

Notes on Compact Quantum Groups

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Abstract

Compact quantum groups have been studied by several authors and from different points of view. The difference lies mainly in the choice of the axioms. In the end, the main results turn out to be the same. Nevertheless, the starting point has a strong influence on how the main results are obtained and on showing that certain examples satisfy these axioms.

In these notes, we follow the approach of Woronowicz and we treat the compact quantum groups in the C^* -algebra framework. This is a natural choice when compact quantum groups are seen as a special case of locally compact quantum groups. A deep understanding of compact quantum groups in this setting is very important for understanding the problems that arise in developing a theory for locally compact quantum groups.

We start with a discussion on locally compact quantum groups in general but mainly to motivate the choice of the axioms for the compact quantum groups. Then we develop the theory. We give the main examples and we show how they fit into this framework.

The paper is an expository paper. It does not contain many new results although some of the proofs are certainly new and more elegant than the existing ones. Moreover, we have chosen to give a rather complete and self-contained treatment so that the paper can also serve as an introductory paper for non-specialists. Different aspects can be learned from these notes and a great deal of insight can be obtained.

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1 Introduction

Quantum groups have been widely studied the last decade. Most of all, they were investigated in a purely algebraic context and examples played an important role. Not so much has been done for a comprehensive theory of locally compact quantum groups. By now, the reason for this has become clear. It is twofold. First of all, it seems to be difficult, both technically and conceptually, to develop such a theory. Secondly, examples that are simple on the algebraic level turn out to get sometimes very complicated when lifted to topological level. Nevertheless there are many good reasons to develop such a theory of locally compact quantum groups. The main advantage of such a theory is the possibility of using the equivalent of the Haar measure for locally compact groups.

In section 2 of these notes, we say a little more about the ideas behind a possible theory of locally compact quantum groups. But since this is not the main purpose of these notes, we do not enter very deeply into it. General theories of locally compact quantum groups exist (see e.g. [2, 6, 17, 16, 18]), but it must be said that they are not yet completely satisfactory. A lot of research still needs to be done there.

The situation is quite different for discrete and compact quantum groups. These two classes are very well understood. In fact there is now also a class of locally compact quantum groups, that contains the discrete and the compact ones, and is self dual in the sense that the Pontryagin dual of a locally compact quantum group of this type is again of this type. These quantum groups are studied in [26, 27, 13].

The compact quantum groups are certainly those locally compact quantum groups that have been studied most. We just mention some of the references here: Woronowicz [32, 34, 35], Kirchberg [11], Dijkhuizen and Koornwinder [4]. These are the places where compact quantum groups were developed. But there are also other authors that treated compact quantum groups in duality with discrete quantum groups. This is e.g. the case in the paper [2] by Baaj and Skandalis. And remark that the paper on compact quantum groups [4] by Dijkhuizen and Koornwinder is very close to the paper [7] by Effros and Ruan on discrete quantum groups. There has indeed been some ambiguity. For instance, Kirchberg called his objects discrete quantum groups while others would speak about compact quantum groups.

When looking at compact quantum groups as compact, locally compact quantum groups, the obvious approach is the one of Woronowicz. He first developed the theory of compact matrix quantum groups in [32]. Later, an important note [34] was published where the axioms could be simplified. In 1992, a preprint, entitled 'Compact Quantum Groups' [35] was widely distributed, but it was not completed for a long time. Now, it seems that the paper is going to appear.

Because of this ambiguity and the different approaches by different authors, it seems desirable to have a text which starts from a natural set of axioms and develops the theory in a fairly complete way. This is what we do in this paper. We work with the definition of

Woronowicz since we view compact quantum groups as compact locally compact quantum groups, and therefore, the C^* -approach is the natural choice. We start by motivating this choice. Then we develop the theory as it can be found in different articles. To a great extend however, we follow the general treatment of Woronowicz in [35]. But we also explain the link with the earlier paper on compact matrix pseudogroups [32] and the note [34], simplifying the axioms for such compact matrix pseudogroups. Moreover, we include the important examples in the beginning. And we end the paper by treating the Pontryagin dual of a compact quantum group using more recent ideas.

This paper is expository. The results are not new. Nevertheless, some of the proofs, also of the more important ones, are original. Some novelty also lies in the structure of our treatment. In any case, we have made an effort to write down notes that are self-contained to a great extend. This should help the reader who wants to study this approach to compact quantum groups without the need to find his way in the different articles on the subject (with different approaches).

The paper is organized as follows. In section 2 we introduce the concept of quantization, as a way to generalize locally compact groups. We describe how this process works in the case of finite quantum groups, and what the problems are when we want to extend this construction to the locally compact case. We then motivate how Woronowicz' treatment of compact quantum groups fits into this scheme. In section 3 we define a compact quantum group. We motivate and discuss this definition and we give some examples. We also introduce the compact matrix quantum groups defined in [34] and show that they indeed satisfy this set of axioms. One of the main features of this definition is that, starting from a simple set of axioms, very strong results can be proven. Such a result is the existence and the uniqueness of the Haar measure. This is proven and discussed in section 4.

In section 5, the concept of representation is translated to the quantum language. Then, like in the classical group case, the Haar measure is used to construct the right regular representation. In section 6, we go deeper into the representation theory. The main results in this section are that any unitary representation can be decomposed into irreducible unitary representations and that all irreducible unitary representations are contained in the regular representation.

In section 7, we look at the subspace spanned by the matrix elements of finite-dimensional unitary representations. We show that it is a dense $*$ -subalgebra and that it can be made into a Hopf $*$ -algebra. It then becomes very straightforward to prove that Woronowicz' new definition of a compact matrix quantum group in [34] is equivalent with the original one in [32]. In the last section we generalize the Pontryagin duality between compact and discrete abelian groups to a duality theory between compact quantum groups and discrete quantum groups.

For the theory of C^* -algebras, we refer to the book of Dixmier [5]. For general results on Hopf algebras, we refer to the books of Abe [1] and Sweedler [20], and for Hopf $*$ -

algebras and the duality between them, to [22]. For any Hilbert space \mathcal{H} , the C*-algebra of bounded linear operators on \mathcal{H} will be denoted by $\mathcal{B}(\mathcal{H})$ and the C*-algebra of compact operators by $\mathcal{B}_0(\mathcal{H})$. For given vectors $\xi, \eta \in \mathcal{H}$, we will denote by $\omega_{\xi, \eta}$ the linear functional on $\mathcal{B}(\mathcal{H})$ such that $\omega_{\xi, \eta}(x) = \langle x\xi, \eta \rangle$ for all $x \in \mathcal{B}(\mathcal{H})$. If A and B are C*-algebras, then the algebraic tensor product will be denoted by $A \odot B$. We will always use the minimal C*-norm and the completion of $A \odot B$ with respect to this norm will be denoted by $A \otimes B$. We will use the notation ι for the identity map. If ω is a continuous linear functional on A , one can consider the linear mapping $(\omega \otimes \iota) : A \odot B \rightarrow B$ defined by $(\omega \otimes \iota)(a \otimes b) = \omega(a)b$ if $a \in A$ and $b \in B$. This mapping is continuous with respect to the minimal C*-norm on $A \odot B$, and hence can be extended to a linear mapping $A \otimes B \rightarrow B$, which will still be denoted by $\omega \otimes \iota$. In the same way $\iota \otimes \omega$ is defined. These maps are called slice maps.

2 Locally compact quantum groups

Let A be a C*-algebra. Let $A \otimes A$ be the C*-tensor product of A with itself with respect to the minimal C*-tensor product norm. Let $M(A \otimes A)$ be the multiplier algebra of $A \otimes A$. A *-homomorphism $\Phi : A \rightarrow M(A \otimes A)$ is called non-degenerate if $\Phi(A)(A \otimes A)$ is dense in $A \otimes A$. Denote by ι the identity map on A and consider the *-homomorphisms $\Phi \otimes \iota$ and $\iota \otimes \Phi$ on the algebraic tensor product $A \odot A$ of A with itself. By continuity, they have unique extensions to *-homomorphisms from $A \otimes A$ to $M(A \otimes A \otimes A)$. If Φ is non-degenerate, there are also unique extensions to *-homomorphisms from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$ (see e.g. [21]). These extensions will still be denoted by $\Phi \otimes \iota$ and $\iota \otimes \Phi$.

Definition 2.1 A non-degenerate *-homomorphism $\Phi : A \rightarrow M(A \otimes A)$ is called coassociative if $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$. Then we call Φ a comultiplication on A .

Let G be a locally compact space. Let A be the C*-algebra $C_0(G)$ of continuous complex functions on G , vanishing at infinity. Here, $A \otimes A$ can be identified with $C_0(G \times G)$ and $M(A \otimes A)$ with the C*-algebra $C_b(G \times G)$ of bounded complex functions on $G \times G$. If G is endowed with a continuous, associative multiplication, then we can define a comultiplication Φ on A by $(\Phi(f))(p, q) = f(pq)$ whenever $f \in C_0(G)$ and $p, q \in G$. It is not hard to see that Φ is indeed non-degenerate. And of course, the coassociativity follows from the associativity of the product on G . So, with any locally compact semi-group, we can associate, in a canonical way, a pair (A, Φ) of an abelian C*-algebra A and a comultiplication Φ on A .

Conversely, suppose that A is an abelian C*-algebra and Φ a comultiplication on A , then A has the form $C_0(G)$ for some locally compact space G and Φ defines a multiplication on G by the formula $f(pq) = (\Phi(f))(p, q)$. The non-degeneracy of Φ must be used to do this. And again, the multiplication is associative because Φ is coassociative. Hence G is a locally compact semi-group.

This shows that there is a one-to-one correspondence between the locally compact semi-groups on the one hand and the pairs (A, Φ) of an abelian C^* -algebra A with a comultiplication Φ on the other hand. The C^* -algebra structure takes care of the underlying topology, while the comultiplication reflects the multiplication.

This brings us to the following notion.

Definition 2.2 A locally compact quantum semi-group is a pair (A, Φ) of a C^* -algebra A and a comultiplication Φ on A .

The next natural question is : What is a locally compact quantum group ? Since a locally compact group is a locally compact semi-group, where the multiplication satisfies some extra conditions, it is most natural that a locally compact quantum group will be a pair (A, Φ) of a C^* -algebra A with a comultiplication Φ on A satisfying some extra properties. What are such conditions ?

To distinguish a group from a semi-group, the existence of a unit and of an inverse for all the elements is required. If G is a locally compact group, the identity gives rise to a $*$ -homomorphism $\varepsilon : C_0(G) \rightarrow \mathbb{C}$ given by $\varepsilon(f) = f(e)$. The inverse gives rise to an anti-homomorphism $\kappa : C_0(G) \rightarrow C_0(G)$ given by $(\kappa(f))(p) = f(p^{-1})$ whenever $p \in G$. The properties of the identity and the inverse can easily be translated into properties of ε and κ . However, as it turns out, to obtain the quantum analogues of the maps ε and κ and their natural properties seems to be rather complicated.

We will explain these difficulties by looking first at the case of a finite group where this whole procedure presents no problems at all.

If G is a finite group, the C^* -algebra $C_0(G)$ is the $*$ -algebra $C(G)$ of all complex functions on G with pointwise operations. It is of course finite-dimensional so that it suffices to look at the algebraic tensor product $C(G) \odot C(G)$. Also the multiplier algebra is this algebraic tensor product. In particular, the comultiplication Φ is a comultiplication in the ordinary, algebraic sense on the algebra $C(G)$.

The properties $ep = pe = p$ and $p^{-1}p = pp^{-1} = e$ for all p , become in terms of ε and κ

$$(\varepsilon \otimes \iota)\Phi(f) = (\iota \otimes \varepsilon)\Phi(f) = f$$

$$m(\kappa \otimes \iota)\Phi(f) = m(\iota \otimes \kappa)\Phi(f) = f$$

where ι denotes the identity map as before, and where m is the multiplication as a map from the tensor product $C(G) \odot C(G)$ to $C(G)$ defined by $m(f \otimes g) = fg$. This means that $(C(G), \Phi)$ is a Hopf algebra. In fact, it is a Hopf $*$ -algebra if we let $f^*(p) = \overline{f(p)}$.

Let us recall the definition of a Hopf $*$ -algebra (see e.g. [22]) :

Definition 2.3 Let A be an associative $*$ -algebra over \mathbb{C} with an identity. Assume that Φ is a $*$ -homomorphism from A to the algebraic tensor product $A \odot A$ satisfying $\Phi(1) = 1 \otimes 1$

and coassociativity $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$. Also assume that there are linear maps $\varepsilon : A \rightarrow \mathbb{C}$ and $\kappa : A \rightarrow A$ such that

$$\begin{aligned}(\varepsilon \otimes \iota)\Phi(a) &= (\iota \otimes \varepsilon)\Phi(a) = a \\ m(\kappa \otimes \iota)\Phi(a) &= m(\iota \otimes \kappa)\Phi(a) = \varepsilon(a)1\end{aligned}$$

for all $a \in A$. Then (A, Φ) is called a Hopf $*$ -algebra.

If they exist as linear maps satisfying the above equations, the maps ε and κ are unique. We have automatically that ε is a $*$ -homomorphism, that κ is an anti-homomorphism and that $\kappa(\kappa(a)^*)^* = a$ (see e.g. [22]).

So, in this setting, it is natural to define a finite quantum group in the following way :

Definition 2.4 A finite quantum group is a pair (A, Φ) of a finite-dimensional C^* -algebra A with a comultiplication Φ such that (A, Φ) is a Hopf $*$ -algebra.

Of course, there are other possible choices. The most general notion would be to take any Hopf algebra (A, Φ) with A finite-dimensional. If we restrict to the case of Hopf $*$ -algebras with a finite-dimensional C^* -algebra for the underlying algebra, we have the property that a finite quantum group is a group if and only if the underlying algebra A is abelian. We will see later in these notes that there are more reasons to define finite quantum groups in this way (within the theory of locally compact quantum groups).

