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ORBIFOLD-HODGE NUMBERS OF HILBERT SCHEMES

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1. Introduction. Let G be a finite group acting on a compact differentiable manifold X. Then in [D*1], [D*2] an orbifold Euler number of X is defined by

$$e(X,G) := \frac{1}{|G|}\sum_{gh=hg} e(X^g \cap X^h),$$

where the sum runs over all commuting pairs in G and X^g denotes the set of fixed points of the action of g. According to [H-H] and [D*2] the formula above can be rewritten as follows: Let $C(g) := \{h \in G \mid hg = gh\}$ be the centralizer of $g \in G$. Then $C(g)$ acts on X^g and

$$e(X,G) = \sum_{[g]} e(X^g/C(g)),$$

where g runs over a system of representatives for the conjugacy classes of G. $e(X,G)$ is expected to coincide with the Euler number of a crepant resolution of the quotient X/G. This fact has been shown in many cases (see e.g. [H-H], [Ro]).

Now let X be a compact Kähler manifold or a Moishezon manifold of complex dimension d. The Hodge polynomial $h(X,x,y)$ is defined by $h(X,x,y) := \sum_{p,q} h^{p,q}(X)x^p y^q$, where the $h^{p,q}(X)$ are the Hodge numbers of X. Then, motivated by [V] and [Z], orbifold Hodge numbers $h^{p,q}(X,G)$ are introduced in [B-D] as follows: For any point $x \in X^g$, the eigenvalues of g on the tangent space $T_{X,x}$ are roots of unity $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_d}$, where $0 \le \alpha_j < 1$ and the α_j are locally constant functions on X^g. Let $X_1(g), \ldots, X_r(g)$ be the connected components of X^g. For $i = 1, \ldots, r$ we put $F_i(g)$ equal to the value of $\sum_{j=1}^d \alpha_j$ on $X_i(g)$ and set

$$h_g^{p,q}(X,G) := \sum_{i=1}^r \dim(H^{p-F_i(g),q-F_i(g)}(X_i(g))^{C(g)}).$$

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Here $C(g)$ is the centralizer of g in G and $(\cdot)^{C(g)}$ denotes the $C(g)$-invariant part. Then the orbifold Hodge numbers are defined by

$$h^{p,q}(X,G) := \sum_{[g]} h_g^{p,q}(X,G),$$

where g runs again over a system of representatives for the conjugacy classes of G. The orbifold Hodge polynomial of X is $h(X,G,x,y) := \sum_{p,q} h^{p,q}(X,G)x^p y^q$.

The $X_i(g)$ are smooth and thus the quotients $X_i(g)/C(g)$ are V-manifolds, thus their Hodge numbers are well defined and satisfy $h^{p,q}(X_i(g)/C(g)) = \dim(H^{p,q}(X_i(g))^{C(g)})$. We get therefore the formula

$$h(X,G,x,y) = \sum_{[g]} \sum_{i=1}^{r} h(X_i(g)/C(g),x,y)(xy)^{F_i(g)}. \tag{1.1}$$

It is conjectured that, for any crepant resolution $\widetilde{X/G}$ of X/G, the orbifold Hodge numbers satisfy $h(X,G,x,y) = h(\widetilde{X/G},x,y)$. In [B-D] this conjecture is related to the "strong MacKay conjecture" ([B-D] conjecture 5.3, see also [Re]). In [B-D] the conjecture is also shown in several cases and the orbifold Hodge numbers are related to "string-theoretic" Hodge numbers.

We want to show this conjecture in two cases, where the acting group is the symmetric group, and the resolutions are (related to) Hilbert schemes of points.

(1) Let S be an algebraic surface. Let $G(n)$ be the symmetric group on n letters acting on S^n by permuting the factors. Then $S^n/G(n)$ is the symmetric power $S^{(n)}$, which has the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points on S as a natural desingularisation via the Hilbert-Chow morphism $\omega_n : \mathrm{Hilb}^n(S) \longrightarrow S^{(n)}$ (see e.g. [Fo1]). It is easy to see that $S^n/G(n)$ is Gorenstein and this resolution is crepant (see e.g. [Be]). In [H-H] it was shown that $e(S^n, G(n)) = e(\mathrm{Hilb}^n(S))$.

