Order of tangency between manifolds

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Introduction

Two plane curves, both nonsingular at a point x^0, are said to have a contact of order at least k at x^0 if, in properly chosen regular parametrisations, those two curves have identical Taylor polynomials of degree k about x^0.

Contact problems have been of both classical and modern interest, particularly in light of Hilbert's 15th problem to make rigorous the classical calculations of enumerative geometry, especially those undertaken by Schubert.

Let's restrict ourselves to families of curves. The situation regarding ordinary contacts between families of curves is now well understood thanks in large measure to the contact formula of Fulton, Kleiman and MacPherson. (Ordinary contact is a point of tangency.) They consider p-parameter family of plane curves together with p individual curves. They compute, in terms of certain "characteristic numbers", the number of members of the family which simultaneously have an ordinary contact with each of the curves.

Apart from contact formulas, an important role is played by the “order of tangency”. Let us discuss this notion for Thom polynomials. Among important properties of Thom polynomials we record their positivity closely related to Schubert calculus. Namely, the order of tangency allows one to define the jets of Lagrangian submanifolds. The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

Little o notation

$f(x) = o(h(x)) \quad$ when $x \to x_0$

means

$\lim_{x \to x_0} \frac{f(x)}{h(x)} = 0.$

"$f(x)$ is much smaller than $h(x)$ for x near x_0."

Two manifolds M and $\widetilde{M}$ in $\mathbb{R}^m$, both of class C^r, $r \geq 1$, and the same dimension p, intersecting at $x^0 \in M \cap \widetilde{M}$, for $k \leq r$, *have at x^0 the order of tangency at least k*, when there exist neighbourhoods $U \ni u^0$ and $\widetilde{U} \ni \tilde{u}^0$ in $\mathbb{R}^p$, parametrisations

$$q\colon (U, u^0) \to (M, x^0), \qquad \tilde{q}\colon (\widetilde{U}, \tilde{u}^0) \to (\widetilde{M}, x^0)$$

and a diffeomorphism $\Phi\colon U \to \widetilde{U}$ (all, naturally, of class C^r) such that

$$\Big(\tilde{q} \circ \Phi - q\Big)(u) \;=\; o\Big(\big|u - u^0\big|^k\Big) \tag{1}$$

when $U \ni u \to u^0$. (Then, clearly, $\tilde{q}(\tilde{u}^0) = x^0$ and $\Phi(u^0) = \tilde{u}^0$.) It is straightforward that this definition does not depend on the choice of local parametrisations q and $\tilde{q}$. We record the following reformulation of (1):

Proposition

The condition (1) is equivalent to

$$T^k_{u^0} q \; = \; T^k_{u^0}\big(\tilde{q} \circ \Phi\big)$$

where $T^k_{u^0}(\cdot)$ means the Taylor polynomial about u^0 of order k.

Indeed, we have

$$\tilde{q} \circ \Phi(u) - q(u) \; = \big(\tilde{q} \circ \Phi(u) - \; T^k_{u^0}\big(\tilde{q} \circ \Phi\big)(u - u^0)\big)$$
$$+ \big(T^k_{u^0}\big(\tilde{q} \circ \Phi\big)(u - u^0) - \; T^k_{u^0}\, q(u - u^0)\big) + \big(T^k_{u^0}\, q(u - u^0) - q(u)\big)\,,$$

where the first and last summands are $o\Big(\big|u - u^0\big|^k\Big)$ by Taylor.

Under (1), so is the middle summand

$$T^k_{u^0}(\tilde{q} \circ \Phi)(u - u^0) - T^k_{u^0}\, q(u - u^0) = o\left(\left|u - u^0\right|^k\right)$$

and our assertion follows from the following general result.

Lemma 1

Let $w \in \mathbb{R}[u_1, u_2, \ldots, u_p]$, $\deg w \leq k$, $w(\mathbf{u}) = o\left(|\mathbf{u}|^k\right)$ when $\mathbf{u} \to 0$ in $\mathbb{R}^p$. Then w is identically zero.

The implication: Proposition $\Rightarrow$ (1) is obvious, because now the middle term on the RHS vanishes, so that the RHS is automatically $o\left(\left|u - u^0\right|^k\right)$.

Assume that

$$s := \max\{k\colon \text{the order of tangency} \geq k\} < r\,. \qquad (2)$$

Here r is the assumed class of smoothness of manifolds, finite or infinite when a category is real. (When $r = \infty$, the condition (2) simply says that s is finite.)

Our second approach uses *pairs of curves* lying, respectively, in M and $\widetilde{M}$. We naturally assume that $T_{x^0}M = T_{x^0}\widetilde{M}$.

We now present the following mini-max procedure.

Theorem 1

Under (2),

$$\min_{v}\Big(\max_{\gamma,\tilde\gamma}\big(\max\big\{l\colon\ |\gamma(t)-\tilde\gamma(t)| = o\big(|t|^l\big)\ \text{when}\ t\to 0\,\big\}\big)\Big)\ =\ s\,. \tag{3}$$

The **minimum** *is taken over all* $0 \neq v \in T_{x^0}M = T_{x^0}\widetilde{M}$*. The* **outer maximum** *is taken over all pairs of curves* $\gamma \subset M$*,* $\tilde\gamma \subset \widetilde{M}$ *such that* $\gamma(0) = x^0 = \tilde\gamma(0)$*, and – both non-zero! – velocities* $\dot\gamma(0)$*,* $\dot{\tilde\gamma}(0)$ *are both parallel to* v*. The* **inner maximum** *is taken over admissible positive integers only.*

Attention. In this theorem the assumption (2) is essential; our proof would not work in the situation $s = r$.