Now we can discuss the main problems that arise if we want to generalize this. First of all, it turns out that the natural candidates for the maps ε and κ in interesting examples, are only densely defined, and not continuous. The counit ε is a $*$ -homomorphism on a dense $*$ -subalgebra. The fact that κ is an anti-homomorphism, related with the involution by the formula $\kappa(\kappa(a)^*)^* = a$ causes some extra difficulties. Then it is no longer obvious how one has to define the maps $\varepsilon \otimes \iota$, $\iota \otimes \varepsilon$, $\kappa \otimes \iota$ and $\iota \otimes \kappa$ on the image of Φ which lies in $M(A \otimes A)$. A last difficulty is the multiplication map m . It is only defined on the algebraic tensor product $A \odot A$ and it is very badly behaved with respect to both the algebraic and the topological structure.

That these maps (the counit ε , the antipode κ and the multiplication m) are badly behaved in the general quantum case, is certainly the main reason why, at this moment, there is not yet a satisfactory theory of locally compact quantum groups (in general).

Then, one might wonder why such a theory is desirable anyway. What kind of properties do we hope to obtain ? We would like to have conditions on Φ such that the following program can be carried out.

First of all, if (A, Φ) comes from a locally compact group as before, then Φ should satisfy the conditions. On the other hand, if A is abelian and Φ satisfies these conditions, one should be back in the group case. Remark that, in the finite case, with the definition 2.4 for a finite quantum group, this is the case. It should also be possible to develop a nice theory for these locally compact quantum groups. For instance we would like to prove

the existence and uniqueness of the Haar measure, to have a nice representation theory... We are also very much interested in a duality theory extending the Pontryagin duality for locally compact abelian groups. It is expected that the dual of a locally compact quantum group will again be such a locally compact quantum group. Finally, the conditions on Φ may not be too strong so that sufficiently many examples fit into this frame.

As we mentioned already in the introduction, there exist notions of locally compact quantum groups. First, we have the Kac algebras (see [6]). They have recently been generalized by Masuda and Nakagami [16] so as to include also the more recent examples where the antipode κ is not a $*$ -map. These two theories are formulated in the von Neumann algebra context. It would be desirable (and natural) to have a theory in the C^* -algebra framework. The work of Masuda, Nakagami and Woronowicz [18] has been announced at several conferences, among them in Warsaw [17], but the paper is still not available.

Generally, it is the opinion of many researchers in the field that there is still a long way to go. But equally strong is the feeling that, since there are several special cases and also nice examples not belonging to these special cases, it should be possible to come up with a comprehensive theory in the future.

Now, what are the special cases that have been studied and are understood? First there is of course the finite case. It is known that for any finite-dimensional Hopf algebra, there exist a left and a right invariant integral (the equivalent of the Haar measure). With our definition of a finite quantum group, the left and right invariant integrals coincide and are positive (like what you expect in the finite case). Of course, there is also a duality (see e.g. [22]). The dual algebra is again a finite-dimensional C^* -algebra. This is an extra motivation for our definition of a finite quantum group. For a simple treatment of finite quantum groups, we refer to [28].

There are two other special cases that are well understood. The first one is the compact case and the second one is the discrete case. Compactness and discreteness are topological properties. Therefore, they are reflected into properties of the underlying C^* -algebra.

In the compact case, this is easy. If G is a compact group, then $C_0(G)$ coincides with the C^* -algebra of all complex continuous functions on G . So $C_0(G)$ has an identity. Therefore, a compact quantum group will be a pair (A, Φ) of a C^* -algebra A with an identity and a comultiplication Φ satisfying certain extra conditions. In section 3 of these notes we will give these conditions and argue about them. And as we will see further in these notes, this notion of a compact quantum group yields a satisfactory theory with nice examples.

In the discrete case, the choice is not so obvious. If G is discrete, then the multiplier algebra $C_b(G)$ of $C_0(G)$ coincides with the double dual of $C_0(G)$. The C^* -algebra of compact operators on a Hilbert space also has this property, and direct sums of these. There are however several reasons to consider only direct sums of full matrix algebras $M_n(\mathbb{C})$ (the compact operators on a finite-dimensional Hilbert space). So, a discrete

quantum group is a pair (A, Φ) of a C^* -algebra A which is a C^* -direct sum of full matrix algebras satisfying certain properties. See [25] for a possible more precise definition. Also here, there is a nice theory with interesting examples.

The compact case is dual to the discrete case in a sense that can be made precise. Using the Haar measure on a compact quantum group, one can construct the dual, and this will be a discrete quantum group. We will show this in section 8 of these notes. There we will also give some more information about the notion of a discrete quantum group. Conversely, starting with a discrete quantum group, one can recover (more or less) the original compact quantum group by a similar process.

Many of the different approaches to compact quantum groups are in one way or another related with this duality between compact and discrete quantum groups. In [11] Kirchberg called a compact quantum group in our sense a discrete quantum group. And in [7] Effros and Ruan introduce discrete quantum groups while other people would claim that in fact, they study compact quantum groups from another point of view. Similar is the approach of Dijkhuizen and Koornwinder in [4]. In most of these other approaches, conditions are formulated, not only on the algebra A and the comultiplication Φ , but also on the dual objects. Very often then, it is easier to get to the main results. This is not at all surprising : Part of the main properties are then already included in the axioms.

We will discuss these different approaches further in these notes. Our philosophy should be clear already. A locally compact group is a locally compact space with a group structure, compatible with the topological structure. So a locally compact quantum group is a locally compact quantum space (a C^* -algebra) with a quantum group structure (a comultiplication with certain properties). Furthermore compactness is a property of the underlying topological space and therefore reflected in a property of the C^* -algebra (and not in extra properties on Φ). And similarly for discreteness.

In a discrete space, there is very little topology. Therefore it is of course not hard (and maybe more natural) to treat the discrete quantum groups outside the C^* -framework. And because of this ambiguity, due to the duality between the compact and the discrete case, it is also quite natural to treat the compact case without the use of C^* -algebra theory, such as in [4]. A more general theory is set up in [26, 27], where duality is studied within the category of Multiplier Hopf Algebras. This framework includes both the discrete and the compact quantum groups and the duality in this category includes the duality between discrete and compact quantum groups. This whole setting is purely algebraic, but can be fitted into the C^* -algebraic framework : In [12] and [13], starting from such an algebraic quantum group, a C^* -algebraic group in the sense of [18] is constructed.

In these notes, we look at the compact and the discrete cases as special cases of the more general locally compact quantum groups, and hence it is more obvious to do this within the C^* -algebra framework. In fact, one could say that the C^* -approach to quantum groups is the study of locally compact quantum groups.

This approach to quantum groups is necessary if we want to look at the Haar measure and if we want to study unitary representations for non-compact quantum groups. If e.g. we consider a non-compact matrix group, the Haar measure will not be defined on the function algebra generated by the matrix elements. These polynomial functions are nicely behaved with respect to the comultiplication (you get a Hopf algebra) but we need e.g. continuous functions tending (fast enough) to zero at infinity in order to use the Haar measure.

3 Compact Quantum Groups : Definition and examples

As we have seen in the previous section, a compact quantum group is a pair (A, Φ) of a C^* -algebra with an identity and a comultiplication Φ on A with some extra properties. We will discuss these extra conditions on Φ in this section. We will also show how these conditions can be verified in examples.

First observe once more that $M(A \otimes A)$ now coincides with $A \otimes A$ because we have an identity. So a comultiplication Φ on A is now a $*$ -homomorphism from A to $A \otimes A$. Such a $*$ -homomorphism is non-degenerate if and only if it is unital. So in this case, we obtain the following, easier characterization of a comultiplication.

Lemma 3.1 *If A is a C^* -algebra with an identity, then a comultiplication on A is a unital $*$ -homomorphism Φ from A to $A \otimes A$ satisfying coassociativity $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$.*

As before, we use the minimal C^* -tensor product. Again ι is the identity map and here $\Phi \otimes \iota$ and $\iota \otimes \Phi$ are the continuous extensions to $A \otimes A$ of the obvious maps on the algebraic tensor product.

Following the ideas of the previous section, a pair (A, Φ) of a C^* -algebra A with an identity and a comultiplication Φ on A is thought of as a compact quantum semi-group.

Now, what are the extra conditions on Φ in order to have a compact quantum group? To require the existence of a counit and an antipode is not the best choice as we indicated already in the previous section. We will come back to this in section 7. Instead, the notion of a compact quantum group, as defined by Woronowicz in [35], is motivated by the following two results.

First we have a group-theoretical result. It can be found in [8], but we include the simple proof for convenience of the reader.

Proposition 3.2 *A compact semi-group G with cancellation is a group.*

PROOF: Take any element $s \in G$. Consider the closed semi-group H generated by s . The intersection of two closed ideals of H is again such an ideal, and because H is generated

by s , it is non-empty. Hence, by compactness, the intersection I of all closed ideals of H is non-empty. So it is the smallest closed ideal in H .

For any $p \in I$ we have $pI \subseteq I$ and pI is still a closed ideal in H . Hence $pI = I$ for all $p \in I$. Given $p \in I$ there is an element $e \in I$ such that $pe = p$. Multiply with any $q \in G$ to the right and cancel p to obtain that $eq = q$. Then multiply with any element $p \in G$ to the left and cancel q to get $pe = p$. So e is the identity of G .

Now, from $se = s$ and the fact that $e \in I$ and that I is an ideal, we must have $s \in I$. Because $sI = I$ and $e \in I$, s has an inverse. So G is a group. ■

Next, we translate the cancellation law in terms of Φ . We formulate the result for locally compact semi-groups.

Proposition 3.3 *Let G be a locally compact semi-group, let $A = C_0(G)$ and $\Phi : A \rightarrow M(A \otimes A)$ be defined by $(\Phi(f))(p, q) = f(pq)$ as before. Then G has the cancellation property if the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense subsets of $A \otimes A$.*

PROOF: First let us assume that the set $\Phi(A)(1 \otimes A)$ is a dense subset of $A \otimes A$. Take elements p, q, r in G and assume $pr = qr$. Then, for all f, g in A we have

$$(\Phi(f)(1 \otimes g))(p, r) = f(pr)g(r)$$

$$(\Phi(f)(1 \otimes g))(q, r) = f(qr)g(r)$$

and by assumption, these numbers are equal. Since this is true for all f, g , again by assumption

$$h(p, r) = h(q, r)$$

for all $h \in A \otimes A$. Therefore $p = q$. So G has right cancellation. Similarly, if $\Phi(A)(1 \otimes A)$ is a dense subset of $A \otimes A$, we have left cancellation. ■

If G is a locally compact group, the maps

$$(p, q) \rightarrow (pq, q)$$

$$(p, q) \rightarrow (p, pq)$$

are homeomorphisms of $G \times G$. They give rise to isomorphisms of $A \otimes A$. These isomorphisms are given by

$$f \otimes g \rightarrow \Phi(f)(1 \otimes g)$$

$$f \otimes g \rightarrow (f \otimes 1)\Phi(g).$$

So, when G is a locally compact group, the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense subspaces of $A \otimes A$.

If we combine this result with proposition 3.2, we find that also the converse of proposition 3.3 holds for a compact semi-group.

All these observations motivate the following definition of Woronowicz (given in [35]).

Definition 3.4 A compact quantum group is a pair (A, Φ) of a C^* -algebra A with identity and a multiplication Φ on A such that the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense in $A \otimes A$.

From the preceding propositions, we see that any compact group G will give rise to a compact quantum group and conversely that any compact quantum group (A, Φ) with A abelian arises in this way. Recall from the previous section that this was one of the properties that we considered desirable for our theory.

Further in the notes, we will show that a nice theory can be developed, starting from these axioms and that this theory is very similar to the classical theory. These are features which we discussed in the previous section as well, and which motivate the choice of this definition.

Later in these notes, we will discuss the connection with the definition of Dijkhuizen and Koornwinder in [4]. For this we refer to section 7.

Of course, it is also important that the axioms are relatively easy to verify in the examples. This is not so obvious for the density conditions. Usually such conditions need the existence of some kind of an antipode. We will show now how this can be done.

We will need the notion of a finite-dimensional representation.

Definition 3.5 Let (A, Φ) be a pair of a C^* -algebra A with identity and a $*$ -homomorphism $\Phi : A \rightarrow A \otimes A$. By a finite-dimensional representation of (A, Φ) we mean a matrix (v_{pq}) in $M_n(A)$ for some n such that

$$\Phi(v_{pq}) = \sum_{k=1}^n v_{pk} \otimes v_{kq}$$

for all $p, q = 1, \dots, n$. If the matrix has an inverse in $M_n(A)$, then the representation is called non-degenerate.

We will come back to this definition, and give some motivation in sections 5 and 6 where we study representations more extensively.

Using this notion, we can give the following important result.

Proposition 3.6 *Let A be a unital C^* -algebra and let $\Phi : A \rightarrow A \otimes A$ be a $*$ -homomorphism. Assume that A is generated, as a normed algebra, by the matrix elements of its non-degenerate finite-dimensional representations. Then (A, Φ) is a compact quantum group.*

PROOF: Let (v_{pq}) be a finite-dimensional representation of (A, Φ) . Then clearly

$$(\iota \otimes \Phi)\Phi(v_{pq}) = (\Phi \otimes \iota)\Phi(v_{pq}) = \sum_{k,l} v_{pk} \otimes v_{kl} \otimes v_{lq}$$

for all $p, q = 1, \dots, n$. This implies that Φ is coassociative.

Now assume that (v_{pq}) is a non-degenerate finite-dimensional representation and that (w_{pq}) is the inverse of (v_{pq}) . Then we have

$$\sum_k \Phi(v_{pk})(1 \otimes w_{kq}) = \sum_{k,l} v_{pl} \otimes v_{lk} w_{kq} = \sum_l v_{pl} \otimes \delta_{lq} 1 = v_{pq} \otimes 1.$$

Here δ denotes the Kronecker delta. So $v_{pq} \otimes 1 \in \Phi(A)(1 \otimes A)$. Now, suppose that

$$\begin{aligned} a \otimes 1 &= \sum_k \Phi(p_k)(1 \otimes q_k) \\ b \otimes 1 &= \sum_l \Phi(r_l)(1 \otimes s_l) \end{aligned}$$

with $p_k, q_k, r_l, s_l \in A$. Then

$$\begin{aligned} ab \otimes 1 &= \sum_k \Phi(p_k)(1 \otimes q_k)(b \otimes 1) \\ &= \sum_k \Phi(p_k)(b \otimes 1)(1 \otimes q_k) \\ &= \sum_{k,l} \Phi(p_k)\Phi(r_l)(1 \otimes s_l)(1 \otimes q_k) \\ &= \sum_{k,l} \Phi(p_k r_l)(1 \otimes s_l q_k). \end{aligned}$$

So, if $a \otimes 1, b \otimes 1 \in \Phi(A)(1 \otimes A)$, then also $ab \otimes 1 \in \Phi(A)(1 \otimes A)$.