Theorem 1. *$h(S^n, G(n), x, y) = h(\mathrm{Hilb}^n(S), x, y)$.*

(2) Let A be an abelian surface. Then the symmetric group $G(n)$ acts on A^{n-1} as follows. A^{n-1} is embedded in A^n as the subvariety

$$A_0^n := \Big\{(x_1,\dots,x_n) \in A^n \;\Big|\; \sum_i x_i = 0 \text{ in the group } A\Big\}.$$

Obviously the action of $G(n)$ on A^n by permuting the factors restricts to an action on A_0^n. The quotient A^{n-1}/G and its desingularisation can be described as follows: Let again $A^{(n)}$ be the n-fold symmetric power of A. Then there is a natural morphism $\sigma : A^{(n)} \longrightarrow A$, associating to a 0-cycle $\sum[x_i]$ its sum $\sum x_i$ in the group A. Then $A^{n-1}/G(n) \simeq \sigma^{-1}(0)$, and a resolution of $A^{n-1}/G(n)$ is the higher order Kummer variety $K_{n-1} := \omega_n^{-1}\sigma^{-1}(0)$, where again $\omega_n : \mathrm{Hilb}^n(A) \longrightarrow A^{(n)}$ is the Hilbert-Chow morphism.

Again it can easily be shown that the quotient $A^{n-1}/G(n)$ is Gorenstein and the resolution is crepant [Be].

Theorem 2. *$h(A^{n-1}, G(n), x, y) = h(K_{n-1}, x, y)$.*

This paper was inspired by the paper [B-D] and a question of one of its authors D. Dais.

2. The formula for Hilbert schemes. We first introduce some notations and definitions on partitions and the symmetric group.

DEFINITION 2.1. Let $P(n)$ be the set of partitions of n. We write an $\alpha \in P(n)$ as $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$, where α_i is the number of repetitions of i in the partition α, and put $|\alpha| := \sum_i \alpha_i$. For $\alpha \in P(n)$ we put $S^\alpha := S^{\alpha_1} \times \ldots \times S^{\alpha_n}$ and $S^{(\alpha)} := S^{(\alpha_1)} \times \ldots \times S^{(\alpha_n)}$.

The cycle type $\alpha(g)$ of $g \in G(n)$ is the partition $(1^{\alpha_1(g)}, \ldots, n^{\alpha_n(g)})$, where $\alpha_i(g)$ is the number of cycles of length i in the representation of g as a product of disjoint cycles. As usual we denote by $(n_1, \ldots, n_r)$ the cycle defined by mapping n_i to n_{i+1} for $i < r$ and n_r to n_1.

By [G-S] (see also [Göl] Thm. 2.3.14) we have

$$h(\mathrm{Hilb}^n(S), x, y) = \sum_{\alpha \in P(n)} (xy)^{n-|\alpha|} h(S^{(\alpha)}, x, y). \tag{2.1}$$

In this section we want to show that (2.1) coincides termwise with (1.1). It is elementary that $\alpha(g)$ only depends on the conjugacy class of g and that the map $[g] \mapsto \alpha(g)$ is a bijection from the conjugacy classes of $G(n)$ to $P(n)$.

LEMMA 2.2. *For $g \in G(n)$ we have $(S^n)^g / C(g) \simeq S^{(\alpha(g))}$.*

Proof. We see that $(S^n)^g$ is the set of all $(x_1, \ldots, x_n) \in S^n$ such that $x_i = x_j$ if there exists a cycle of g containing i and j. Thus we get $(S^n)^g = S^{\alpha(g)}$. The centralizer $C(g)$ acts on the elements of $[g]$ by permuting the cycles of the same length. Therefore it acts on $S^{\alpha(g)}$ by permuting the factors of the $S^{\alpha_i(g)}$, and the result follows. ■

In order to show Theorem 1 we therefore only have to show the following.

