PROJECTIVE MODELS OF NIKULIN ORBIFOLDS

Joint with A. Garbagnati, G. Kapustka, M. Kapustka

1. INTRODUCTION

symplectic

X IHSM = cpt Kähler $\pi_1(X)=0$ $\quad H^{2,0}(X) = \mathbb{C}\omega_X$

$X \sim K3^{[n]}$ = deformation of $\mathrm{Hilb}^n(S)$ $\quad S$ smooth K3

$G = \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$ $\quad \iota$ symplectic inv. $=$ $(\iota^*\omega_X = \omega_X)$

<u>$\dim X = 2,\ n=1$:</u> $\quad X$ K3 $\quad$ [Nikulin] $\mathrm{Fix}\,\iota = \{p_1, \dots, p_8\}$

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ \widetilde{X/\iota} \arrow[r] \arrow[d] & X/\iota \\ {} & \end{tikzcd}$$

min. resolution is a smooth K3, called a <u>Nikulin surface</u>

<u>$\dim X \geq 4$:</u> $\quad \forall$ known (X, ι) $\quad \mathrm{Fix}\,\iota \supset$ isolated fixed pts

$\Rightarrow \nexists$ crepant res. of X/ι

e.g. $\dim X = 4$, $n=2$: $X \sim K3^{[2]}$

Fix $\Lambda = \{p_1, \dots, p_{28}\} \cup 1$ smooth K3 W [–, Mongardi]

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[\rho]{} & X/\iota \end{array}$$

Y = partial res. of X/ι over $\pi(W)$

$\mathrm{Sing}(Y) = 28$ pts $\frac{1}{2}(1,1,1,1)$

Def: Y is called the Nikulin orbifold (of dim 4) corr. to (X, ι).

If Y is a deformation of a Nikulin orbifold we call it an orbifold of Nikulin type.

Idea: enlarge class of IHSM by allowing some singularities, in such a way that:

- ω_Y symplectic form
- $(H^2(Y,\mathbb{Z}), q_Y)$ is a lattice
- MMP
- moduli spaces & period map
- Global Torelli Thm [Bakker-Lehn ISV; Menet ISO]

(i) proj. ISV

(ii) irred. sympl. orbifolds (ISO) Y = cpt kähler orbifold Y s.t. $\pi_1(Y \setminus \operatorname{Sing} Y) = 0$ and
$H^{2,0}(Y) = \mathbb{C}\omega_Y$

RMK: • orbifolds of Nikulin type over ISO

• proj. ISO $\Longrightarrow$ ISV

PROP. $X \sim K3^{[n]}$, $\exists\, \iota \in \mathrm{Aut}(X)$ sympl. inv. $\Longleftrightarrow$ $\underset{\underset{\left(H^2(X,\mathbb{Z})^{\iota^*}\right)^{\perp}}{\parallel}}{E_8(-2)} \subset_{\text{prim.}} \mathrm{Pic}(X)$

[Nikulin $n=1$, Mongardi $n \geq 2$]

PROP. [Nikulin - Garbagnati - Sarti] S is a Nikulin surface
$\Longleftrightarrow$ the Nikulin lattice $N \subset \mathrm{Pic}(S)$ prim.

NOTATION: $\mathcal{M}_n := \{(X,\iota) \,/\, X \sim K3^{[n]}, \iota \text{ sympl. inv.}\}/_{\cong}$

RMK: $(X,\iota) \in \mathcal{M}_n$ v.g. $\Longrightarrow$ $\mathrm{Pic}(X) \cong E_8(-2)$, X not algebraic

THM [vGeemen - Sarti] $(S,\iota) \in \mathcal{M}_1$

S projective $\Longleftrightarrow$ (*) $\begin{cases} \mathrm{Pic}(S) \supset \Lambda_{2d} := \langle 2d \rangle \overset{\perp}{\oplus} E_8(-2) \\ \text{or} \\ d \equiv 0\ (2) \text{ and } \mathrm{Pic}(S) \supset \widetilde{\Lambda_{2d}} \text{ overlattice} \\ \text{of } \Lambda_{2d} \supset_{\text{prim}} E_8(-2) \end{cases}$

Q1: alg. NL locus inside $\mathcal{M}_2$?

Q2: what are the corresp. Nikulin orbifolds?

↳ relation between Y and W?

Q3: proj. models of Nikulin orbifolds?