Because $v_{pq} \otimes 1 \in \Phi(A)(1 \otimes A)$ when (v_{pq}) is a non-degenerate representation and these matrix elements generate A , we must have that $A \otimes 1$ is contained in the closure of $\Phi(A)(1 \otimes A)$. Then of course, $\Phi(A)(1 \otimes A)$ is dense in $A \otimes A$. Similarly $(A \otimes 1)\Phi(A)$, and by taking adjoints also $\Phi(A)(A \otimes 1)$ is dense in $A \otimes A$. \blacksquare

Because also the converse of this proposition is true (we will show this in section 7), one can hope that the result will be useful to construct examples.

The following proposition is an easy consequence of the previous one.

Proposition 3.7 *Let (A, Φ) be as in the previous proposition. Now, assume that A is generated, as a C^* -algebra, by the matrix elements of its non-degenerate, finite-dimensional representations (v_{pq}) so that also (v_{pq}^*) is an invertible matrix in $M_n(A)$. Then (A, Φ) is a compact quantum group.*

PROOF: Remark that

$$\Phi(v_{pq}^*) = \sum_k v_{pk}^* \otimes v_{kq}^*$$

so that (v_{pq}^*) is also a non-degenerate representation. Then the result follows from the previous proposition. \blacksquare

We will have a closer look at this representation in section 6. It will be called the adjoint representation. We will see that it is automatically non-degenerate, if we already have that (A, Φ) is a compact quantum group.

There is an obvious notion of direct sum of finite-dimensional representations. Therefore, when A is generated by the matrix elements of a finite number of non-degenerate finite-dimensional representations, it is in fact generated by the matrix elements of one non-degenerate finite-dimensional representation. In this case we have a compact matrix quantum group, as in [32] and [34]. Note that in [32] this is called a compact matrix pseudogroup.

For such a compact matrix quantum group we find the following simple characterization as a consequence of proposition 3.7. It says that a compact matrix quantum group as defined in [34] indeed satisfies the axioms of a compact quantum group given in [35].

Proposition 3.8 *Let A be a C^* -algebra with identity and $\Phi : A \rightarrow A \otimes A$ a $*$ -homomorphism. Assume that A is generated, as a C^* -algebra, by elements (u_{pq}) such that*

$$\Phi(u_{pq}) = \sum_k u_{pk} \otimes u_{kq}$$

and such that the matrix (u_{pq}) and the transpose matrix (u_{qp}) are invertible. Then (A, Φ) is a compact quantum group.

PROOF: The transpose matrix (u_{qp}) is the conjugate of the matrix (u_{pq}^*) , so the invertibility of the matrix (u_{qp}) in $M_n(A)$ is equivalent with the invertibility of (u_{pq}^*) . Now the result follows from proposition 3.7. \blacksquare

Let us now have a look at some examples. The first example we consider is $SU_\lambda(n)$ (see e.g. [33] and [22]).

Example 3.9 Fix $\lambda \in \mathbb{R}$. Let A_0 be the universal unital $*$ -algebra generated by elements $\{u_{pq} \mid p, q = 1, \dots, n\}$ such that

$$\begin{aligned} \sum_k u_{kp}^* u_{kq} &= \delta_{pq} 1 \\ \sum_k u_{pk} u_{qk}^* &= \delta_{pq} 1 \end{aligned}$$

and such that

$$\sum_{k_i} E(k_1, k_2, \dots, k_n) u_{l_1 k_1} u_{l_2 k_2} \dots u_{l_n k_n} = E(l_1, l_2, \dots, l_n) 1.$$

Here $E(k_1, k_2, \dots, k_n) = 0$ when two indices are equal and otherwise

$$E(k_1, k_2, \dots, k_n) = (-\lambda)^{I(k_1, k_2, \dots, k_n)}$$

where $I(k_1, k_2, \dots, k_n)$ is the number of inversed pairs in the permutation (k_1, k_2, \dots, k_n) .

We can define a comultiplication Φ on A_0 by $\Phi(u_{pq}) = \sum u_{pk} \otimes u_{kq}$. It is a straightforward calculation to verify that the candidates for $\Phi(u_{pq})$ satisfy the same relations in $A_0 \odot A_0$ as the elements u_{pq} . In this way (A_0, Φ) becomes a Hopf *-algebra. This was first shown by Woronowicz in [33]. However in [22] a simpler (and more direct) proof is given. The counit ε is given by $\varepsilon(u_{kl}) = \delta_{kl}$ and the antipode κ by $\kappa(u_{kl}) = u_{lk}^*$. It is clear that ε respects the relations, and that it extends to the counit, satisfying $(\varepsilon \otimes \iota)\Phi = (\iota \otimes \varepsilon)\Phi = \iota$. The main problem is κ : it is not a *-anti-homomorphism. But it can be shown that the adjoints u_{pq}^* belong to the algebra generated by the u_{kl} , and therefore κ is completely determined. The next thing then is to show that this candidate for the antipode respects the defining relations. The formulas

$$m(\kappa \otimes \iota)\Phi(u_{pq}) = \varepsilon(u_{pq})1 \quad \text{and} \quad m(\iota \otimes \kappa)\Phi(u_{pq}) = \varepsilon(u_{pq})1$$

are nothing but the first two defining relations.

In our setting, we do not really need the fact that (A_0, Φ) is a Hopf *-algebra. Observe that (u_{pq}) is a unitary. Hence it is invertible in $M_n(A_0)$. Also (see e.g. lemma 4.7 in [22]) the matrix (u_{pq}^*) is invertible.

So we just have to argue that there is a C*-norm for which Φ is continuous. For $a \in A_0$, let

$$\|a\|_0 = \sup\{\|\pi(a)\| \mid \pi \text{ is a } *- \text{representation of the algebra } A_0\}.$$

Note that the set of *-representations of A_0 is non-empty, take e.g. $\pi(u_{pq}) = \delta_{pq}$. Let π be any *-representation of the algebra A_0 on a Hilbert space \mathcal{H} . Since (u_{pq}) is unitary, we have $\sum_k u_{kq}^* u_{kq} = 1$. Hence for all $\xi \in \mathcal{H}$,

$$\|\xi\|^2 = \langle \xi, \xi \rangle = \sum_k \langle \pi(u_{kq}^* u_{kq}) \xi, \xi \rangle = \sum_k \langle \pi(u_{kq}) \xi, \pi(u_{kq}) \xi \rangle \geq \|\pi(u_{pq}) \xi\|^2.$$

So $\|\pi(u_{pq})\| \leq 1$. This implies that $\|a\|_0$ is finite for all $a \in A$. So we have a C*-seminorm on A_0 . Note that Φ is continuous with respect to this seminorm. Let $I = \{a \in A_0 \mid \|a\|_0 = 0\}$. Then I is a two-sided ideal in A_0 . Define A to be the C*-completion of A_0/I for the C*-norm $\|\cdot\|_0$. Remark that Φ gives a well-defined comultiplication on A_0/I . By continuity we can extend this comultiplication to the whole of A . We will denote this extension again by Φ . Proposition 3.7 now gives that (A, Φ) is a compact quantum group.

It is a priori not obvious that this quantum group is non-trivial. However, it can be shown that there are enough $*$ -representations of the algebra A_0 (see below).

We will have a closer look at the case $n = 2$. From lemma 4.6 of [22] it follows that $u_{12}^* = -\lambda u_{21}$ and that $u_{11}^* = u_{22}$. If we denote u_{11} by α and u_{21} by γ , we get

$$u = \begin{pmatrix} \alpha & -\lambda\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

and the defining relations become

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= 1 & \alpha\alpha^* + \lambda^2\gamma\gamma^* &= 1 \\ \alpha\gamma &= \lambda\gamma\alpha & \alpha\gamma^* &= \lambda\gamma^*\alpha & \gamma\gamma^* &= \gamma^*\gamma. \end{aligned}$$

The comultiplication now is given by

$$\begin{aligned} \Phi(\alpha) &= \alpha \otimes \alpha - \lambda\gamma^* \otimes \gamma \\ \Phi(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma. \end{aligned}$$

Details about the quantum group $SU_\lambda(2)$ can be found in [31]. It is shown in [31] that this $*$ -algebra can be faithfully represented by bounded operators on a Hilbert space. In fact, these representations of $SU_\lambda(2)$ can be used to construct enough $*$ -representations of $SU_\lambda(n)$ to assure that this quantum group is non-trivial.

Example 3.10 This example is the object of study in [29]. Let $Q \in M_n(\mathbb{C})$ be any invertible matrix. For a matrix $u = (u_{pq})$ with entries in a $*$ -algebra, we denote by u^t the transpose of u , by \bar{u} the matrix (u_{ij}^*) , and $u^* = \bar{u}^t$ the usual conjugation of u . We use the notation I for the $n \times n$ matrix with entries $\delta_{ij}1$, where 1 is the identity of the algebra.

Let $A_u(Q)$ be the universal unital C^* -algebra generated by elements $\{u_{pq} \mid p, q = 1, \dots, n\}$ subject to the following two sets of relations:

$$uu^* = I = u^*u$$

and

$$u^t Q \bar{u} Q^{-1} = I = Q \bar{u} Q^{-1} u^t.$$

One can define a comultiplication Φ on $A_u(Q)$ by $\Phi(u_{pq}) = \sum_k u_{pk} \otimes u_{kq}$. It is shown in [29] that the candidates for $\Phi(u_{pq})$ satisfy the same relations in $A_u(Q) \otimes A_u(Q)$ as the elements u_{pq} . The second set of relations for the u_{pq} 's implies that the matrix u^t is invertible. So by proposition 3.8, $(A_u(Q), \Phi)$ is a compact quantum group.

Also here, there is a problem with the C^* -norm. For some matrices Q , it is in principle possible that the requirement of having a C^* -norm introduces more relations among the generators so that e.g. in the extreme case, the result is a trivial C^* -algebra. Fortunately, for enough matrices Q , we get non-trivial C^* -algebras (see also [29]).

These compact matrix quantum groups are universal in the sense that any compact matrix quantum group is a quantum subgroup of some of these. Here a compact quantum group B is called a quantum subgroup of another compact quantum group A if there is a surjective morphism of C^* -algebras from A to B that preserves the coproducts. We will give a sketch of the proof of this universality property. For more details, we refer to [29].

Let $G = (A, \Phi)$ be a compact matrix quantum group, generated by the matrix elements of a non-degenerate finite-dimensional representation u . From theorem 6.4 it will follow that we may assume that u is unitary. This gives the first set of relations in the definition of the universal compact matrix quantum groups $A_u(Q)$.

Let h be the Haar measure on (A, Φ) (see section 4). Put $Q = (\iota \otimes h)u^t\bar{u}$. Then Q is an invertible matrix in $M_n(A)$ (see e.g. the proof of proposition 6.4, in section 6 of this paper, for $v = \bar{u}$). It is shown in [29] that it satisfies the second set of defining relations in the definition of the universal compact matrix quantum groups.

4 The Haar measure

A very deep and non-trivial result in the theory of locally compact groups is the existence and uniqueness (up to a scalar) of the Haar measure. One of the main difficulties in finding a satisfactory definition for a locally compact quantum group seems to be exactly this ability to deduce the existence of (the equivalent of) the Haar measure from the set of axioms. The fact that this can be done in the case of compact quantum groups shows again the strength of Woronowicz' definition. However, in his proof, Woronowicz assumed the extra condition that the quantum group admits a faithful state (which is fulfilled in particular when it is separable). In [24], it is shown that this condition is not necessary. We will follow this approach here. The basic idea of the proof is the same as in [35], and is essentially already given in [32]. Also the techniques here were already used by Woronowicz.

For a compact group G , left invariance of the Haar measure h means that for any $f \in C(G)$

$$\int_G f(ts) dh(s) = \int_G f(s) dh(s) \quad (4.1)$$

for all $t \in G$. This notion can be translated to the corresponding C^* -algebra $C(G)$. The Haar measure defines a linear functional on $C(G)$ which we will again denote by h . Then we have for $F \in C(G \times G)$ that

$$((\iota \otimes h)F)(t) = \int_G F(t, s) dh(s)$$

for any $t \in G$. So equation (4.1) can be expressed as

$$(\iota \otimes h)\Phi(f) = h(f)1$$

for any $f \in C(G)$. Similarly, right invariance of h will amount to having

$$(h \otimes \iota)\Phi(f) = h(f)1$$

for any $f \in C(G)$. This justifies the following definition.

Definition 4.1 Let $G = (A, \Phi)$ be a compact quantum group. A linear functional h on A is called left invariant if $(\iota \otimes h)\Phi(a) = h(a)1$ for all a in A . A linear functional h' on A is called right invariant if $(h' \otimes \iota)\Phi(a) = h'(a)1$ for all a in A .

Let (A, Φ) be a compact quantum group. We will denote the space of bounded linear functionals on A by A^* . The comultiplication on A induces a product on A^* by

$$(\omega_1 \omega_2)(a) = (\omega_1 \otimes \omega_2)\Phi(a)$$

when $a \in A$ and $\omega_1, \omega_2 \in A^*$. We will show that there is a state h on A such that $\omega h = h\omega = \omega(1)h$ for any $\omega \in A^*$. First, we prove two lemmas.

