there are cofinally many x in $\mathbb{I}$".

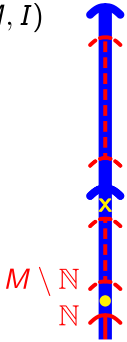
Example

If M is a nonstandard model of PA, then $(M, \mathbb{N}) \models$ regularity.

Cofinally extending the cut

Theorem

Let $(M, I) \models \mathsf{PA}^{\mathsf{cut}} +$ regularity in which I is nonstandard.
If (M, I) is countable, then there is a countable $(K, J) \succ (M, I)$ such that $J \supset_{\mathsf{cf}} I$.



Preserving the cut

Theorem

For any countable $(M, I) \models \mathsf{PA}^{\mathsf{cut}}$, the following are equivalent.

(a) (M, I) satisfies the contraregularity scheme.

(b) There is $(K, I) \succ (M, I)$ in which $\mathrm{rev}(K \setminus I) \not\supset_{\mathrm{cf}} \mathrm{rev}(M \setminus I)$.

(c) There is $(K, I) \succ (M, I)$ in which I has uncountable downward cofinality.

Contraregularity scheme

For each formula $\theta(x, y)$ in $\mathscr{L}_{\mathsf{cut}}$,

$$\forall x {\in} \mathbb{I} \;\; \exists y {>} \mathbb{I} \;\; \theta(x, y) \rightarrow \exists b {>} \mathbb{I} \;\; \forall x {\in} \mathbb{I} \;\; \exists y {>} b \;\; \theta(x, y).$$

Remark

Every model of PA has an elementary extension K such that $(K, \mathbb{N}) \models$ contraregularity.

Downward cofinally extending the complement

Theorem

For any countable $(M, I) \models \mathsf{PA}^{\mathsf{cut}} + \text{regularity}$, the following are equivalent.

(a) (M, I) satisfies the contraregularity scheme.

(b) (M, I) satisfies the *weak contraregularity scheme*, and there is a countable $(K, J) \succ (M, I)$ such that $J \supset_{\mathsf{e}} I$ and $\mathrm{rev}(K \setminus J) \supset_{\mathsf{cf}} \mathrm{rev}(M \setminus I)$.

(c) There is $(K, J) \succ (M, I)$ such that J is ω_1-like and $K \setminus J$ has *countable* downward cofinality.

Standard systems

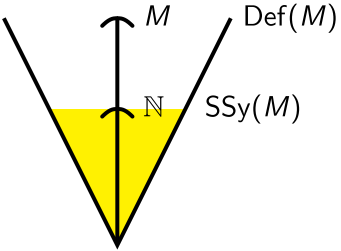
Definition

For a structure $\mathfrak{M}$, denote by $\mathrm{Def}(\mathfrak{M})$ the collection of all *parametrically* definable subsets of $\mathfrak{M}$.

Definition (Tennenbaum 1959, Friedman 1973, ...)

For a nonstandard $M \models \mathrm{PA}$,

$$\mathrm{SSy}(M) = \{X \cap \mathbb{N} : X \in \mathrm{Def}(M)\}.$$



Second order arithmetic

- *Second order arithmetic* lives in first order logic.
- It has a *number sort* and a *set sort*.
- Models of second order arithmetic consist of $(M, \mathscr{X})$, where M is the universe for the number sort, and $\mathscr{X} \subseteq \mathcal{P}(M)$ is the universe for the set sort.

Observation

$(\mathbb{N}, \mathrm{SSy}(M))$ is a model of second order arithmetic whenever $M \models \mathrm{PA}$.

Fact

$(M, \mathbb{N})$ uniformly interprets $(\mathbb{N}, \mathrm{SSy}(M))$ for nonstandard $M \models \mathrm{PA}$.

Second order strength

Problem

Given an $\mathscr{L}_{\mathsf{cut}}$ theory T, what is

$$\mathsf{Th}_{\mathbb{N}}(T) = \bigcap \left\{ \mathsf{Th}(\mathbb{N}, \mathsf{SSy}(M)) : (M, \mathbb{N}) \models \mathsf{PA}^{\mathsf{cut}} + T \right\}?$$

In particular, where does it sit relative to the Big Five theories

$$\mathsf{RCA}_0, \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}_0$$

of reverse mathematics?

Strength of regularity

$$\mathrm{Th}_{\mathbb{N}}(T) = \bigcap \{\mathrm{Th}(\mathbb{N}, \mathrm{SSy}(M)) : (M, \mathbb{N}) \models \mathrm{PA}^{\mathrm{cut}} + T\}$$

Regularity scheme

For each formula $\theta(x, y)$ in $\mathscr{L}_{\mathrm{cut}}$,

$$\forall a \in \mathbb{I}\ \big(\mathrm{Q}x \in \mathbb{I}\ \ \exists y < a\ \ \theta(x, y) \to \exists y < a\ \ \mathrm{Q}x \in \mathbb{I}\ \ \theta(x, y)\big),$$

where $\mathrm{Q}x \in \mathbb{I}$ means "there are cofinally many x in $\mathbb{I}$".

Proposition

$\mathrm{Th}_{\mathbb{N}}(\text{regularity}) = \mathrm{WKL}_0$.