$\Downarrow$

1 complete proj. family of orbifolds of Nikulin type

2. PROJECTIVE K3^[2] WITH A SYMPLECTIC INVOLUTION

<u>PROP</u> [CGKK] $(X,\iota)\in\mathcal{M}_2$

X proj $\iff$ $(*)$ holds

<u>But</u> it is no longer the case that Λ_{2d} admits a unique embedding inside $H^2(X,\mathbb{Z})\cong L_X := U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle -2\rangle$

<u>THM 1</u> [CGKK]. If $(X,\iota)\in\mathcal{M}_2$ proj. then (X,ι) is (Λ,j)-polarized (i.e. $\Lambda\subset \mathrm{Pic}(X)$) with (Λ, j) as follows:

$$\begin{tikzcd} \Lambda \arrow[r, phantom, "\subset"] \arrow[dr, "j"'] & \mathrm{Pic}(X) \arrow[d, hook] \\ & L_X \end{tikzcd}$$

Condition on d	Embed. $\mathrm{Pic}(X)\subset L$	$\mathrm{Pic}(X)$	T_X
$\forall d\in\mathbb{N}$	j_1	Λ_{2d}	$T_{2d,1} := U^{\oplus 2}\oplus E_8(-2)\oplus\langle -2d\rangle\oplus\langle -2\rangle$
$d\equiv 1 \mod 2$	j_2	Λ_{2d}	$T_{2d,2} := U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^5$
$d\equiv 3 \mod 4$	j_3	Λ_{2d}	$T_{2d,3} := U^{\oplus 2}\oplus E_8(-2)\oplus K_d$
$d\equiv 0 \mod 2$	$\widetilde{j}$	$\widetilde{\Lambda}_{2d}$	$\widetilde{T}_{2d} := U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^5$

EXAMPLES

d=1 : j_1 : double EPW sextics $X \xrightarrow[2:1]{} Y_A \subset \mathbb{P}^5 \circlearrowleft I$

$\mathrm{Pic}(X) \supset \langle 2 \rangle \oplus E_8(-2)$ ++++ - -

j_2: $S_1^{[2]}$ $\quad \tau \circlearrowright S_1 \xrightarrow[2:1]{} dP_2 \quad \rho(S_1) \geq 7$

$\iota = \tau^{[2]} \circ \beta \qquad \beta =$ Beauville's non-nat. involution on $S_1^{[2]}$

d=2: j_1: ?

$\tilde{j}$: $\mathrm{Pic}(X) \supset \widetilde{\Lambda_4} > \langle 4 \rangle \oplus E_8(-2)$

$X \sim K3^{[2]}$ ass. to Verra 4-fold $\quad V \quad$ [IKKR]

$$\begin{array}{c} V \\ \downarrow 2:1 \\ D_{(2,2)} \subset \mathbb{P}^2 \times \mathbb{P}^2 \\ \sigma \end{array}$$

$\underline{d=3}$: j_1, j_2 : ?

j_3 : $F(Z)$ for $Z \subset \mathbb{P}^5$ cubic 4-fold

$\circlearrowleft$

I

RMK: $\forall d \ \forall j \quad X \underset{\text{bir}}{\Longrightarrow} \mathcal{M}_\sigma\left(\begin{smallmatrix} S_d \\ Z_d \end{smallmatrix}, \beta\right)$

with S_d, Z_d with $\rho \geq 7$

⚠ ι is not induced by an inv. on S_d or Z_d

3. NIKULIN ORIBIFOLDS

THM [Menet] $(H^2(Y,\mathbb{Z}),q_Y) \cong L_Y := U(2)^{\oplus 3} \oplus E_8(-1) \oplus \underset{\delta_1}{\langle -2\rangle} \oplus \underset{\delta_2}{\langle -2\rangle}$

Moreover, the exceptional divisor Σ of $\rho: Y \to X/\iota$ is $\Sigma = \delta_1 - \delta_2$ and $q_Y(\Sigma) = -4$.

COR. $\mathrm{Pic}(X) \cong E_8(-2) \Rightarrow \mathrm{Pic}(Y) = \mathbb{Z}\Sigma \cong \langle -4\rangle$.