Lemma 4.2 *Let ω be a state on A . Then there is a state h such that $h\omega = \omega h = h$.*

PROOF: Define

$$\omega_n = \frac{1}{n}(\omega + \omega^2 + \dots + \omega^n).$$

Then ω_n is a state for all n . Consider a weak *-limit point of the sequence (ω_n) . It is still a state because A has an identity. As

$$\|\omega_n \omega - \omega_n\| = \frac{1}{n} \|\omega^{n+1} - \omega\| \leq \frac{2}{n},$$

we get in the limit $h\omega = h$. Similarly $\omega h = h$. ■

Lemma 4.3 *Let ω and h be states on A such that $h\omega = \omega h = h$. If $\rho \in A^*$ and $0 \leq \rho \leq \omega$, then also $\rho h = \rho(1)h$.*

PROOF: Let $a \in A$ and define $b = (\iota \otimes h)\Phi(a)$. Then

$$(\iota \otimes \omega)\Phi(b) = (\iota \otimes \omega h)\Phi(a) = (\iota \otimes h)\Phi(a) = b.$$

So

$$\begin{aligned} & (\iota \otimes \omega)((\Phi(b) - b \otimes 1)^*(\Phi(b) - b \otimes 1)) \\ &= (\iota \otimes \omega)(\Phi(b^*b)) + b^*b - (\iota \otimes \omega)(\Phi(b^*)(b \otimes 1)) - (\iota \otimes \omega)((b^* \otimes 1)\Phi(b)) \\ &= (\iota \otimes \omega)\Phi(b^*b) - b^*b. \end{aligned}$$

If we apply h to the above equation, we get

$$(h \otimes \omega)((\Phi(b) - b \otimes 1)^*(\Phi(b) - b \otimes 1)) = 0.$$

Because $0 \leq \rho \leq \omega$, also

$$(h \otimes \rho)((\Phi(b) - b \otimes 1)^*(\Phi(b) - b \otimes 1)) = 0.$$

By the Cauchy-Schwarz inequality, we obtain

$$(h \otimes \rho)((c \otimes 1)(\Phi(b) - b \otimes 1)) = 0$$

for all $c \in A$. If now we use that $b = (\iota \otimes h)\Phi(a)$, we get

$$(h \otimes \rho \otimes h)((c \otimes 1 \otimes 1)(\Phi \otimes \iota)\Phi(a)) = \rho(1)(h \otimes h)((c \otimes 1)\Phi(a)).$$

Now

$$\begin{aligned} (c \otimes 1 \otimes 1)(\Phi \otimes \iota)\Phi(a) &= (c \otimes 1 \otimes 1)(\iota \otimes \Phi)\Phi(a) \\ &= (\iota \otimes \Phi)((c \otimes 1)\Phi(a)). \end{aligned}$$

By the definition of a compact quantum group, linear combinations of elements of the form $(c \otimes 1)\Phi(a)$ are dense in $A \otimes A$. So we can in the above equality replace $(c \otimes 1)\Phi(a)$ by $1 \otimes q$. We obtain

$$(h \otimes \rho \otimes h)((\iota \otimes \Phi)(1 \otimes q)) = \rho(1)(h \otimes h)(1 \otimes q),$$

and hence, because $h(1) = 1$, we get

$$(\rho \otimes h)(\Phi(q)) = \rho(1)h(q)$$

for any $q \in A$. This completes the proof. ■

The existence of an invariant functional now follows easily by using a compactness argument.

Theorem 4.4 *There is a unique left invariant state on A . It is also right invariant.*

PROOF: For any $\omega \geq 0$, define

$$K_\omega = \{h \in A^* \mid h \text{ is a state and } \omega h = \omega(1)h\}.$$

This is a compact subset of A^* for the weak $*$ -topology. It is non-empty because of lemma 4.2. By lemma 4.3, we have that $K_\omega \subseteq K_\rho$ when $0 \leq \rho \leq \omega$. It follows that

$K_{\omega_1+\omega_2} \subseteq K_{\omega_1} \cap K_{\omega_2}$ for any pair of positive linear functionals ω_1, ω_2 . Therefore the intersection of all K_ω is non-empty. This proves the existence of a left invariant state.

Similarly, or by applying this result to the opposite comultiplication Φ' , obtained from Φ by using the flip, there is a state h' such that $h'\rho = \rho(1)h'$ for all $\rho \in A^*$. Then $h'h = h$ by the first formula and $h'h = h'$ by the second formula. So $h' = h$ and the result is proved. ■

So, as in the classical case of a compact group, the left and right invariant state coincide. It is called the Haar measure on (A, Φ) .

For an example of an explicit calculation of the Haar measure on a concrete compact quantum group, we refer to [32] and to [19], where the Haar measures on $SU_\lambda(2)$ and on $SU_\lambda(n)$ respectively are calculated.

In general, h needs not to be faithful. The following example gives a compact quantum group with a Haar measure which is not faithful.

Example 4.5 Let G be a discrete group. Denote by $C^*(G)$ the full group C^* -algebra of G . Because G is discrete, this is a unital C^* -algebra. For every $s \in G$, we can define $U(s) \in M(C^*(G))$ such that

$$(U(s)f)(t) = f(s^{-1}t)$$

for $f \in L^1(G)$. The mapping $s \mapsto U(s)$ defines a strictly continuous group representation of G as multipliers on $C^*(G)$. Also the mapping $s \mapsto U(s) \otimes U(s)$ is a strictly continuous group representation of G on $M(C^*(G) \otimes C^*(G))$. Note that in this context, usually the maximal C^* -norm is used. In this paper we have chosen to use the minimal C^* -tensor product, for which the construction of this example remains valid. It can be shown that one obtains a unique non-degenerate C^* -homomorphism $\Phi : C^*(G) \rightarrow M(C^*(G) \otimes C^*(G))$ such that

$$\Phi(U(s)) = U(s) \otimes U(s)$$

for all $s \in G$. Because $(U(s) \otimes U(s)) \otimes U(s) = U(s) \otimes (U(s) \otimes U(s))$, Φ is coassociative. It is clear that the sets $\Phi(C^*(G))(1 \otimes C^*(G))$ and $\Phi(C^*(G))(C^*(G) \otimes 1)$ are dense in $C^*(G) \otimes C^*(G)$. Hence $(C^*(G), \Phi)$ is a compact quantum group. For the details of this construction, we refer to [15]. In the case that G is a discrete, non-amenable group, the Haar measure on $(C^*(G), \Phi)$ is not faithful.

We will come back to the question of faithfulness of the Haar measure in section 7.

5 The right regular representation

Let us first define what is meant by a unitary representation of a compact quantum group (A, Φ) .

Let \mathcal{H} be a Hilbert space and denote by $\mathcal{B}_0(\mathcal{H})$ the C^* -algebra of all compact operators on \mathcal{H} . Consider the multiplier algebra $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$. This algebra has two natural embeddings into the multiplier algebra $M(\mathcal{B}_0(\mathcal{H}) \otimes A \otimes A)$. The first one is obtained by extending the map $x \mapsto x \otimes 1$ from $\mathcal{B}_0(\mathcal{H}) \otimes A$ to $\mathcal{B}_0(\mathcal{H}) \otimes A \otimes A$. The second one is obtained by composing this map with the flip on the last two factors. It maps $x \otimes a$ to $x \otimes 1 \otimes a$ when $x \in \mathcal{B}_0(\mathcal{H})$ and $a \in A$. We will write $v_{(12)}$ and $v_{(13)}$ for the images of an element $v \in M(\mathcal{B}_0(\mathcal{H}) \otimes A)$ by these two maps. This is the common *leg numbering notation*. We use brackets to make a distinction with matrix elements.

Definition 5.1 Let (A, Φ) be a compact quantum group. A representation of (A, Φ) on a Hilbert space \mathcal{H} is an element v in $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$ such that

$$(\iota \otimes \Phi)(v) = v_{(12)}v_{(13)}. \quad (5.2)$$

If v is unitary, we call it a unitary representation.

Here $\iota \otimes \Phi$ is the unique extension to $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$ of $\iota \otimes \Phi$ on the algebraic tensor product.

To see where this definition comes from, consider a strongly continuous unitary representation $p \mapsto u(p)$ of a compact group on a Hilbert space \mathcal{H} . Then u is a continuous map from G to $\mathcal{B}(\mathcal{H})$ where $\mathcal{B}(\mathcal{H})$ is considered with the strong operator topology. Because u is unitary, u is also continuous when we take $\mathcal{B}(\mathcal{H})$ with the strict topology, as the multiplier algebra of $\mathcal{B}_0(\mathcal{H})$. This means that u can be considered as an element in $M(\mathcal{B}_0(\mathcal{H}) \otimes C(G))$. We can identify elements in $M(\mathcal{B}_0(\mathcal{H}) \otimes C(G) \otimes C(G))$ with strictly continuous $\mathcal{B}(\mathcal{H})$ -valued functions on $G \times G$. Then we have

$$\begin{aligned} u_{(12)}(p, q) &= u(p) \\ u_{(13)}(p, q) &= u(q). \end{aligned}$$

Moreover, when Φ is the comultiplication on $C(G)$ given by $(\Phi f)(p, q) = f(pq)$ as before, we also get

$$((\iota \otimes \Phi)(u))(p, q) = u(pq).$$

So we see that $(\iota \otimes \Phi)(u) = u_{(12)}u_{(13)}$ means nothing else but $u(pq) = u(p)u(q)$ for all $p, q \in G$.

A few words about the terminology : When the algebra A is non-commutative, it does not correspond anymore to functions on a compact group. But we could think of it as the set of continuous functions on an imaginary (compact) geometric object. When we speak

of representations of the quantum group $G = (A, \Phi)$, we refer to representations of this underlying geometric object. These are in fact the *corepresentations* of the *coalgebra*.

Note that, when v is a representation of a compact quantum group (A, Φ) on a finite-dimensional Hilbert space \mathcal{H} , we can write $v = \sum e_{pq} \otimes v_{pq}$ where (e_{pq}) are matrix units in $\mathcal{B}(\mathcal{H})$. In that case (5.2) means $\Phi(v_{pq}) = \sum_{k=1}^n v_{pk} \otimes v_{kq}$. So we get the notion of finite-dimensional representation as we defined it in section 3.

Now the Haar measure h on the compact quantum group (A, Φ) , obtained in the previous section, will be used to construct the right regular representation. This construction and the proofs in this section are essentially copied from [35].

Consider the G.N.S. representation of A associated with h . We denote the Hilbert space simply by \mathcal{H} , the cyclic vector by ξ_0 and we will omit π_h in the sense that we will write $a\xi$ for $\pi_h(a)\xi$ when $a \in A$ and $\xi \in \mathcal{H}$.

Let \mathcal{K} be another Hilbert space and assume that A acts faithfully and non-degenerately on \mathcal{K} . Also here we will write $a\xi$ for the action of an element $a \in A$ on a vector $\xi \in \mathcal{K}$.

Proposition 5.2 *There is a unitary operator u on $\mathcal{H} \otimes \mathcal{K}$ defined by $u(a\xi_0 \otimes \eta) = \Phi(a)(\xi_0 \otimes \eta)$ when $a \in A$ and $\eta \in \mathcal{K}$.*

PROOF: If $a_1, a_2, \dots, a_n \in A$ and $\eta_1, \eta_2, \dots, \eta_n \in \mathcal{K}$, we have

$$\begin{aligned} \left\| \sum_i \Phi(a_i)(\xi_0 \otimes \eta_i) \right\|^2 &= \sum_{i,j} \langle \Phi(a_j^* a_i) \xi_0 \otimes \eta_i, \xi_0 \otimes \eta_j \rangle \\ &= \sum_{i,j} \langle (h \otimes \iota) \Phi(a_j^* a_i) \eta_i, \eta_j \rangle \\ &= \sum_{i,j} h(a_j^* a_i) \langle \eta_i, \eta_j \rangle \\ &= \left\| \sum_i a_i \xi_0 \otimes \eta_i \right\|^2. \end{aligned}$$

It follows that there is an isometric operator u on $\mathcal{H} \otimes \mathcal{K}$ given by $u(a\xi_0 \otimes \eta) = \Phi(a)(\xi_0 \otimes \eta)$ whenever $a \in A$ and $\eta \in \mathcal{K}$.

Let $a \in A$ and $\eta \in \mathcal{K}$. From the density of $\Phi(A)(1 \otimes A)$ in $A \otimes A$ it follows that $a\xi_0 \otimes \eta$ can be approximated by linear combinations of elements $\Phi(b)(\xi_0 \otimes c\eta)$ with $b, c \in A$. Now $\Phi(b)(\xi_0 \otimes c\eta) = u(b\xi_0 \otimes c\eta)$. Since this holds for any $a \in A$ and ξ_0 is cyclic, we see that the range of u is dense. Therefore u is unitary. \blacksquare

We will now show that u is a multiplier of $\mathcal{B}_0(\mathcal{H}) \otimes A$. Because $\mathcal{B}_0(\mathcal{H}) \otimes A$ acts faithfully and non-degenerately on $\mathcal{H} \otimes \mathcal{K}$, and because finite rank operators are dense in $\mathcal{B}_0(\mathcal{H})$, it will be sufficient to show that $u(x \otimes 1)$ and $(x \otimes 1)u$ are in $\mathcal{B}_0(\mathcal{H}) \otimes A$ for all finite rank operators x .

Proposition 5.3 *The element u is a multiplier of $\mathcal{B}_0(\mathcal{H}) \otimes A$.*

PROOF: Let x be a rank one operator of the form $x\xi = \langle \xi, \xi_1 \rangle a\xi_0$ where $a \in A$ and $\xi_1, \xi \in \mathcal{H}$. Then, for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, we have

$$\begin{aligned} u(x \otimes 1)(\xi \otimes \eta) &= \langle \xi, \xi_1 \rangle u(a\xi_0 \otimes \eta) \\ &= \langle \xi, \xi_1 \rangle \Phi(a)\xi_0 \otimes \eta. \end{aligned}$$

Now, we can approximate $\Phi(a)$ by elements of the form $\sum b_i \otimes c_i$. We have

$$\langle \xi, \xi_1 \rangle (\sum b_i \otimes c_i)(\xi_0 \otimes \eta) = (\sum y_i \otimes c_i)(\xi \otimes \eta),$$

where y_i is the rank one operator given by $y_i\xi = \langle \xi, \xi_1 \rangle b_i\xi_0$. One can check that

$$\left\| u(x \otimes 1) - \sum y_i \otimes c_i \right\| \leq \|\xi_1\| \|\xi_0\| \left\| \Phi(a) - \sum b_i \otimes c_i \right\|.$$

This proves that $u(x \otimes 1) \in \mathcal{B}_0(\mathcal{H}) \otimes A$. It follows by the density of $A\xi_0$ in \mathcal{H} that this will be true for all rank one operators x .