Proof

- Every $(M, \mathbb{N}) \models \mathrm{PA}^{\mathrm{cut}}$ satisfies the regularity scheme.
- Scott (1962) says a countable $\mathscr{X} \subseteq \mathcal{P}(\mathbb{N})$ realizes as $\mathrm{SSy}(M)$ for some $M \models \mathrm{PA}$ if and only if $(\mathbb{N}, \mathscr{X}) \models \mathrm{WKL}_0$. □

Strength of contraregularity

$$\mathsf{Th}_{\mathbb{N}}(T) = \bigcap \{\mathsf{Th}(\mathbb{N}, \mathsf{SSy}(M)) : (M, \mathbb{N}) \models \mathsf{PA}^{\mathsf{cut}} + T\}$$

Contraregularity scheme

For each formula $\theta(x, y)$ in $\mathscr{L}_{\mathsf{cut}}$,

$$\forall x{\in}\mathbb{I}\ \ \exists y{>}\mathbb{I}\ \ \theta(x, y) \rightarrow \exists b{>}\mathbb{I}\ \ \forall x{\in}\mathbb{I}\ \ \exists y{>}b\ \ \theta(x, y).$$

Theorem

$\mathsf{Th}_{\mathbb{N}}(\text{contraregularity}) \supseteq \mathsf{ACA}_0$.

Proof

Via the Kirby–Paris notion of *strong cuts*:

$$\forall f\ \ \exists b{>}\mathbb{I}\ \ \forall x{\in}\mathbb{I}\ \ \big(f(x) > \mathbb{I} \rightarrow f(x) > b\big).$$

□

Saturation

$$\mathsf{Th}_{\mathbb{N}}(T) = \bigcap \{\mathsf{Th}(\mathbb{N}, \mathsf{SSy}(M)) : (M, \mathbb{N}) \models \mathsf{PA}^{\mathsf{cut}} + T\}$$

Strong standard systems implies strong *saturation conditions* when the model is recursively saturated.

Theorem (Wilmers 1975)

A countable $\mathscr{X} \subseteq \mathcal{P}(\mathbb{N})$ realizes as $\mathsf{SSy}(M)$ for some recursively saturated $M \models \mathsf{PA}$ if and only if $(\mathbb{N}, \mathscr{X}) \models \mathsf{WKL}_0$.

Definition (Kaye, Kossak, Kotlarski, Schmerl, ... 1990s)

A recursively saturated $M \models \mathsf{PA}$ is *arithmetically saturated* if $(\mathbb{N}, \mathsf{SSy}(M)) \models \mathsf{ACA}_0$.

Transplendency

$$\mathrm{Th}_{\mathbb{N}}(T) = \bigcap \big\{ \mathrm{Th}(\mathbb{N}, \mathrm{SSy}(M)) : (M, \mathbb{N}) \models \mathrm{PA}^{\mathrm{cut}} + T \big\}$$

Engström and Kaye (2012) introduced a notion of transplendency which ensures the existence of expansions omitting suitably consistent types.

Theorem (Engström–Kaye 2012)

Transplendent $M \models \mathrm{PA}$ make $(\mathbb{N}, \mathrm{SSy}(M)) \prec (\mathbb{N}, \mathcal{P}(\mathbb{N}))$.
In particular, $\mathrm{Th}_{\mathbb{N}}(\text{transplendency}) = \mathrm{Th}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Disadvantage

Transplendency may not be axiomatizable in $\mathscr{L}_{\mathrm{cut}}$.

Fullness

$$\mathrm{Th}_{\mathbb{N}}(T) = \bigcap\{\mathrm{Th}(\mathbb{N}, \mathrm{SSy}(M)) : (M, \mathbb{N}) \models \mathrm{PA}^{\mathrm{cut}} + T\}$$

Definition

Let $\mathrm{SSy}(M, \mathbb{N}) = \{X \cap \mathbb{N} : X \in \mathrm{Def}(M, \mathbb{N})\}$ for $(M, \mathbb{N}) \models \mathrm{PA}^{\mathrm{cut}}$.

Definition

A nonstandard $M \models \mathrm{PA}$ is *full* if $\mathrm{SSy}(M, \mathbb{N}) \subseteq \mathrm{SSy}(M)$.

Observation

Fullness of a model $M \models \mathrm{PA}$ is a first order property of $(M, \mathbb{N})$.

Example

If $M \models \mathrm{PA}$ such that $\mathrm{SSy}(M) = \mathcal{P}(\mathbb{N})$, then M is full.

Theorem

$\mathrm{Th}_{\mathbb{N}}(\text{fullness}) \supseteq \mathrm{CA}$.

Conclusion

Summary

- $\dfrac{\text{regularity}}{\text{cut}} = \dfrac{\text{contraregularity}}{\text{model} - \text{cut}}$.
- $\text{strength}_{\mathbb{N}}(\text{regularity}) = \text{WKL}_0$.
- $\text{strength}_{\mathbb{N}}(\text{contraregularity}) \supseteq \text{ACA}_0$.

Some further topics

- Definable sets in a model of PA^{cut}
- Elementary extensions of models of set theory