LEMMA 2.3. *For any $g \in G(n)$ we have $F(g) = n - |\alpha(g)|$.*

Proof. We can write

$$S^n = S^{\alpha_1(g)} \times (S^{\alpha_2(g)})^2 \times \ldots \times (S^{\alpha_n(g)})^n,$$

and g operates on S^n by permuting cyclically the factors $S^{\alpha_i(g)}$ of each $(S^{\alpha_i(g)})^i$. From this it follows easily that $F(g) = \sum_{i=1}^r F(g_i)$, where $g = g_1 \ldots g_r$ is the representation of g as a product of disjoint cycles, and the g_i are considered as acting on S^i. Therefore it is enough to show the result in the case that g has cycle type (n^1), and we can assume that $g = (1, 2, 3, \ldots, n)$.

Let $s \in S$ and x, y local coordinates on S centered at s. We denote by x_j, y_j the pullback of x, y from the j^{th} factor of S^n. $x_1, \ldots, x_n, y_1, \ldots, y_n$ obviously give local coordinates of S^n near the point $p = (s, \ldots, s) \in (S^n)^g$. For $k = 1, \ldots, n$ we put

$$w_k := \sum_{j=1}^n x_j \cdot e^{2\pi i k j/n}, \ z_k := \sum_{j=1}^n y_j \cdot e^{2\pi i k j/n}.$$

Then also $(w_1, \ldots, w_n, z_1, \ldots, z_n)$ form a system of local coordinates, and we get

$$g(w_k) = e^{-2\pi i k/n} \cdot w_k, \ g(z_k) = e^{-2\pi i k/n} \cdot z_k.$$

Therefore the action of g on the tangent space $T_{S^n, p}$ is given by

$$g(\tfrac{\partial}{\partial w_k}) = e^{2\pi i k/n} \tfrac{\partial}{\partial w_k}, \ g(\tfrac{\partial}{\partial z_k}) = e^{2\pi i k/n} \tfrac{\partial}{\partial z_k}.$$

So we get $F(g) = 2 \cdot (1 + 2 + \ldots + n - 1 + 0 + 0)/n = n - 1$, and the result follows. ■

3. The formula for higher order Kummer varieties. By [G-S] (see also [Gö1] 2.4.17) we have

$$h(K_{n-1},x,y)=\sum_{\alpha\in P(n)}(xy)^{n-|\alpha|}\gcd(\alpha)h(A^{(\alpha)},x,y)/h(A,x,y). \tag{3.1}$$

Here we denote for a partition $\alpha=(1^{\alpha_1},\dots,n^{\alpha_n})$ of n by $\gcd(\alpha)$ the greatest common divisor of the numbers k with $\alpha_k\neq 0$. Again we want to show that (3.1) coincides termwise with (1.1).

Let $g\in G(n)$ of cycle type $\alpha=(1^{\alpha_1},\dots,n^{\alpha_n})$. Then $(A_0^n)^g=(A^n)^g\cap A_0^n=K^\alpha$, where we denote by K^α the subvariety

$$K^\alpha:=\Big\{\Big((x_j^1)_{j=1}^{\alpha_1},\dots,(x_j^n)_{j=1}^{\alpha_n}\Big)\in A^\alpha\ \Big|\ \sum i\cdot x_j^i=0\Big\}.$$

Then in the proof of Theorem 6 in [G-S] the following is shown:

LEMMA 3.1.
(1) *If* $\gcd(\alpha)=1$*, then* $A\times K^\alpha\simeq A^\alpha$*.*
(2) *In general we have*

$$K^\alpha=\coprod_{y\in A(\gcd(\alpha))}K_y^\alpha.$$

Here $A(\gcd(\alpha))$ *is the set of* $\gcd(\alpha)$*-division points of* A *and*

$$K_y^\alpha:=\left\{\Big((x_j^1)_{j=1}^{\alpha_1},\dots,(x_j^n)_{j=1}^{\alpha_n}\Big)\in A^\alpha\ \Big|\ \sum_{i,j}\frac{i}{\gcd(\alpha)}x_j^i=y\right\}.$$

Furthermore we have an isomorphism $K_y^\alpha\simeq K^{(\beta)}$*, where* $\beta:=(1^{\beta_1},\dots,n^{\beta_n})$ *with* $\beta_i:=\alpha_{\gcd(\alpha)\cdot\beta}$ *for all* i*.*

It is easy to see that both isomorphisms $A\times K^\alpha\simeq A^\alpha$ and $K_y^\alpha\simeq K^\beta$ commute with the $G(n)$-action.

As above we see that the centralizer $C(g)$ acts on $A^\alpha=A^{\alpha_1}\times\dots\times A^{\alpha_n}$ by permuting the factors of each factor A^{α_i}. So we get

$$(A_0^n)^g/C(g)=K^{((\alpha))}:=\left\{\Big(\sum_{j=1}^{\alpha_1}[x_j^1],\dots,\sum_{j=1}^{\alpha_n}[x_j^n]\Big)\in A^{(\alpha)}\ \Big|\ \sum_{i,j}i\cdot x_j^i=0\right\}.$$

Again by the proof of Theorem 6 in [G-S] we have:

LEMMA 3.2.
(1) *If* $\gcd(\alpha)=1$*, then* $h(A\times K^{((\alpha))},x,y)=h(A^{(\alpha)},x,y)$*.*
(2) *In general we have*

$$K^{((\alpha))}=\coprod_{y\in A(\gcd(\alpha))}K_y^{((\alpha))}.$$

Here

$$K_y^{(\alpha)}:=\left\{\Big(\sum_{j=1}^{\alpha_1}[x_j^1],\dots,\sum_{j=1}^{\alpha_n}[x_j^n]\Big)\in A^{(\alpha)}\ \Big|\ \sum_{i,j}\frac{i}{\gcd(\alpha)}x_j^i=y\right\}.$$

Furthermore we have an isomorphism $K_y^{((\alpha))}\simeq K^{((\beta))}$*, where* $\beta:=(1^{\beta_1},\dots,n^{\beta_n})$ *with* $\beta_i:=\alpha_{\gcd(\alpha)\cdot\beta}$ *for all* i*.*

If we put these results together, it is enough (for the proof of Theorem 2) to show:

LEMMA 3.3. *For the action of g on A_0^n we have $F(g) = n - |\alpha|$.*

Proof. The action of $G(n)$ on A_0^n is just the restriction of the action on A^n. Therefore in the same way as in the proof of Lemma 2.3 we can assume that $g = (1, 2, \ldots, n)$. Let again (x_i, y_i) be pullbacks of local coordinates from the i^{th} of A^n near a point

$$p \in (A_0^n)^g := \{(a, \ldots, a) \mid a \in A(n)\}.$$

We can assume that a is the neutral element of the group A. Now we define local coordinates $w_1, \ldots, w_n$, $z_1, \ldots, z_n$ on A^n by the same formulas as in the proof of Lemma 2.3. Then the subvariety $A_0^n \subset A^n$ is near p defined by the equations $w_n = 0$, $z_n = 0$, and $w_1, \ldots, w_{n-1}$, $z_1, \ldots, z_{n-1}$ are local coordinates on A_0^n. We get as above $g(\frac{\partial}{\partial w_j}) = e^{2\pi i j/n}\frac{\partial}{\partial w_j}$, $g(\frac{\partial}{\partial z_j}) = e^{2\pi i j/n}\frac{\partial}{\partial z_j}$. Therefore we obtain $F(g) = 2 \cdot (1+2+\ldots+(n-1))/n = n-1$, and the result follows. ■

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