Again let x be of the form $x\xi = \langle \xi, \xi_1 \rangle a\xi_0$ and now consider $u^*(x \otimes 1)$. For $x \in \mathcal{H}$ and $\eta \in \mathcal{K}$ we get

$$u^*(x \otimes 1)(\xi \otimes \eta) = \langle \xi, \xi_1 \rangle u^*(a\xi_0 \otimes \eta).$$

By density of $\Phi(A)(1 \otimes A)$, we can approximate $a \otimes 1$ by elements of the form $\sum_i \Phi(b_i)(1 \otimes c_i)$. Now

$$\begin{aligned} \langle \xi, \xi_1 \rangle u^* \sum \Phi(b_i)(1 \otimes c_i)(\xi_0 \otimes \eta) &= \langle \xi, \xi_1 \rangle \sum b_i \xi_0 \otimes c_i \eta \\ &= \left(\sum y_i \otimes c_i \right) (\xi \otimes \eta), \end{aligned}$$

where $y_i\xi = \langle \xi, \xi_1 \rangle b_i\xi_0$. Again, one can check that

$$\left\| u^*(x \otimes 1) - \sum y_i \otimes c_i \right\| \leq \|\xi_1\| \|\xi_0\| \left\| a \otimes 1 - \sum \Phi(b_i)(1 \otimes c_i) \right\|.$$

It follows that $u^*(x \otimes 1) \in \mathcal{B}_0(\mathcal{H}) \otimes A$ and again by density, that this holds for all rank one operators x .

By the remark preceding the proposition, this completes the proof. ■

Now, we can consider $(\iota \otimes \Phi)(u)$ and prove that u is a representation.

Theorem 5.4 *We have $(\iota \otimes \Phi)(u) = u_{(12)}u_{(13)}$ and therefore, u is a unitary representation of the compact quantum group (A, Φ) .*

PROOF: Let $b, c \in A$ and $\eta_1, \eta_2 \in \mathcal{K}$. Then

$$\begin{aligned} u_{(12)}(b \otimes 1 \otimes c)(\xi_0 \otimes \eta_1 \otimes \eta_2) &= \Phi(b)(\xi_0 \otimes \eta_1) \otimes c\eta_2 \\ &= (\Phi \otimes \iota)(b \otimes c)(\xi_0 \otimes \eta_1 \otimes \eta_2). \end{aligned}$$

Let $a \in A$. Again because we can approximate $\Phi(a)$ by elements of the form $\sum b_i \otimes c_i$, we can replace $b \otimes c$ in this equation by $\Phi(a)$. We get

$$u_{(12)}u_{(13)}(a\xi_0 \otimes \eta_1 \otimes \eta_2) = ((\Phi \otimes \iota)\Phi(a))(\xi_0 \otimes \eta_1 \otimes \eta_2).$$

On the other hand, we have for any $y \in \mathcal{B}(\mathcal{H})$ and $b \in A$ that

$$(\iota \otimes \Phi)(y \otimes b)(a\xi_0 \otimes \eta_1 \otimes \eta_2) = ya\xi_0 \otimes \Phi(b)(\eta_1 \otimes \eta_2).$$

Let $x \in \mathcal{B}_0(\mathcal{H})$ be any compact operator on \mathcal{H} . Then $(x \otimes 1)u \in \mathcal{B}_0(\mathcal{H})$ can be approximated by linear combinations of elements $y \otimes b$. Hence we can replace in the previous formula $y \otimes b$ by $(x \otimes 1)u$. We get

$$(x \otimes 1 \otimes 1)((\iota \otimes \Phi)(u))(a\xi_0 \otimes \eta_1 \otimes \eta_2) = (x \otimes 1 \otimes 1)((\iota \otimes \Phi)\Phi(a))(\xi_0 \otimes \eta_1 \otimes \eta_2).$$

Because x was arbitrary, we can cancel it. By the coassociativity of Φ , we get $(\iota \otimes \Phi)(u) = u_{(12)}u_{(13)}$ and this proves the result. \blacksquare

It is not hard to see that in the group case, we indeed get the right regular representation here. If G is a compact group, let $A = C(G)$ and Φ be as before. In this case, $\mathcal{H} = L^2(G)$, the Hilbert space of (classes of) square integrable functions on G with respect to the Haar measure, and $C(G)$ acts on \mathcal{H} by multiplication. Classically, the right regular representation $u : G \rightarrow \mathcal{B}(\mathcal{H})$ of G is given by

$$(u(q)\xi)(p) = \xi(pq)$$

when $p, q \in G$ and $\xi \in L^2(G)$. We will consider u as an element in $M(\mathcal{B}_0(\mathcal{H}) \otimes C(G))$, and let this algebra act on $\mathcal{H} \otimes \mathcal{H}$. Then we have for $\xi, \eta \in \mathcal{H}$ and $p, q \in G$ that

$$\begin{aligned} u(\xi \otimes \eta)(p, q) &= \xi(pq)\eta(q) \\ &= (\Phi(\xi)(1 \otimes \eta))(p, q). \end{aligned}$$

Remembering that here the cyclic vector $\xi_0 = 1$, we see that the notion of right regular representation for a compact quantum group indeed corresponds to the classical one in the case of a compact group.

For the explicit calculation of the regular representation in an example, we refer to [14], where Lance constructed the regular representation of $SU_\lambda(2)$.

Next, we prove a technical result which will be very useful in the following sections.

Proposition 5.5 *The set $\{(\omega \otimes \iota)(u) \mid \omega \in \mathcal{B}_0(\mathcal{H})^*\}$ is dense in A .*

PROOF: Let $a, b \in A$. Recall that for given vectors $\xi_1, \xi_2 \in \mathcal{H}$, we denote by ω_{ξ_1, ξ_2} the linear functional on $\mathcal{B}(\mathcal{H})$ such that $\omega_{\xi_1, \xi_2}(x) = \langle x\xi_1, \xi_2 \rangle$ for all $x \in \mathcal{B}(\mathcal{H})$. We have

$$\begin{aligned} \langle ((\omega_{a\xi_0, b\xi_0} \otimes \iota)u)\eta_1, \eta_2 \rangle &= \langle u(a\xi_0 \otimes \eta_1), b\xi_0 \otimes \eta_2 \rangle \\ &= \langle \Phi(a)(\xi_0 \otimes \eta_1), b\xi_0 \otimes \eta_2 \rangle \\ &= \langle (h \otimes \iota)((b^* \otimes 1)\Phi(a))\eta_1, \eta_2 \rangle \end{aligned}$$

for any $\eta_1, \eta_2 \in \mathcal{K}$. So

$$(\omega_{a\xi_0, b\xi_0} \otimes \iota)u = (h \otimes \iota)((b^* \otimes 1)\Phi(a)).$$

By the axioms of a compact quantum group, linear combinations of elements of the form $(b^* \otimes 1)\Phi(a)$ with $a, b \in A$ are dense in $A \otimes A$. It follows that $\{(\omega_{a\xi_0, b\xi_0} \otimes \iota)(u) \mid a, b \in A\}$ is dense in A . Remark also that by density of $A\xi_0$ in \mathcal{H} , it follows that $(\omega \otimes \iota)(u) \in A$ for any ω of the form ω_{ξ_1, ξ_2} with $\xi_1, \xi_2 \in \mathcal{H}$. And because linear combinations of these functionals are dense in $\mathcal{B}_0(\mathcal{H})^*$, we get that $(\omega \otimes \iota)(u) \in A$ for any $\omega \in \mathcal{B}_0(\mathcal{H})^*$. ■

Now we show that u implements Φ .

Proposition 5.6 *For all $a \in A$ we have that $\Phi(a) = u(a \otimes 1)u^*$.*

PROOF: When $a, b \in A$ and $\eta \in \mathcal{K}$ we have

$$\begin{aligned} u(a \otimes 1)(b\xi_0 \otimes \eta) &= \Phi(ab)(\xi_0 \otimes \eta) \\ &= \Phi(a)\Phi(b)(\xi_0 \otimes \eta) \\ &= \Phi(a)u(b\xi_0 \otimes \eta). \end{aligned}$$

Hence $u(a \otimes 1) = \Phi(a)u$ and because u is unitary, $\Phi(a) = u(a \otimes 1)u^*$. ■

This result remains true if we take $\mathcal{K} = \mathcal{H}$. And this will imply that the element u is a multiplicative unitary in the sense of [2]. We can deduce for u , which is in $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$, that

$$(\iota \otimes \Phi)(u) = (1 \otimes u)(u \otimes 1)(1 \otimes u^*).$$

On the other hand we know that $(\iota \otimes \Phi)(u) = u_{(12)}u_{(13)}$. So we have

$$(1 \otimes u)(u \otimes 1)(1 \otimes u^*) = u_{(12)}u_{(13)},$$

or equivalently,

$$u_{(23)}u_{(12)} = u_{(12)}u_{(13)}u_{(23)}.$$

So u is a multiplicative unitary.

Using the arguments of the proofs of propositions 5.2 and 5.3 one can show that there exists a unitary $v \in M(\mathcal{B}_0(\mathcal{H}) \otimes A)$ such that

$$v^*(a\xi_0 \otimes \eta) = \Phi'(a)(\xi_0 \otimes \eta)$$

when $a \in A$ and $\eta \in \mathcal{K}$. Here Φ' denotes the opposite comultiplication $\sigma \circ \Phi$, where σ is the flip map. Again, v is a representation and will be called the left regular representation of (A, Φ) . Also here, it is not hard to see that in the group case, we indeed get the left regular representation.

6 Representation theory

In the previous section we have constructed the right regular representation by using the Haar measure. The main result in this section is that any unitary representation can be decomposed into irreducible unitary representations and that all irreducible unitary representations are contained in the regular representation.

Let (A, Φ) be a compact quantum group. Let v be a representation of (A, Φ) on a Hilbert space \mathcal{H} . A closed subspace \mathcal{H}_1 of \mathcal{H} is said to be invariant if $(e \otimes 1)u(e \otimes 1) = u(e \otimes 1)$, where e is the orthogonal projection onto this subspace. The representation v is called irreducible if the only invariant subspaces are $\{0\}$ and \mathcal{H} . Like in the classical case, also here, when u is a unitary representation and \mathcal{H}_1 is an invariant subspace, also the orthogonal complement of \mathcal{H}_1 will be invariant. We will first obtain this result. Throughout this section, we will denote the Haar measure on (A, Φ) by h .

Proposition 6.1 *Let v be a unitary representation of (A, Φ) on a Hilbert space \mathcal{H} . Denote by \mathcal{B} the normclosure in $\mathcal{B}(\mathcal{H})$ of the set of elements $(\iota \otimes h)(v(1 \otimes a))$ where $a \in A$. Then \mathcal{B} is a non-degenerate C^* -subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover $v \in M(\mathcal{B} \otimes A)$.*

PROOF: Take $a \in A$ and define an element $x \in \mathcal{B}(\mathcal{H})$ by $x = (\iota \otimes h)(v(1 \otimes a))$. Using the left invariance of h and the fact that v is a representation, one can easily show that

$$v^*(x \otimes 1) = (\iota \otimes h)(v_{(13)}(1 \otimes \Phi(a))) \tag{6.1}$$

in $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$.

When $a, b \in A$ and x, y are defined by

$$\begin{aligned} x &= (\iota \otimes h)(v(1 \otimes a)) \\ y &= (\iota \otimes h)(v(1 \otimes b)), \end{aligned}$$

using result (6.1), we get

$$y^*x = (\iota \otimes h \otimes h)(v_{(13)}(1 \otimes (b \otimes 1)\Phi(a))).$$

It follows that $y^*x = (\iota \otimes h)(v(1 \otimes c))$ where $c = (h \otimes \iota)((b \otimes 1)\Phi(a))$. This implies that $\mathcal{B}^*\mathcal{B} \subseteq \mathcal{B}$. But as $(A \otimes 1)\Phi(A)$ is dense in $A \otimes A$, it also implies that $\mathcal{B}^*\mathcal{B}$ is dense in \mathcal{B} . As a consequence, \mathcal{B} is self-adjoint and $\mathcal{B}\mathcal{B} \subseteq \mathcal{B}$. Hence \mathcal{B} is a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$.

To prove that \mathcal{B} acts non-degenerately on \mathcal{H} , take any $x \in \mathcal{B}_0(\mathcal{H})$ and $a \in A$. Then $(\iota \otimes h)(v(x \otimes a)) \subseteq \mathcal{B}\mathcal{B}_0(\mathcal{H})$. Because v is a unitary multiplier of $M(\mathcal{B}_0(\mathcal{H}) \otimes A)$, we have that $v(\mathcal{B}_0(\mathcal{H}) \otimes A)$ is dense in $\mathcal{B}_0(\mathcal{H}) \otimes A$. It follows that $\mathcal{B}\mathcal{B}_0(\mathcal{H})$ is dense in $\mathcal{B}_0(\mathcal{H})$. This proves that \mathcal{B} acts non-degenerately.

Finally, we show that $v \in M(\mathcal{B} \otimes A)$. Again using the formula (6.1), we find for every $a, b \in A$ that

$$v^*(x \otimes b) = (\iota \otimes \iota \otimes h)(v_{(13)}(1 \otimes (\Phi(a)(b \otimes 1))))$$

when $x = (\iota \otimes h)(v(1 \otimes a))$. The right hand side belongs to $\mathcal{B} \otimes A$ as $\Phi(a)(b \otimes 1) \in A \otimes A$. Moreover, as $\Phi(A)(A \otimes 1)$ is dense in $A \otimes A$, we have that $v^*(\mathcal{B} \otimes A)$ is dense in $\mathcal{B} \otimes A$. Then also $v(\mathcal{B} \otimes A) \subseteq \mathcal{B} \otimes A$ and we obtain that $v \in M(\mathcal{B} \otimes A)$. \blacksquare

Now we can prove that also orthogonal complements of invariant subspaces are invariant.