THM 2. [CGKK] Families of proj. Nikulin orbifolds corr. to those of THM 1 are:

Embedding $\mathrm{Pic}(X) \subset L$	$\mathrm{NS}(Y)$	T_Y
j_1	$\langle 4d\rangle \oplus \langle -4\rangle$	$U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4d\rangle \oplus \langle -4\rangle$
$j_2,\ d \equiv 1 \mod 2$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$
$j_3,\ d \equiv 3 \mod 4$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$
$\widetilde{j},\ d \equiv 0 \mod 2$	$\langle d\rangle \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus \langle -d\rangle \oplus N \oplus \langle -4\rangle$

Proof: THM 1 + study of $\pi_* : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$

Generators of $\mathrm{Pic}(Y)$ are obtained by $\pi_* j(h)$ (where $q_X(h) = 2d$ on X) and Σ

e.g. $j = \tilde{j}$, $d \equiv 0 \ (2)$ $\quad \mathrm{Pic}(Y) = \left\langle \frac{\pi_* j(h)}{2}, \Sigma \right\rangle$

$$\langle d \rangle \overset{\perp}{\oplus} \langle -4 \rangle \quad \blacksquare$$

RMK: $(S,\iota) \in \mathcal{M}_1 \Rightarrow (S^{[2]}, \iota^{[2]}) \in \mathcal{M}_2$ "in codim 1"

Fix $\iota^{[2]} \supset W \cong \widetilde{S/\iota}$ Nikulin surface

$\Rightarrow$ family of W $\supset$ family of Nikulin surfaces

CONJ. $T_W \cong T_Y$ isometry over $\mathbb{Z}$

PROP. [CGKK] Conj. holds if:

(i) $(S^{[2]}, L^{[2]})$ for $(S,L) \in \mathcal{M}_1$ v.g.

(ii) some families for $d=2,3$

(iii) $T_W \otimes \mathbb{Q} \cong_{\mathbb{Q}} T_Y \otimes \mathbb{Q}$.

Dim: (iii) $\nu : W \subset X$ $\quad \nu^* \omega_X = \omega_W$

$\rightsquigarrow$ isom. of HS $\quad T_W \otimes \mathbb{Q} \cong_{HS} T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$

[Voisin] $\langle \nu^* x, \nu^* y \rangle_W = x.y.[W]$ $\quad$ ~ sympl. $\forall x,y \in T_X \otimes \mathbb{Q}$

[Menet] + int. theory $\Rightarrow T_W \otimes \mathbb{Q} \cong_{\mathbb{Q}} T_Y(4) \otimes \mathbb{Q} \cong_{\mathbb{Q}} T_Y \otimes \mathbb{Q}$ $\blacksquare$

4. PROJECTIVE MODELS

$(X,\iota)\in \mathcal{M}_2 \qquad h\in \mathrm{Pic}(X) \qquad q_X(h)=2d \qquad \iota^* h = h$

$$
\begin{tikzcd}
 & X \arrow[loop left, "\iota"] \arrow[r, "\varphi_{|h|}"] \arrow[d, "\pi"'] & \mathbb{P}^n \arrow[loop right, "I"] \arrow[d, dashed, "p_+"'] \arrow[dr, dashed, "p_-"] & \\
Y \arrow[r, "\rho"] \arrow[rr, dashed, bend right, "\varphi_{|D_+|}"] \arrow[rrr, dashed, bend right=40, "\varphi_{|D_-|}"'] & X/\iota \arrow[r, dashed] & \mathbb{P}U_+ \cong \mathbb{P}^{n_1} & \mathbb{P}U_- \cong \mathbb{P}^{n_2}
\end{tikzcd}
$$

I involution with eigensp. $U_\pm$

$\pi^* \rho_* D_\pm = h$

In general, $D_\pm$ are Weil $\mathbb{Q}$-Cartier s.t. $2D_\pm \in \mathrm{Pic}(Y)$

ORBIFOLD RR FORMULA [Bloch, Buckley-Reid-Zhou]

D $\mathbb{Q}$-Cartier on Y Nikulin orbifold

$$\begin{tikzcd} E_W \subset \widetilde{X} \arrow[r, "r"] \arrow[d, "q", "2:1"'] & & X \arrow[d, "\pi"] \\ \widetilde{Y} \arrow[r, "\beta"'] & Y \arrow[r, "\rho"'] & X/\iota \\ & D \arrow[u] & \end{tikzcd}$$

s.t. $q^*\beta^* D = r^* H + k E_W$

with $H \in \mathrm{Pic}(X)$

THM 3 [CGKK]

$$\chi(Y,D) = \frac{1}{48}H^4 + \frac{1}{48}H^2.c_2(X) + (\frac{1}{16} - \frac{k^2}{8})(H_{|W})^2 + 3 - \frac{N}{16} + \frac{k^4}{4} - \frac{3k^2}{2}.$$

where $N = \#$ sing. pts where D is not Cartier (≤ 28)

COR. If $D \in \mathrm{Pic}(Y)$, $\chi(Y,D) = \frac{1}{4}\left(q_Y(D)^2 + 6 q_Y(D) + 12\right)$.

e.g. $d=2$, $j=\tilde{j}$, $D_+ \in \mathrm{Pic}(Y)$, $\chi(D_+) = h^0(D_+) = 7$.

