# Counting external sets in models of arithmetic

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**Abstract**

I report on a number of ideas and results related to counting in models of arithmetic. These include: (a) the arithmetic of the Dedekind completion of a model (adding, subtracting, multiplication and division of cuts) and differentiation of cuts; (b) the methods of counting an external bounded set and their properties; and (c) possible extensions of the theory to a satisfactory notion of 'measurable' subset and measure theory over a model of PA.

## 1 Introduction

This reports on a number of ideas and results related to counting in models of arithmetic. Some of the ideas originated through discussions with Henryk Kotlarski and Roman Kossak on work we did on automorphisms of models of arithmetic [4].

The main idea was presented by Allsup [1] and developed further by Reading [6].

**Definition 1.1.** Given a bounded set $X \subseteq M$ we define

$$\overline{\operatorname{card}}\, X = \inf\{\operatorname{card} a : a \in M \text{ and } a \supseteq X\}$$

and

$$\underline{\operatorname{card}}\, X = \sup\{\operatorname{card} a : a \in M \text{ and } a \subseteq X\}.$$

If these cuts are equal we say that $X$ is *$M$-countable* and write $\operatorname{card} X$ for this cut.

One of the surprises is that quite complicated sets (complicated from the point of view of definability) are $M$-countable. This suggests a possibly useful category of 'nice' external sets with interesting model theoretic properties.

In the case of a countable recursively saturated $M \models \mathrm{PA}$, given a complete type $p(x)$ realised in $M$ one can count $p(M)$, the set of elements of $M$ realising $p(x)$. The observation that $\operatorname{Aut}(M, a)_{a \in p(M)} = \operatorname{Aut}(M, i)_{i \in \overline{\operatorname{card}}\, p(M)}$ was one of the starting points of the study of closed normal subgroups of $\operatorname{Aut}(M)$.

We restarted work in this area after results by Allsup and Kaye [2] on nonstandard symmetric groups and by Reading and Kaye [6] on nonstandard cyclic groups where similar techniques were used to classify subgroups etc.

In hindsight, the definitions above are very natural: they are nothing else than the version for models of arithmetic of Loeb's approach to measure in nonstandard analysis.

## 2 Arithmetic and calculus of cuts

It soon became obvious that if cuts are to behave in any way like cardinalities then the operations of arithmetic on cuts—addition, subtraction, multiplication etc.—should be explored. It is folklore that there are such operations, but no good account of this is available.

To see that there are genuine difficulties note that there are in fact two addition operations.

**Definition 2.1.**

$$I + J = \sup\{i + j : i \leqslant I, j \leqslant J\}$$

and

$$I \oplus J = \inf\{i' + j' : i' \geqslant I, j' \geqslant J\}.$$

These are not always the same. Similar remarks go for subtraction, multiplication, division etc.

It is not entirely straightforward to solve even 'easy' equations in cuts such as $X + I = J$ (for an unknown $X$ and known cuts $I, J$.) The solution to this problem involves the derivative of a cut.

**Definition 2.2.**

$$\partial I = \inf\{i' - i : i \leqslant I \leqslant i'\}.$$

This section will give a number of results on arithmetic and differentiation of cuts from new work by the present author [3]. All of the basic arithmetic operations are given in a context where the various dualities are clear, and a number of algebraic properties and examples will be given. Also, differentiation is explored. To a certain extent, rules for differentiation mimic standard rules such as the sum and product rules in the calculus.

## 3 Exploration of the notion of $M$-countable cuts

This section will give a brief survey of work by Reading and the author [5, 6] following from Allsup's work, in the general theory of 'countability' of bounded external subsets of $M \vDash \mathrm{PA}$.

The conclusion will be that although the theory is quite elegant and interesting and potentially useful it suffers a number of shortcomings, and is not a particularly good analogue to the successful notion of Lebesgue (or Loeb) measurability. For instance, it is not true in general that the intersection or union of two $M$-countable sets is $M$-countable.

## 4 Application: Lagrange's theorem

If time permits, I will sketch the ideas required for the recent analogue [6] of Lagrange's theorem for external $M$-countable groups $H < G < g$ interpreted in

$M$, where $g$ is a *bona fide* $M$-finite group defining the multiplication operation. Essentially the result states that $[G{:}H] = \mathrm{card}\,G/\mathrm{card}\,H$ and in this case there is a $M$-countable transversal $T$ of $H$ in $G$. However we currently need some rather mild closure conditions on $\mathrm{card}\,G$ and $\mathrm{card}\,H$ for this to work, and we do not know whether these conditions are necessary.

## 5 Measure theory over models of PA?

This section will survey some attempts to build the idea of '$M$-countable sets' into a satisfactory measure theory for models of PA. There are a number of dead-ends, one or two avenues still unexplored, and a large number of open questions.

## References


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[5] Richard Kaye and Alan Reading. Internally countable subsets of a model of arithmetic. In preparation, 2012.

[6] Alan Reading. Unpublished M.Phil thesis, University of Birmingham, 2012.