Model Theory and Proof Theory of Arithmetic
Booklet of Abstracts

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Full Satisfaction Classes in a General Setting (Tutorial)

Ali Enayat and Albert Visser

A *full satisfaction class* on an $\mathcal{L}$-structure $\mathcal{M}$ with sufficient coding apparatus decides the 'truth' of every $\mathcal{L}$-formula with parameters in $\mathcal{M}$, *including the nonstandard formulas in* $\mathcal{M}$, while obeying the usual recursive Tarski conditions for a satisfaction predicate. In this tutorial we present a robust technique for building a wide variety of full satisfaction classes using *model-theoretic* ideas, in the setting of a flexible notion of 'base theory' that encompasses base theories as weak as bounded arithmetic and as strong as Zermelo-Fraenkel set theory. Our model-theoretic construction is also shown to be implementable in the fragment WKL_0 of Second Order Arithmetic, which in turn implies that the conservativity of B + "S is a full satisfaction class" over B can be verified in Primitive Recursive Arithmetic for every r.e. base theory B. We also investigate interpretability issues connected to satisfaction classes. In particular, we show that B + "S is a full satisfaction class" is *interpretable* in B for all *inductive* base theories B, such as B = Peano arithmetic, or B = Zermelo-Fraenkel set theory.

New unprovability results in the Infinite-Dimensional Ramsey Theory and other recent developments in metamathematics

Andrey Bovykin
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Infinite-Dimensional Ramsey Theory is a rich source of Unprovability and Arithmetical Strength, beyond predicativity. It is difficult to think of a statement in this part of mathematics that would be provable or refutable in weak theories.

Firstly, I am going to discuss several variants of the Halpern Lauchli Theorem on products of trees.

Secondly, I will talk about Ramsey Theory on the rational line and the strength of Devlin's theorems.

Thirdly, I will sketch the current state of the other simultaneous projects in the study of Unprovability: the Atlas of all possibilities and modifying Friedman's Boolean Relation Theory (Proposition B) to extract short unprovability proofs.

In the end I am planning to spend some 10 minutes sharing new thoughts about Arithmetical Splitting because Arithmetical Splitting is, by far, the most important issue in metamathematics.

This work has been done in Federal University of Rio de Janeiro in 2012, and has been supported by a visiting research grant of CNPq.

Three results related to the Paris-Harrington principle

Lorenzo Carucci
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I present three results related to the Paris-Harrington principle. The first is a proof-complexity analysis of propositional tautologies encoding the Paris-Harrington therem for bicolorings of graphs. I show a non-trivial quasipolynomial upper bound in Resolution and a conditional superpolynomial lower bound in Res(2). (This is joint work with Galesi and Lauria). Next I present a computability-theoretic and Reverse Mathematics analysis of the infinite Ramsey Theorem for bicolorings of "relatively large sets" (in the sense of Paris and Harrington). The theorem is due to Pudlak and Rodl and independently to Farmaki. I show the theorem is equivalent over RCA_0 to closure under the omega-Turing Jump. (This is joint work with Zdanowski). Finally I present a result on "positional games" (in the sense of Beck) on graphs and hypergraphs based on the Paris-Harrington principle. I show elementary upper bounds on the game numbers for Paris-Harrington games on hypergraphs of any finite dimension. These contrast with the Ackermannian lower bounds on the witnessing function of the corresponding combinatorial principles. (This is joint work in progress with Lauria).

On Local Induction and Collection Principles.

Part I: Basic notions and applications to reflection principles.

Andrés Cordón-Franco

Part II: Inference rules and applications to parameter free induction.

F. Félix Lara-Martín

In these two talks we shall discuss properties of fragments of first order arithmetic axiomatized by Local Induction and Collection schemes. These are axiom schemes in which for each formula, the conclusion of the induction or collection axiom is granted only for elements having some characteristic property (for instance, elements definable by some existential formula). These schemes are motivated by natural model-theoretic considerations, and enjoy interesting connections with the classical fragments given by the standard induction or collection principles. Remarkably, they are useful tools in the study of conservation results for the classical fragments.

In the first talk we introduce the Local Induction and Collection schemes, study their basic properties, and discuss some applications to two different topics: i) axiomatizations of the Σ_n-consequences of the fragments of Peano Arithmetic; and ii) Local Reflection Principles in arithmetic.

In the second talk we shall discuss variants of the Local Induction schemes formulated by means of inference rules. As an application we obtain new conservation results for parameter free Π_n-induction as well as new proofs of other results already known. In particular, we improve a conservation result obtained by L. Beklemishev and show that the scheme of parameter free Π_2-induction is Π_3-conservative over the scheme of Σ_1-induction. Thereby, a direct characterization of the provably total computable functions of this theory is derived, avoiding the metamathematical machinery involved in previous proofs.

Recursively saturated real closed fields

Paola D'Aquino

I will consider integer parts of real closed fields and analyze those of recursively saturated real closed fields. If time permits I will present a valuation theoretical analysis of recursively saturated real closed fields.

