Nonstandard Analysis: a New Way to Compute

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Abstract

Constructive Analysis, introduced in 1967 by Errett Bishop ([1]), is the redevelopment of Mathematics based on *algorithm* and *proof* along the lines of the famous BHK (Brouwer-Heyting-Kolmogorov) interpretation. *Constructive Reverse Mathematics* is a spin-off from Harvey Friedman's famous *Reverse Mathematics* program ([8]), based on Constructive Analysis ([3, 4]).

We identify a fragment of Nonstandard Analysis ([6]) which captures Bishop's Constructive Analysis. The counterparts of *algorithm* and *proof* in Nonstandard Analysis are played by *Ω-invariance* and *Transfer*. Transfer expresses Leibniz' law that $\mathbb{N}$ and its nonstandard extension $^*\mathbb{N}$ satisfy the same properties. Furthermore, an object is *Ω-invariant* if it does not depend on the *choice* of the infinitesimal used in its definition. Incidentally, the latter is exactly the way infinitesimals are used in Physics. Less incidentally, the latter discipline tends to limit itself to Mathematics formalizable in Constructive Analysis ([2]).

We obtain a large number of equivalences from Constructive Reverse Mathematics in our constructive version of Nonstandard Analysis and discuss implications of our results. In particular, we discuss how our approach is the dual of Palmgren and Moerdijk ([5]) towards *Reuniting the Antipodes* ([7]).

References

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