Proposition 6.2 *Let v be a unitary representation of (A, Φ) on \mathcal{H} . If \mathcal{H}_1 is an invariant subspace, then also the orthogonal complement of \mathcal{H}_1 is invariant.*

PROOF: Denote the orthogonal projection of \mathcal{H} onto \mathcal{H}_1 by e . Since $v(e \otimes 1) = (e \otimes 1)v(e \otimes 1)$, we will have $xe = exe$ for every $x \in \mathcal{B}$, where \mathcal{B} is the algebra defined in the previous proposition. As \mathcal{B} is self-adjoint, this implies $xe = ex$ for all $x \in \mathcal{B}$. As \mathcal{B} acts non-degenerately on \mathcal{H} , it follows that $xe = ex$ for all $x \in M(\mathcal{B})$. Then also $v(e \otimes 1) = (e \otimes 1)v$. \blacksquare

The theory of representations of a compact quantum group is very similar to the one of compact groups. The result above illustrates this. In the classical situation, the proofs are of course easier. But except for the result above, in most cases, the general theory is not much more complicated than the classical one.

Just as for compact groups, also for compact quantum groups, it is possible to define the direct sum and the tensor product of two representations. The definition for the direct sum is more or less obvious. The tensor product of two representations v and w of (A, Φ) on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively, is defined as $v \otimes w = v_{(13)}w_{(23)}$. Then $v \otimes w \in M((\mathcal{B}_0(\mathcal{H}_1) \otimes \mathcal{B}_0(\mathcal{H}_2)) \otimes A) \cong M(\mathcal{B}_0(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes A)$, and one can check that it is again a representation.

Let v, w be representations of (A, Φ) on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. An intertwiner between v and w is an element $x \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $(x \otimes 1)v = w(x \otimes 1)$.

We will denote the set of intertwiners between v and w by $\text{Mor}(v, w)$. Two representations are said to be equivalent if there is an invertible intertwiner. They are unitarily equivalent if the intertwiner can be chosen to be unitary.

We will use the Haar measure, as in the classical theory, to construct intertwiners.

Lemma 6.3 *Let v and w be representations of (A, Φ) on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. Let x be any compact operator from \mathcal{H}_1 to \mathcal{H}_2 . Define*

$$y = (\iota \otimes h)(w^*(x \otimes 1)v).$$

Then y is a compact operator from \mathcal{H}_1 to \mathcal{H}_2 such that

$$w^*(y \otimes 1)v = y \otimes 1.$$

PROOF: Denote by $\mathcal{B}_0(\mathcal{H}_1, \mathcal{H}_2)$ the compact operators from \mathcal{H}_1 to \mathcal{H}_2 . Because $x \in \mathcal{B}_0(\mathcal{H}_1, \mathcal{H}_2)$ and $v \in M(\mathcal{B}_0(\mathcal{H}_1) \otimes A)$ we have $(x \otimes 1)v \in \mathcal{B}_0(\mathcal{H}_1, \mathcal{H}_2) \otimes A$ and also $w^*(x \otimes 1)v \in \mathcal{B}_0(\mathcal{H}_1, \mathcal{H}_2) \otimes A$. Therefore y is well defined and $y \in \mathcal{B}_0(\mathcal{H}_1, \mathcal{H}_2)$.

We also have

$$(\iota \otimes \Phi)(w^*(x \otimes 1)v) = w_{(13)}^* w_{(12)}^*(x \otimes 1 \otimes 1)v_{(12)}v_{(13)}.$$

If we apply $\iota \otimes h \otimes \iota$ to the left hand side of this equation we get $y \otimes 1$ by the invariance of h . If we apply the same map to the right hand side, we obtain $w^*(y \otimes 1)v$. This proves the lemma. ■

If w is a unitary representation, then $(y \otimes 1)v = w(y \otimes 1)$ and y will be an intertwiner between v and w .

Proposition 6.4 *Any non-degenerate finite-dimensional representation is equivalent with a unitary representation.*

PROOF: So, let \mathcal{H} be a finite-dimensional Hilbert space and assume that v is a non-degenerate representation on \mathcal{H} . Let

$$y = (\iota \otimes h)(v^*v).$$

By the previous lemma, $v^*(y \otimes 1)v = y \otimes 1$. Now $y \geq 0$ and because v is invertible, $v^*v \geq \varepsilon 1$ for some $\varepsilon > 0$ and so $y \geq \varepsilon 1$. This implies that y is invertible. If we let

$$w = (y^{1/2} \otimes 1)v(y^{-1/2} \otimes 1)$$

we get a representation, equivalent with v . Because $w^*w = 1$ and w is invertible, we have that w is unitary. This proves the proposition. ■

The result is still true for infinite-dimensional representations. Then we must use an extended version of lemma 6.3 with x a bounded operator.

Next we show that any unitary representation decomposes into irreducible finite-dimensional representations.

Theorem 6.5 *Let v be a unitary representation of (A, Φ) on a Hilbert space \mathcal{H} . Then there is a set $\{e_\alpha \mid \alpha \in I\}$ of mutually orthogonal finite-dimensional projections with sum 1 and satisfying*

$$v(e_\alpha \otimes 1) = (e_\alpha \otimes 1)v$$

and $v(e_\alpha \otimes 1)$, considered as an element in $\mathcal{B}(e_\alpha \mathcal{H}) \otimes A$ is a finite-dimensional irreducible unitary representation of (A, Φ) .

PROOF: Consider the set \mathcal{B} of operators y in $\mathcal{B}_0(\mathcal{H})$ satisfying $v(y \otimes 1) = (y \otimes 1)v$. This is clearly a norm closed subalgebra of $\mathcal{B}_0(\mathcal{H})$. Because v is unitary, it is also a self-adjoint subset of $\mathcal{B}_0(\mathcal{H})$. Therefore \mathcal{B} is a C^* -subalgebra of $\mathcal{B}_0(\mathcal{H})$.

By lemma 6.3 we see that

$$(\iota \otimes h)(v^*(x \otimes 1)v) \in \mathcal{B}$$

for all $x \in \mathcal{B}_0(\mathcal{H})$. If we consider an increasing net (x_λ) of positive elements in $\mathcal{B}_0(\mathcal{H})$ with $\sup x_\lambda = 1$ then we get an increasing net (y_λ) of positive elements in \mathcal{B} such that

$$\sup y_\lambda = (\iota \otimes h)(v^*v) = h(1)1 = 1.$$

It follows that \mathcal{B} acts non-degenerately on \mathcal{H} . Then, taking a maximal family of mutually orthogonal, minimal projections in \mathcal{B} , we get the desired result. \blacksquare

So any irreducible unitary representation of a compact quantum group is finite-dimensional. Now we can easily prove Schur's lemma:

Lemma 6.6 *Let v, w be irreducible unitary representations of (A, Φ) on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. Then either v is not equivalent to w and $\text{Mor}(v, w) = 0$, or v is equivalent to w and there is an invertible $x \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\text{Mor}(v, w) = \{\lambda x \mid \lambda \in \mathbb{C}\}$. In particular, if u is an irreducible unitary representation of G , then $\text{Mor}(u, u) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.*

PROOF: Notice that \mathcal{H}_1 and \mathcal{H}_2 are finite-dimensional. It is straightforward to check that the kernel of an intertwiner between v and w is a v -invariant subspace, and that the image of an intertwiner is a w -invariant subspace.

Suppose there is an $x \in \text{Mor}(v, w)$ with $x \neq 0$. So the kernel of x is a v -invariant subspace, which is not the whole of \mathcal{H} . Because v is irreducible, it follows that $\ker(x) = \{0\}$. Similarly, the image of x is a non-zero w -invariant subspace and hence must be the whole of \mathcal{K} . So x is invertible.

Now let $y \in \text{Mor}(v, w)$. For any $\lambda \in \mathbb{C}$, it is clear that $\lambda x - y$ is again an intertwiner between v and w . Take λ such that $\det(\lambda x - y) = 0$. Then the kernel of $\lambda x - y$ is non-zero, and hence equals \mathcal{H} . So $y = \lambda x$. \blacksquare

We can apply the result of theorem 6.5 to the right regular representation, constructed in the previous section. The next theorem says that all irreducible unitary representations will be obtained. Until now the proofs of this section are the ones given in [35]. The next results were also present in [35], but the proofs given here are different (and simpler), using the technique of lemma 6.3.

Theorem 6.7 *Every irreducible unitary representation is contained in the regular representation.*

PROOF: Let u be the right regular representation, acting on \mathcal{H} and let v be any irreducible unitary representation, acting on \mathcal{K} . Take any compact operator x from \mathcal{H} to \mathcal{K} and consider, as before

$$y = (\iota \otimes h)(v^*(x \otimes 1)u).$$

By lemma 6.3 this is an operator from \mathcal{H} to \mathcal{K} such that

$$(y \otimes 1)u = v(y \otimes 1).$$

Let e be the projection on the range of y . Then

$$(e \otimes 1)v(y \otimes 1) = (ey \otimes 1)u = (y \otimes 1)u = v(y \otimes 1)$$

so that

$$(e \otimes 1)v(e \otimes 1) = v(e \otimes 1).$$

Because v is assumed to be irreducible, we have $e = 0$ or $e = 1$. This means that $y = 0$ or y is surjective.

If y is surjective, it follows that v is equivalent with a subrepresentation of u . Now, suppose that $y = 0$ for all x . Consider any x of the form $x\xi = \langle \xi, \xi_1 \rangle \eta_1$ where $\xi_1 \in \mathcal{H}$ and $\eta_1 \in \mathcal{K}$. Note that for any $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$

$$(v^*(x \otimes 1)u)(\xi \otimes \eta) = (v^*(1 \otimes a))(\eta_1 \otimes \eta)$$

with $a = (\omega_{\xi, \xi_1} \otimes \iota)u$. So $y = 0$ implies $(\iota \otimes h)(v^*(1 \otimes a))(\eta_1) = 0$. Because this holds for any η_1 we have

$$(\iota \otimes h)(v^*(1 \otimes a)) = 0$$

for any a of the form $(\omega_{\xi, \xi_1} \otimes \iota)u$. Now we have seen in proposition 5.5 that such elements are dense in A . Therefore

$$(\iota \otimes h)(v^*(1 \otimes a)) = 0$$

for all $a \in A$. If we multiply to the right with any $x \in \mathcal{B}(\mathcal{K})$ and use linearity we find that also $(\iota \otimes h)(v^*v) = 0$. But this is a contradiction as $v^*v = 1$. ■

Theorem 6.5 and 6.7 give the main results in this section. However, in the next section, we will also need the adjoint of a unitary representation.

Definition 6.8 Let v be a representation on a finite-dimensional Hilbert space \mathcal{H} . If we denote the matrix units in $\mathcal{B}(\mathcal{H})$ by (e_{pq}) , we can write $v = \sum e_{pq} \otimes v_{pq}$. Define $\bar{v} = \sum e_{pq} \otimes v_{pq}^*$. Then \bar{v} is still a representation. It is called the adjoint of v .

The definition of \bar{v} depends on the choice of the matrix units. But a different choice will give an equivalent representation. So, the adjoint is only defined up to equivalence.

In the classical theory, the adjoint of a unitary representation is again a unitary representation. In the quantum case, this needs no longer be true. We will have however that it is equivalent to a unitary and that it is still irreducible. To obtain this result, we would need e.g. that \bar{v} is invertible when v is unitary. This seems to be not so obvious. We have to give a proof using the basic technique of lemma 6.3 again.

Lemma 6.9 *If v is a finite-dimensional irreducible representation, then \bar{v} is also irreducible.*

PROOF: Let v act on \mathcal{H} . Choose matrix units (e_{pq}) in $\mathcal{B}(\mathcal{H})$ and define the anti-homomorphism γ of $B(\mathcal{H})$ by $\gamma(e_{pq}) = e_{qp}$. Remark that $\bar{v} = (\gamma \otimes \iota)v^*$. Now let e be a projection in $B(\mathcal{H})$ such that

$$\bar{v}(e \otimes 1) = (e \otimes 1)\bar{v}(e \otimes 1).$$

If we apply $\gamma \otimes \iota$, we get

$$(\gamma(e) \otimes 1)v^* = (\gamma(e) \otimes 1)v^*(\gamma(e) \otimes 1)$$

and if we take adjoints, we obtain

$$v(f \otimes 1) = (f \otimes 1)v(f \otimes 1)$$

where $f = \gamma(e)^*$. Then f is also a projection and because v is irreducible, it is either 0 or 1. This implies that also $e = 0$ or $e = 1$. ■

Proposition 6.10 *If v is an irreducible unitary representation, then \bar{v} is equivalent with a unitary representation.*

PROOF: Let u be the right regular representation, acting on \mathcal{H} . Let v act on the finite-dimensional space \mathcal{K} . Take any compact operator x from \mathcal{K} to \mathcal{H} and define as before

$$y = (\iota \otimes h)(u^*(x \otimes 1)\bar{v}),$$

where h denotes again the Haar measure. This is an operator from \mathcal{K} to \mathcal{H} such that $(y \otimes 1)\bar{v} = u(y \otimes 1)$. We will show that $y \neq 0$ for some x . Because \bar{v} is irreducible, this will imply that \bar{v} is equivalent with a subrepresentation of u .

So assume that $y = 0$ for all x . This means

$$(\iota \otimes h)(u^*(x \otimes 1)\bar{v}) = 0$$

for all operators x from \mathcal{K} to \mathcal{H} . As in the proof of theorem 6.7, this implies

$$(\iota \otimes h)(u^*(1 \otimes b)) = 0$$

where $b = (\psi \otimes \iota)\bar{v}$ for any $\psi \in \mathcal{B}(\mathcal{K})^*$ (recall that \mathcal{K} is finite-dimensional, so any operator on \mathcal{K} has finite rank). By applying any $\omega \in \mathcal{B}_0(\mathcal{H})^*$, we get

$$h(ab) = 0$$

where $a = (\omega \otimes \iota)u^*$. By lemma 5.5, elements a of this form are dense in A . So we obtain

$$(\psi \otimes \iota)(\iota \otimes h)((1 \otimes a)\bar{v}) = 0$$

for any $\psi \in \mathcal{B}(\mathcal{K})^*$. Hence

$$(\iota \otimes h)((1 \otimes a)\bar{v}) = 0$$

for all $a \in A$. By applying transposition γ we get

$$(\iota \otimes h)((1 \otimes a)v^*) = 0$$

for all $a \in A$. Then also

$$(\iota \otimes h)((x \otimes a)v^*) = 0$$

for all $x \in \mathcal{B}(\mathcal{K})$ and $a \in A$, and by linearity

$$(\iota \otimes h)(vv^*) = 0.$$

This is a contradiction as $vv^* = 1$. ■

Remark that it follows that \bar{v} is non-degenerate whenever v is non-degenerate.