$q_Y(D_+) = d = 2$.

THM 4 [CBKK]. $\chi(D_\pm) = h^0(D_\pm)$

and $H^0(X, h) = \pi^* H^0(Y, D_+) \oplus \pi^* H^0(Y, D_-)$

$N_+ \quad N_-$

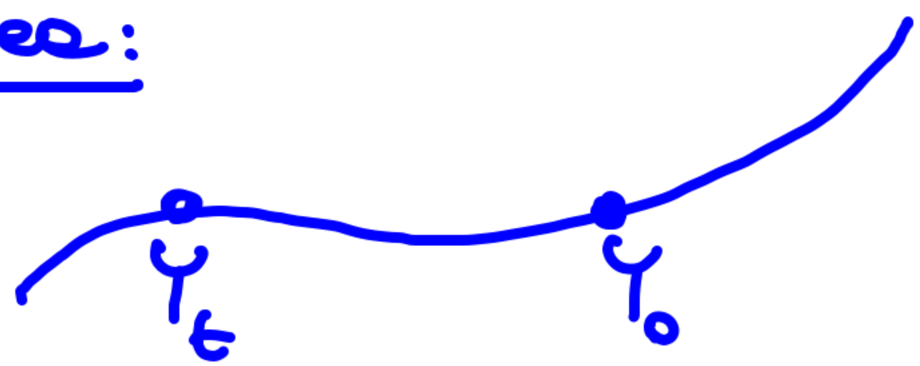
Embedding $\mathrm{Pic}(X) \subset L$	(N_1, N_2)	m_1	m_2
$j_1,\ d \equiv 1 \mod 2$	$(12, 16)$	$\frac{d^2}{4} + \frac{3d}{2} + \frac{5}{4}$	$\frac{d^2}{4} + d - \frac{1}{4}$
$j_1,\ d \equiv 0 \mod 2$	$(16, 12)$	$\frac{d^2}{4} + \frac{3d}{2} + 1$	$\frac{d^2}{4} + d$
$j_2,\ d \equiv 1 \mod 2$	$(28, 0)$	$\frac{d^2}{4} + \frac{3d}{2} + \frac{1}{4}$	$\frac{d^2}{4} + d + \frac{3}{4}$
$j_3,\ d \equiv 3 \mod 4$	$(28, 0)$	$\frac{d^2}{4} + \frac{3d}{2} + \frac{1}{4}$	$\frac{d^2}{4} + d + \frac{3}{4}$
$\widetilde{j},\ d \equiv 0 \mod 2$	$(0, 28)$	$\frac{d^2}{4} + \frac{3d}{2} + 2$	$\frac{d^2}{4} + d - 1$

THM 5. A complete proj. family of orbifolds of Nikulin type Y s.t. $\mathrm{Pic}(Y) \supset \mathbb{Z}D$ with $q_Y(D)=2$ (and $\mathrm{div}=1$) is given by

$$Y \xrightarrow[\mathcal{L}_{|D|}]{2:1} J \subset \mathbb{P}^6$$

and $J := \mathcal{L}_{|D|}(Y)$ is a (special) complete intersection $(3,4)$, ramified along a surface of degree 48.

Idea:



Y_0 = Nikulin orbifold corr. to $(\underline{X,\iota})$ associated to the symm. Verra 4-folds

$\mathrm{Pic}(Y_t) = \mathbb{Z}D_t = \langle 2 \rangle$

$\mathrm{Pic}(Y_0) = \mathbb{Z}D_0 \oplus \mathbb{Z}\Sigma$
$\underline{\langle 2 \rangle \oplus \langle -4 \rangle}$

$$Y_0 = \underbrace{Z_3}_{\downarrow} \cap \underbrace{T_4}_{\to \text{ EPW quartics}}$$

cone over the det. cubic in $\mathbb{P}^5$

$$\overset{\nearrow 28\text{-dim}}{A} \subset 3\Lambda^5 V_7 \oplus \Lambda^3 V_7 \oplus 3V_7 \quad \text{isotropic}$$

wrt $\quad b : 3\Lambda^3 V_7 \oplus \Lambda^3 V_7 \oplus 3V_7 \longrightarrow \Lambda^6 V_7$