Self-embeddings of Models of Arithmetic, Redux

Ali Enayat

This talks reports on recent joint work with Volodya Shavrukov on self-embeddings (endomorphisms) of nonstandard models of both first order and second arithmetic onto initial segments of themselves, with special attention to controlling fixed-points of self-embeddings.

Polish G-spaces similar to logic G-spaces of continuous structures

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We extend the concept of nice topologies of H.Becker to the general case of Polish G-spaces (Becker assumed that $G < Sym(\omega)$). Our apprach is based on continuous first order logic.

Let $(\mathbf{Y}, d)$ be a Polish space and $Iso(\mathbf{Y})$ be the corresponding isometry group endowed with the pointwise convergence topology. Then $Iso(\mathbf{Y})$ is a Polish group.

It is worth noting that any Polish group G can be realised as a closed subgroup of the isometry group $Iso(\mathbf{Y})$ of an appropriate Polish space $(\mathbf{Y}, d)$. Moreover it is shown by J.Melleray that G can be chosen as the automorphism group of a continuous metric structure on $\mathbf{Y}$ which is approximately ultrahomogeneous.

For any countable continuous signature L the set $\mathbf{Y}_L$ of all continuous metric L-structures on $(\mathbf{Y}, d)$ can be considered as a Polish $Iso(\mathbf{Y})$-space. We call this action *logic* and show that it is universal for Borel reducibility of orbit equivalence relations of Polish G-spaces with closed $G \leq Iso(\mathbf{Y})$.

We investigate Polish G-spaces $\mathbf{X}$ for such G. Note that for any tuple $\bar{s} \in \mathbf{Y}$ the map $g \to d(\bar{s}, g(\bar{s}))$ can be considered as a graded subgroup of G. We consider G together with an appropriate family of such subgroups. Distinguishing some family $\mathcal{B}$ of graded subsets of $\mathbf{X}$ we arrive at the situation very similar to the logic space $\mathbf{Y}_L$. For example treating elements of $\mathcal{B}$ as continuous formulas we obtain topological generalisations of several theorems from logic, for example Ryll-Nardzewski theorem.

Counting external sets in models of arithmetic

Richard Kaye
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I report on a number of ideas and results related to counting in models of arithmetic. These include: (a) the arithmetic of the Dedekind completion of a model (adding, subtracting, multiplication and division of cuts) and differentiation of cuts; (b) the methods of counting an external bounded set and their properties; and (c) possible extensions of the theory to a satisfactory notion of 'measurable' subset and measure theory over a model of PA.

A pesky fragment of bounded arithmetic

Leszek A. Kołodziejczyk

The two major open problems on the frontier of research in bounded arithmetic are: (1) separating levels of the relativized bounded arithmetic hierarchy with low-complexity sentences and (2) proving relevant independence results for relativized bounded arithmetic with a parity quantifier. I plan to discuss recent work suggesting that a single theory appears in the role of "serious obstacle" in the context of both of these problems. The theory in question is the stronger of two theories defined by Jerabek for the purpose of formalizing approximate counting. The talk will be based on two recent papers, one joint with Buss and Thapen and the other with Buss and Zdanowski.

Automorphisms and cofinal extensions

Roman Kossak

In my last paper with Henryk, "More on extending automorphisms of models of Peano arithmetic," Fund. Math. 200(2008), no. 2, pp. 133–143 we were trying to extend basic and useful results about extending automorphisms of countable recursively saturated models of PA to their elementary end extensions to the case of cofinal extensions. There are some partial results and interesting obstacles and open problems. I will talk about the results from the paper and some later developments.

Truth in the limit

Marcin Mostowski

We consider sl–semantics in which first order sentences are interpreted in potentially infinite domains. A potentially infinite domain is a growing sequence of finite models. We prove the completeness theorem for first order logic under this semantics. Additionally we characterize the logic of such domains as having a learnable, but not recursive, set of axioms.

The work is a part of author's research devoted to computationally motivated foundations of mathematics.

Properties of automorphisms of saturated models of arithmetic

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We will be presenting our joint with Erez Shochat research [3] on properties of automorphisms of saturated model of arithmetic. Let M be a model of Peano Arithmetic and let f be an authomorphism of M. Define $I_{\mathsf{fix}}(f)$ to be the largest cut in M pointwise fixed by f. Smoryński [2] showed that if M is a countable recursively saturated model of PA then a cut $I \subset M$ is an exponentially closed iff there is an automorphism g of M such that $I_{\mathsf{fix}}(g) = I$. We will show the similar result for saturated models of PA.

We can consider the automorphism group of M as a topological group by letting the stabilizers of subsets of M of cardinality less than cardinality of M to be the basic open subgroups. Let I be a cut in a model M. We say I is *invariant* if for every f automorphism of M: $f(I) = I$. Kaye [1] shows that if M is a countable recursively saturated models of PA then H is a closed normal subgroup in automorphism group of M iff there is an invariant cut I such that H is the pointwise stabilizers of I. We will discuss the analog of Kaye's theorem for saturated models of PA.