7 The Hopf *-algebra of matrix elements.

Let A_0 be the subspace of A spanned by the matrix elements of finite-dimensional unitary representations. We will show in this section that A_0 is a dense *-subalgebra of A , that $\Phi(A_0)$ is contained in the algebraic tensor product $A_0 \odot A_0$ and that (A_0, Φ) is a Hopf *-algebra. This result comes from [35], but the proofs given here are somewhat different.

In [4], Dijkhuizen and Koornwinder start from this Hopf *-algebra A_0 to define a compact quantum group. We will have a look at their definition in remark 7.7. In this section we will also show the equivalence between Woronowicz' two definitions of a compact matrix quantum group. We end the section with a discussion about the faithfulness of the Haar measure.

Proposition 7.1 *A_0 is a dense *-subalgebra of A .*

PROOF: The product of two matrix elements of finite-dimensional unitary representations is a matrix element of the tensor product of these two representations. Therefore A_0 is a subalgebra.

Because the adjoint of a finite-dimensional unitary representation is equivalent with a unitary representation, A_0 is *-invariant.

Let u be the right regular representation, acting on the Hilbert space \mathcal{H} . From theorem 6.5 it follows that u decomposes into irreducible finite-dimensional representations $\{u_\alpha \mid \alpha \in I\}$ acting on subspaces $\{\mathcal{H}_\alpha \mid \alpha \in I\}$. For $\alpha \in I$, let $n(\alpha)$ be the dimension of \mathcal{H}_α and let $\{\xi_1^\alpha, \xi_2^\alpha, \dots, \xi_{n(\alpha)}^\alpha\}$ be an orthonormal basis for \mathcal{H}_α . Define $\omega_{pq}^{\alpha\beta} \in \mathcal{B}_0(\mathcal{H})^*$ by $\omega_{pq}^{\alpha\beta}(x) = \langle x\xi_p^\alpha, \xi_q^\beta \rangle$ for $x \in \mathcal{B}_0(\mathcal{H})$. Then the linear span of

$$\{\omega_{pq}^{\alpha\beta} \mid \alpha, \beta \in I, 1 \leq p \leq n(\alpha), 1 \leq q \leq n(\beta)\}$$

is dense in $\mathcal{B}_0(\mathcal{H})^*$. Hence it follows from proposition 5.5 that the linear span of the set

$$\{(\omega_{pq}^{\alpha\beta} \otimes \iota)u \mid \alpha, \beta \in I, 1 \leq p \leq n(\alpha), 1 \leq q \leq n(\beta)\}$$

is dense in A . Now $(\omega_{pq}^{\alpha\beta} \otimes \iota)u = 0$ unless $\alpha = \beta$ and the elements $(\omega_{pq}^{\alpha\alpha} \otimes \iota)u$ are exactly the matrix elements of the representation u_α . This proves the proposition. \blacksquare

Remark that $1 \in A_0$ because 1 is a representation.

Take a complete set $\{u^\alpha \mid \alpha \in I\}$ of mutually inequivalent, irreducible unitary representations. We want to show that the elements $\{u_{pq}^\alpha \mid \alpha \in I, 1 \leq p, q \leq n(\alpha)\}$ form a basis for A_0 . In order to prove the linear independence, we first deduce the following orthogonality relations.

Proposition 7.2 *Let h be the Haar measure on (A, Φ) . For any $\alpha \in I$, there is a positive invertible operator F^α acting on \mathcal{H}_α such that for any $\alpha, \beta \in I$ and $1 \leq j, q \leq n(\alpha)$, $1 \leq i, p \leq n(\beta)$*

$$h((u_{ip}^\beta)^* u_{jq}^\alpha) = \delta_{\alpha\beta} \delta_{pq} F_{ij}^\alpha.$$

PROOF: We apply lemma 6.3 for u^α and u^β , with x the operator mapping the j th basis element of \mathcal{H}_α to the i th basis element of \mathcal{H}_β and the other basis elements to zero. We obtain that the matrix with elements

$$y_{pq} = h((u_{ip}^\beta)^* u_{jq}^\alpha)$$

intertwines u^α with u^β . Because u^α and u^β are irreducible we have $y_{pq} = 0$ if $\alpha \neq \beta$. In the case $\alpha = \beta$ there is $\lambda \in \mathbb{C}$ such that $y_{pq} = \lambda \delta_{pq}$ for all p, q . This λ depends on α and i, j . Let us write

$$F_{ij}^\alpha \delta_{pq} = h((u_{ip}^\alpha)^* u_{jq}^\alpha).$$

Now we apply lemma 6.3 for $\overline{u^\alpha}^*$ and $(u^\alpha)^t$. We obtain that $(\iota \otimes h)(\overline{u^\alpha}(u^\alpha)^t)$ is an intertwiner between $\overline{u^\alpha}$ and $(u^\alpha)^t$. Because $(u^\alpha)^t = \overline{u^\alpha}^*$, it is positive and invertible. In fact the i, j -th element is precisely

$$\sum_{k=1}^{n(\alpha)} h((u_{ik}^\alpha)^* u_{jk}^\alpha) = n(\alpha) F_{ij}^\alpha.$$

It follows that F^α is positive, invertible and intertwines $((u^\alpha)^t)^{-1}$ and $\overline{u^\alpha}$. ■

We will investigate the operators F^α further in section 8.

Proposition 7.3 *The elements $\{u_{pq}^\alpha \mid \alpha \in I, 1 \leq p, q \leq n(\alpha)\}$ form a basis for A_0 .*

PROOF: We still have to show that these elements are linearly independent.

Let $\sum_{\alpha, i, j} \lambda_{ij}^\alpha u_{ij}^\alpha$ be any finite linear combination of the elements u_{ij}^α , and suppose that $\sum_{\alpha, i, j} \lambda_{ij}^\alpha u_{ij}^\alpha = 0$. Let $\beta \in I$ and $1 \leq r, s \leq n(\beta)$. Then

$$0 = \sum_{\alpha, i, j} \lambda_{ij}^\alpha h((u_{rs}^\beta)^* u_{ij}^\alpha) = \sum_i \lambda_{is}^\beta F_{ri}^\beta.$$

Because F^β is invertible, this implies that $\lambda_{is}^\beta = 0$ for all i . As s and β were arbitrary, this completes the proof. ■

Proposition 7.4 *We have $\Phi(A_0) \subseteq A_0 \odot A_0$ where $A_0 \odot A_0$ is the algebraic tensor product of A_0 with itself.*

PROOF: Let v be any finite-dimensional unitary representation, acting on a Hilbert space \mathcal{H} . Fix an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ for \mathcal{H} . For any pair of vectors $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}\Phi((\omega_{\xi, \eta} \otimes \iota)v) &= (\omega_{\xi, \eta} \otimes \iota \otimes \iota)(v_{(12)}v_{(13)}) \\ &= \sum_{i=1}^n (\omega_{\xi, \xi_i} \otimes \iota \otimes \iota)(v_{(12)})(\omega_{\xi_i, \eta} \otimes \iota \otimes \iota)(v_{(13)}) \\ &= \sum_{i=1}^n (\omega_{\xi, \xi_i} \otimes \iota)(v) \otimes (\omega_{\xi_i, \eta} \otimes \iota)(v).\end{aligned}$$

This proves the result. ■

We now want to show that (A_0, Φ) is a Hopf $*$ -algebra. We will use the notations ε and κ for the counit and the antipode. Multiplication in A_0 will sometimes be considered as a linear map $m : A_0 \odot A_0 \rightarrow A_0$ defined by $m(a \otimes b) = ab$. We will also use the summation convention, which is standard in Hopf algebra theory (see [1] and [20]) :

$$\begin{aligned}\Phi(a) &= \sum_{(a)} a_{(1)} \otimes a_{(2)}, \\ \Phi^{(2)}(a) &= (\Phi \otimes \iota)\Phi(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)},\end{aligned}$$

and so on.

In this setting, it is most easy to give the explicit construction of the counit ε and the antipode κ .

Define linear maps $\varepsilon : A_0 \rightarrow \mathbb{C}$ and $\kappa : A_0 \rightarrow A_0$ by

$$\begin{aligned}\varepsilon(u_{pq}^\alpha) &= \delta_{pq} \\ \kappa(u_{pq}^\alpha) &= (u_{qp}^\alpha)^*.\end{aligned}$$

Lemma 7.5 *For all $a \in A$ we have*

$$\begin{aligned}(\varepsilon \otimes \iota)\Phi(a) &= (\iota \otimes \varepsilon)\Phi(a) = a \\ m(\kappa \otimes \iota)\Phi(a) &= m(\iota \otimes \kappa)\Phi(a) = \varepsilon(a)1.\end{aligned}$$

PROOF: For all α, p, q , we have

$$(\iota \otimes \varepsilon)\Phi(u_{pq}^\alpha) = \sum_k u_{pk}^\alpha \varepsilon(u_{kq}^\alpha) = u_{pq}^\alpha,$$

so that, by linearity, $(\iota \otimes \varepsilon)\Phi(a) = a$ for all $a \in A_0$. Similarly $(\varepsilon \otimes \iota)\Phi(a) = a$ for all a .

Again for all α, p, q we get

$$m(\iota \otimes \kappa)\Phi(u_{pq}^\alpha) = \sum_k u_{pk}^\alpha \kappa(u_{kq}^\alpha) = \sum_k u_{pk}^\alpha (u_{qk}^\alpha)^* = \delta_{pq}1 = \varepsilon(u_{pq}^\alpha)1.$$

And once more by linearity, we have $m(\kappa \otimes \iota)\Phi(a) = \varepsilon(a)1$ for all $a \in A_0$. Similarly $m(\iota \otimes \kappa)\Phi(a) = \varepsilon(a)1$ for all a . ■

Theorem 7.6 (A_0, Φ) is a Hopf *-algebra.

PROOF: We have that A_0 is a *-algebra with a comultiplication Φ . Moreover, $\varepsilon : A_0 \rightarrow \mathbb{C}$ is a linear map such that

$$(\varepsilon \otimes \iota)\Phi = (\iota \otimes \varepsilon)\Phi = \iota$$

and $\kappa : A_0 \rightarrow A_0$ is a linear map such that

$$m(\kappa \otimes \iota)\Phi(a) = m(\iota \otimes \kappa)\Phi(a) = \varepsilon(a)1.$$

This automatically implies that ε is a *-homomorphism, and that κ is an anti-homomorphism satisfying $\kappa(\kappa(a^*)^*) = a$ for all $a \in A$ (see e.g. proposition 2.4 in [22]). So A_0 is a Hopf *-algebra. ■

Remark 7.7 In [4], Dijkhuizen and Koornwinder look at a compact quantum group through the algebraic object A_0 . They define a CQG algebra as a Hopf *-algebra A_0 which is spanned by the coefficients of its finite-dimensional unitary (irreducible) corepresentations. In this setting, the existence of the Haar measure becomes trivial : Let $\{u^\alpha \mid \alpha \in I\}$ be a complete set of mutually inequivalent, irreducible unitary representations. Then the u_{pq}^α are linearly independent. Denote by 1 the unique α such that u^α is equivalent to the one-dimensional corepresentation (1). The linear functional $h : A \rightarrow \mathbb{C}$ defined by

$$h(u_{pq}^\alpha) = \begin{cases} 0 & \alpha \neq 1 \\ 1 & \alpha = 1 \end{cases}$$

is easily shown to be the unique invariant functional on A_0 satisfying $h(1) = 1$. As a final observation, it is proven that such a CQG algebra admits a C*-algebra completion. However, this completion is not unique. We will discuss this problem at the end of this section.

With the material we have gathered now, it becomes very straightforward to prove that Woronowicz' new definition of a compact matrix quantum group in [34] is equivalent with the original one in [32].

In [32], a compact matrix quantum group (compact matrix pseudogroup) is defined to be a pair (A, u) of a unital C*-algebra A and a matrix u in $M_n(A)$ satisfying

1. The $*$ -subalgebra A_0 of A , generated by the matrix elements of u , is dense in A .
2. There exists a $*$ -homomorphism $\Phi : A \rightarrow A \otimes A$ such that

$$\Phi(u_{pq}) = \sum_k u_{pk} \otimes u_{kq}$$

for all $p, q = 1, \dots, n$.

3. There exists a linear antimultiplicative mapping $\kappa : A_0 \rightarrow A_0$ such that $\kappa(\kappa(a^*)^*) = a$ for all $a \in A_0$ and

$$\begin{aligned} \sum_k \kappa(u_{pk})u_{kq} &= \delta_{pq}1 \\ \sum_k u_{pk}\kappa(u_{kq}) &= \delta_{pq}1 \end{aligned}$$

for all $p, q = 1, \dots, n$.

In [34], this definition is adapted and a compact matrix quantum group is defined to be a pair (A, u) of a unital C^* -algebra A and a matrix u in $M_n(A)$ satisfying the axioms 1 and 2 above and the following condition:

- 3'. The matrix (u_{pq}) and the transpose matrix (u_{qp}) are invertible.

Proposition 3.8 says that such a compact matrix quantum group is a compact quantum group. Hence, by the previous theorem, it satisfies also the third of the original axioms above.

Conversely, suppose (A, u) satisfies the original conditions 1, 2 and 3. By applying κ^{-1} to the first equation of axiom 3, we get that for all $p, q = 1, \dots, n$

$$\sum_k \kappa^{-1}(u_{kq})u_{pk} = \delta_{pq}1.$$

If we apply the conjugation $*$ and use that $\kappa^{-1}(u_{kq})^* = \kappa(u_{kp}^*)$, we find

$$\sum_k u_{pk}^* \kappa(u_{kq}^*) = \delta_{pq}1.$$

In the same way, by applying κ^{-1} to the second equation of axiom 3, we obtain

$$\sum_k \kappa(u_{pk}^*)u_{kq}^* = \delta_{pq}1.$$

This proves that the matrix (u_{pq}^*) is invertible. Hence also its conjugate, the transpose matrix (u_{qp}) is invertible. So (A, u) satisfies axiom 3' as well. This proves the equivalence of Woronowicz' original definition of a compact matrix quantum group in [32], and his adapted definition in [34].

We now come back to the faithfulness of the Haar measure. We saw in example 4.5 that in general, the Haar measure h on (A, Φ) needs not to be faithful. But let us consider the restriction of h to A_0 . It gives a Haar measure on the Hopf algebra (A_0, Φ) , which will be faithful.

Proposition 7.8 *The Haar measure h is faithful on A_0 .*

PROOF: Let $a \in A_0$ and suppose that $h(a^*a) = 0$. It follows from the Cauchy-Schwarz inequality that then $h(a^*b) = 0$ for all $b \in A_0$. We can write $a^* = (\rho \otimes \iota)v$, where v is a finite-dimensional unitary representation of (A, Φ) on a Hilbert space \mathcal{H} and ρ is a linear functional on $\mathcal{B}(\mathcal{H})$. For any $b \in A_0$ we have

$$\rho((\iota \otimes h)(v(1 \otimes b))) = h(a^*b) = 0.$$

If we denote the C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ defined in proposition 6.1 by \mathcal{B} , then we obtain $\rho(\mathcal{B}) = 0$. Because $v \in \mathcal{B} \otimes A$ this gives us $a^* = (\rho \otimes \iota)v = 0$. ■

In fact we have the following more general property (proposition 3.4 from [26]): Let (A_0, Φ) be any Hopf algebra and h a non-zero linear functional on A_0 which is left invariant. Then h is left faithful in the sense that $h(ba) = 0$ for all $b \in A_0$ implies $a = 0$. If (A_0, Φ) is a Hopf algebra with an invertible antipode, we also have faithfulness on the other side. In the case we consider, (A_0, Φ) is a Hopf $*$ -algebra, so κ is automatically invertible. If moreover h is positive, then left or right faithfulness in the above sense are of course equivalent with the usual notion : $h(a^*a) = 0 \Rightarrow a = 0$.

On the C^* -algebra level, we have the following property :

Proposition 7.9 *Let $a \in A$. Then $h(a^*a) = 0$ if and only if $h(aa^*) = 0$.*

PROOF: We will use the result of proposition 3.12 of [26]. It says that if A_0 is a regular multiplier Hopf algebra with a left invariant functional h , then there is an automorphism σ of A_0 such that $h(ab) = h(b\sigma(a))$ for all $a, b \in A_0$. In our case, A_0 is a Hopf $*$ -algebra and hence automatically regular.

We only prove one implication, the other one is completely analogous. Suppose that $a \in A$ and that $h(a^*a) = 0$. Then by the Cauchy-Schwarz inequality, $h(ba) = 0$ for all $b \in A$. Using the result mentioned above and the continuity of h , we obtain $h(ba) = h(a\sigma(b)) = 0$ for all $b \in A_0$. Now we can approximate a^* by linear combinations of elements of the form $\sigma(b)$ with $b \in A_0$. So we obtain that $h(aa^*) = 0$. ■

So $I = \{a \in A \mid h(a^*a) = 0\}$ is a closed two-sided ideal in A . Hence we can consider the quotient $A_r = A/I$. It is clear that (A_r, Φ) is again a compact quantum group, where now the Haar measure is faithful. Of course, the associated Hopf $*$ -algebra is still the same. So the process of taking the quotient amounts to considering another C^* -norm on A_0 and taking the completion with respect to this norm. This also means that both these compact quantum groups have the same representations, which will imply that they have the same dual (see section 8). These observations suggest that we should consider (A, Φ) and (A_r, Φ) as the same object, as different forms of the same compact quantum group.

8 The dual of a compact quantum group

Let G be an abelian locally compact group. The set of characters of G can be made into an abelian locally compact group in a natural way. It is called the dual group of G , and the Pontryagin-Van Kampen duality theorem says that its dual is again (isomorphic to) the original group G . For compact groups, the Tannaka-Krein duality generalises this duality theory to the non-abelian case. Several authors have discussed a duality theory in the quantum framework: see for instance [2, 6, 9, 10, 16, 17, 18, 26, 27, 30, 33]. In these theories, the unitary irreducible representations play the role the characters have in the case of an abelian group. The idea is that the representations contain sufficient information to fully recover the (quantum) group.

In this section, we will show how the irreducible unitary representations of a compact quantum group give rise to a discrete quantum group in the sense of [25]. We want to mention that this duality between compact and discrete quantum groups is a special case of the duality theory developed in [26] and [27].

Let (A, Φ) be a compact quantum group. To construct the dual, we start with a complete set $\{u^\alpha | \alpha \in I\}$ of mutually non-equivalent, irreducible unitary representations. Let A_0 be the associated Hopf $*$ -algebra as in section 7. Define B_0 to be the space of linear functionals on A defined by $x \mapsto h(ax)$ where $a \in A_0$. So B_0 is a subspace of the dual A^* . For $\alpha \in I$, let F^α be the positive invertible operator obtained in proposition 7.2.

Lemma 8.1 *Let*

$$a = \sum_{k=1}^{n(\alpha)} ((F^\alpha)^{-1})_{pk} (u_{kq}^\alpha)^*$$

and let ω_{pq}^α be the linear functional on A defined by $\omega_{pq}^\alpha(x) = h(ax)$. Then

$$\omega_{pq}^\alpha(u_{pq}^\alpha) = 1$$

and ω_{pq}^α is 0 on all other matrix elements.

PROOF: We have

$$\omega_{pq}^\alpha(u_{rs}^\beta) = \sum_{k=1}^{n(\alpha)} ((F^\alpha)^{-1})_{pk} h((u_{kq}^\alpha)^* u_{rs}^\beta) = \delta_{\alpha\beta} \delta_{qs} \sum_{k=1}^{n(\alpha)} ((F^\alpha)^{-1})_{pk} F_{kr}^\alpha = \delta_{\alpha\beta} \delta_{qs} \delta_{pr}.$$

■

It is clear that B_0 is spanned by the elements $\{\omega_{pq}^\alpha | \alpha \in I, 1 \leq p, q \leq n(\alpha)\}$.

As before, we consider the product on A^* induced by the comultiplication on A . This is defined by $(\omega\psi)(a) = (\omega \otimes \psi)\Phi(a)$. We have

$$\begin{aligned} \omega_{pq}^\alpha \omega_{rs}^\beta(u_{ij}^\gamma) &= \sum_l \omega_{pq}^\alpha(u_{il}^\gamma) \omega_{rs}^\beta(u_{lj}^\gamma) \\ &= \delta_{\alpha\gamma} \delta_{\beta\gamma} \delta_{pi} \delta_{sj} \delta_{qr} = \delta_{\alpha\beta} \delta_{qr} \omega_{ps}^\alpha(u_{ij}^\gamma). \end{aligned}$$

It follows that B_0 is a subalgebra of A^* and that this subalgebra is the algebraic direct sum $\sum_{\alpha \in I} \oplus M_{n(\alpha)}$.

The antipode κ induces a $*$ -structure on B_0 by

$$(\omega_{pq}^\alpha)^*(x) = \omega_{pq}^\alpha(\kappa(x)^*)^-$$

for $x \in A_0$. Now, as $\kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*$, we get $(\omega_{pq}^\alpha)^* = \omega_{qp}^\alpha$. So we get the usual $*$ -algebra structure on B_0 . Remark that B_0 has no identity: $\varepsilon(u_{pq}^\alpha) = \delta_{pq}$ but ε does not belong to B_0 .

Now we want to make B_0 into a discrete quantum group as in [25]. Let us recall the definition:

Definition 8.2 A discrete quantum group is a pair (B_0, Φ) where B_0 is a direct sum of full matrix algebras and Φ is a comultiplication on B_0 making B_0 into a multiplier Hopf $*$ -algebra.

So we have to make B_0 into a multiplier Hopf $*$ -algebra (see definitions 2.3 and 2.4 in [23]). The multiplier algebra of $B_0 \odot B_0$ consists of course of all elements in $\prod_{\alpha, \beta} M_{n(\alpha)} \otimes M_{n(\beta)}$. Define Φ on B_0 by

$$\Phi(\omega)(a \otimes b) = \omega(ab),$$

then we get a linear map from B_0 into this multiplier algebra $M(B_0 \odot B_0)$. It is automatically a $*$ -homomorphism :

$$\begin{aligned} \Phi(\omega\psi)(a \otimes b) &= (\omega\psi)(ab) \\ &= (\omega \otimes \psi) \sum_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} \\ &= \sum_{(a)(b)} \omega(a_{(1)}b_{(1)})\psi(a_{(2)}b_{(2)}) \\ &= \sum_{(a)(b)} \Phi(\omega)(a_{(1)} \otimes b_{(1)})\Phi(\psi)(a_{(2)} \otimes b_{(2)}) \\ &= \Phi(\omega)\Phi(\psi)(a \otimes b). \end{aligned}$$

Similarly,

$$\begin{aligned} \Phi(\omega^*)(a \otimes b) &= \omega^*(ab) \\ &= \omega(\kappa(ab)^*)^- \\ &= \omega(\kappa(a)^*\kappa(b)^*)^- \\ &= \Phi(\omega)(\kappa(a)^* \otimes \kappa(b)^*)^- \\ &= \Phi(\omega)^*(a \otimes b). \end{aligned}$$

Now we can define the antipode. We know that $\kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*$ and that for $\alpha \in I$ there is $\bar{\alpha} \in I$ such that $\overline{u^\alpha}$ is unitarily equivalent with $u^{\bar{\alpha}}$. Hence κ maps B_0 to B_0 . In

fact $\kappa(B_\alpha) \subseteq B_{\bar{\alpha}}$ and $\kappa(B_{\bar{\alpha}}) \subseteq B_\alpha$. So we could use the antipode κ on A to define the antipode on the dual, and prove in this way that B_0 is a multiplier Hopf algebra. But the following method is easier.

Recall that, because (A_0, Φ) is a Hopf algebra, we have that

$$\begin{aligned} T_1 : x \otimes y &\mapsto \Phi(x)(1 \otimes y) \\ T_2 : x \otimes y &\mapsto (x \otimes 1)\Phi(y) \end{aligned}$$

extend to bijections from the algebraic tensor product of $A_0 \odot A_0$ to itself. Now, for ω, ψ in B_0 we have

$$\begin{aligned} ((\omega \otimes 1)\Phi(\psi))(x \otimes y) &= \sum_{(x)} \omega(x_{(1)})\Phi(\psi)(x_{(2)} \otimes y) \\ &= \sum_{(x)} \omega(x_{(1)})\psi(x_{(2)}y) \\ &= (\omega \otimes \psi)(\Phi(x)(1 \otimes y)). \end{aligned}$$

Let $a, b \in A_0$ such that ω and ψ are the functionals given by $x \mapsto h(ax)$ and $x \mapsto h(bx)$ respectively. As we have already mentioned in section 7, the Hopf algebra A_0 is automatically regular, because it is a Hopf *-algebra. This means that the antipode is invertible, which is equivalent to the condition that (A_0, Φ') is also a Hopf algebra, where Φ' denotes the opposite comultiplication. So we can write $a \otimes b = \sum(1 \otimes p_i)\Phi(q_i)$ with $p_i, q_i \in A_0$. Now

$$\begin{aligned} ((\omega \otimes 1)\Phi(\psi))(x \otimes y) &= (h \otimes h)((a \otimes b)\Phi(x)(1 \otimes y)) \\ &= (h \otimes h)(\sum(1 \otimes p_i)\Phi(q_i x)(1 \otimes y)) \\ &= \sum h(q_i x)h(p_i y). \end{aligned}$$

It follows that $(\omega \otimes 1)\Phi(\psi) \in B_0 \odot B_0$. Similarly,

$$(\Phi(\omega)(1 \otimes \psi))(x \otimes y) = (\omega \otimes \psi)((x \otimes 1)\Phi(y))$$

and $\Phi(\omega)(1 \otimes \psi) \in B_0 \odot B_0$. Here we need also the fact that for a regular (multiplier) Hopf algebra A_0 with left invariant functional h , the sets $\{h(a \cdot) \mid a \in A_0\}$ and $\{h(\cdot a) \mid a \in A_0\}$ are equal (proposition 3.11 in [26]).

In the same way, the inverses of the maps T_1, T_2 have duals; and these are the inverses of the duals of T_1 and T_2 . This implies that (B_0, Φ) is a multiplier Hopf algebra.

As B_0 is a direct sum of full matrix algebras, the C*-norm on B_0 exists and is unique. The completion B gives the discrete quantum group on the C*-algebra level.

The element h in proposition 3.1 of [25] is of course the Haar functional h itself. Let Tr_α denote the trace on B_α and consider it as a linear map on all of B_0 . It is proven

in [25] that the operators $K_\alpha = (\text{Tr}_\alpha \otimes \iota)\Phi(h)$ implement κ , i.e. $\kappa^2(\omega) = K_\alpha^{-1}\omega K_\alpha$ for all $\omega \in B_\alpha$. What are these operators here ? We have that $\text{Tr}_\alpha = \text{Tr}_{\bar{\alpha}} \circ \kappa$. So

$$\begin{aligned} \langle (\text{Tr}_\alpha \otimes \iota)\Phi(h), u_{pq}^\beta \rangle &= \langle (\text{Tr}_{\bar{\alpha}} \circ \kappa \otimes \iota)\Phi(h), u_{pq}^\beta \rangle \\ &= \sum_{k=1}^{n(\alpha)} h((u_{kk}^{\bar{\alpha}})^* u_{pq}^\beta) = \delta_{\beta\bar{\alpha}} F_{qp}^{\bar{\alpha}}. \end{aligned}$$

In this way one can construct the dual quantum group out of the unitary representations of a compact quantum group. Conversely, starting from a discrete quantum group, one can recover the Hopf *-algebra A_0 . The C*-norm on A_0 is not unique, but as we discussed at the end of section 7, we could consider the different possible C*-completions as the same compact quantum group.

It is clear that this duality generalizes the Pontryagin duality between compact and discrete abelian groups: If $A = C(G)$ is the function algebra of a compact abelian group, then the irreducible unitary representations are given by the characters of G . The dual discrete quantum group B_0 is then the function algebra of the character group.

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