References

[1] Richard Kaye, A Galois correspondence for countable recursively saturated models of Peano arithmetic, in Kaye and Macpherson (eds) Automorphisms of first-order structures, OUP 1994, 293–312

[2] C. Smoryński, Back and forth inside a recursively saturated model of arithmetic, *Logic Colloquium '80, ed. D. van Dallen*, North-Holland, Amsterdam, pp. 273–278, 1982.

[3] Ermek S. Nurkhaidarov, Erez Shochat, Automorphisms of saturated and boundedly saturated models of arithmetic. Notre Dame Journal of Formal Logic 52 (3), 2010, 315-329

Well-ordering principles, omega models and beta models

Michael Rathjen

The purpose of this talk is to present a general methodology which in many cases allows one to establish an equivalence between two types of statements. The first type is concerned with the existence of omega models of a theory whereas the second type asserts that a certain (usually well-known) elementary operation on orderings preserves the property of being well-ordered. The primordial example is Friedman's characterization of the theory ATR_0 by means of a Π^1_2 sentence of the form "if X is well ordered then $f(X)$ is well ordered", where f is a standard proof theoretic function from ordinals to ordinals. The approach taken here, however, is rather different in that the methods used are proof-theoretic, involving search trees and cut elimination theorems in infinitary logic with ordinal bounds. One could perhaps generalize and say that every cut elimination theorem in ordinal-theoretic proof theory encapsulates a theorem of this type. It is also interesting to ponder the question whether the technique has the potential for generalization, namely whether it can be extended to beta-models and functors acting on ordinal functions.

Characterizing the Existence of Optimal Proof Systems and Complete Sets for Promise Classes

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We investigate the following two questions: Q1: Do there exist optimal proof systems for a given language L? Q2: Do there exist complete problems for a given promise class C? For concrete languages (such as TAUT or SAT) and concrete promise classes (such as UP, disjoint NP-pairs etc.) these questions have been intensively studied during last years, and a number of characterizations have been obtained. Here we provide new characterizations for Q1 and Q2 that apply to almost all promise classes C and languages L, thus creating a unifying framework for the study of these practically relevant questions. Joint work with Olaf Beyersdorff.

Automorphism Groups of models of Peano Arithmetic

James Schmerl

Automorphism groups of models of PA have been studied for the last two decades. Kotlarski contributed some of the earlier and important results. I will cover this topic from his results for recursively saturated models up to the recent results of mine with Nurkhaidarov on saturated models.

Axiom schema for a model of arithmetic with a cut

Tin Lok Wong

Consider first-order structures of the form (M, I), in which a nonstandard model of Peano arithmetic M is expanded by a cut I of M as a new predicate. What interesting first-order properties can we extract from these structures (M, I)? I will present a few obvious natural answers, as well as some more exotic ones, for which 'interesting' can be explained in terms of characteristic elementary extensions and saturation conditions.

This contains some ongoing joint work with Richard Kaye (Birmingham, UK) and Roman Kossak (City University of New York, USA).

Local cut-interpretability in PA^-

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At first sight, cut-interpretability is more well-behaved than ordinary interpretability. For one thing, if we start with a true arithmetical theory, we can only interpret true things on cuts. Nevertheless, in the basic theory PA^-, we can locally cut-interpret $2^{\aleph_0}$ different theories that are incompatible in the sense that the union of any two of them proves that exponentiation is total (and is, thus, not interpretable in PA^-).

The main focus of this presentation is the theory $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$. This theory is axiomatized over PA^- by axioms of the form $\vdash S \to S^I$, where S is $\Sigma_{1,1}$, i.e. it is of the form $\exists x\, \forall y < x\, \exists z\, S_0(x,y,z)$, for S_0 in Δ_0. We show how to locally cut-interpret $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ in PA^-. In many ways $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ is a PA-analogue. We discuss various surprising features of this theory.

- $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ extends $\mathrm{I}\Pi_1^-$.
- $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ proves $\mathrm{B}\Sigma_1$. (Jeřábek)
- $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ proves $\Box_k A \to A$, for each arithmetical sentence A. Here $\Box_k$ stands for provability where the proofs are restricted to formulas of complexity at most k. Hence $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ is not contained in any of the $\mathrm{I}\Sigma_n$.
- Each sequential theory is mutually locally interpretable with $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]+\Theta$, for a Δ_2-sentence Θ.
- The interpretability logic of $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ is ILM.

We will discuss the notion of *model interpretability*. We will show that we can locally cut-model-interpret $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ using a more 'greedy' sequence of approximating theories.

Finally there is a model theoretic characterization of $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$. Consider a countable recursively saturated model $\mathcal{M}$ of PA^-. Let $\mathcal{I}$ be the intersection of the definable cuts in $\mathcal{M}$. Then, $\mathcal{M}$ satisfies $\mathrm{I}\Sigma_\infty[\Sigma_{1,1}]$ iff there is an initial embedding F of $\mathcal{M}$ in $\mathcal{I}$. (Of course, if we demand that F is definable, we get a model of PA.)

Some applications of alpha large sets

Andreas Weiermann

I will discuss the following theme of the research of Henryk Kotlarski: alpha large sets and their applications to Ramsey theory and unprovability theory.