To begin the proof, take the integer s defined in (2). Then it is quick to show that the integer on the left hand side of equality (3) is at least s. Indeed, for every fixed vector v as above, $v = dq(u^0)\mathbf{u}$ (without loss of generality, $\mathbf{u} \in \mathbb{R}^p$, $|\mathbf{u}| = 1$). We now take $\delta(t) = q(u^0 + t\mathbf{u})$ and $\tilde{\delta}(t) = \tilde{q}\big(\Phi(u^0 + t\mathbf{u})\big)$. Then

$$|\delta(t) - \tilde{\delta}(t)| \;=\; o\big(|t\mathbf{u}|^s\big) \;=\; o\big(|t|^s\big)$$

and so, in that equality,

$$\max_{\gamma,\tilde{\gamma}}\big(\max\big\{\, l\colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\big) \;\geq\; s\,.$$

In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's on equality's left hand side.

The opposite inequality is more involved. We give a sketch of the argument. We construct first a certain vector **v** in $T_{x^0}M = T_{x^0}\widetilde{M}$ depending on q and $\tilde{q}$. Then we consider two curves $\delta(t)$ and $\tilde{\delta}(t)$ in the manifolds, both having at $t = 0$ non-zero speeds parallel to the vector **v**. We estimate from above (by s) the LHS of (3): $|\delta(t) - \tilde{\delta}(t)| \neq o\big(|t|^{s+1}\big)$. It is here where the assumption $s \leq r - 1$ is needed. We note that the produced couple of curves δ and $\tilde{\delta}$ is completely general for that chosen vector **v**. Hence it follows that for this precise vector **v** the quantity

$$\max_{\gamma,\tilde{\gamma}}\Big(\max\Big\{\,l\colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^{l}\big) \text{ when } t \to 0\Big\}\Big)$$

does not exceed s. So does the minimum of such quantities over all v's in $T_{x^0}M = T_{x^0}\widetilde{M}$.

Our third approach is based on a tower of consecutive Grassmannians attached to a local C^r parametrisation q. However, to allow for a recursive definition of tower's members, a more general framework is needed.

To every C^1 *immersion* $H : N \to N'$, N – an n-dimensional manifold, N' – an n'-dimensional manifold, we attach the so-called image map $\mathcal{G}H\colon N \to G_n(N')$ of the tangent map $d\,H$: for $s \in N$,

$$\mathcal{G}H(s) \;=\; dH(s)(T_sN)\,,$$

where $G_n(N')$ is the Grassmann *bundle*, with base N', of all n-planes tangent to N'.

We use as previously the pair of parametrisations q and $\tilde{q}$, but now dispense with a local diffeomorphism Φ. So we are now given the mappings

$$\mathcal{G}\, q :\ U \longrightarrow G_p(M)\,, \qquad \mathcal{G}\, \tilde{q} :\ U \longrightarrow G_p\Big(\widetilde{M}\Big)\,.$$

Putting $M^{(0)} = M$, $\widetilde{M}^{(0)} = \widetilde{M}$, $\mathcal{G}^{(1)} = \mathcal{G}$, we get two sequences of recursively defined mappings

$$\mathcal{G}^{(l)} q :\ U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \quad l \geq 2\,, \quad \mathcal{G}^{(l)} q = \mathcal{G}\Big(\mathcal{G}^{(l-1)} q\Big)$$

and

$$\mathcal{G}^{(l)} \tilde{q} :\ U \longrightarrow G_p\Big(\widetilde{M}^{(l-1)}\Big)\,, \quad l \geq 2\,, \quad \mathcal{G}^{(l)} \tilde{q} = \mathcal{G}\Big(\mathcal{G}^{(l-1)} \tilde{q}\Big),$$

where $M^{(l)} = G_p\big(M^{(l-1)}\big)$, $\quad \widetilde{M}^{(l)} = G_p\Big(\widetilde{M}^{(l-1)}\Big)$.

Theorem 2

C^r manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k $(1 \leq k \leq r)$ iff for every parametrisations q and $\tilde{q}$ of the vicinities of x^0 in, respectively, M and $\widetilde{M}$, there holds

$$\mathcal{G}^{(k)} q\left(u^0\right) = \mathcal{G}^{(k)} \tilde{q}\left(u^0\right).$$

(Observe that $q(u^0) = x^0 = \tilde{q}(u^0)$.)

In what follows, of interest for us will be the situation when H above is the graph of a C^1 mapping $h\colon \mathbb{R}^p \supset U \to \mathbb{R}^t$. That is, for $u \in U$, $H(u) = \big(u,\, h(u)\big) \in \mathbb{R}^{p+t} = \mathbb{R}^p \times \mathbb{R}^t$. Then $\mathcal{G}H(u)$ equals

$$\Big(u,\, h(u)\,;\; d\big(u,\, h(u)\big)(u)\Big) = \Big(u,\, h(u);\; \mathrm{span}\{\partial_j + h_j(u)\}\Big)$$

where $j = 1, \ldots, p$ and the symbol h_j means the partial derivative of a vector mapping h with respect to the indeterminate u_j. Moreover, $\partial_j + h_j(u)$ denotes the partial derivative of the vector mapping

$$(\iota,\, h)\colon\, U \to \mathbb{R}^p\big(u_1, \ldots, u_p\big) \times \mathbb{R}^t$$

with respect to u_j. ($\iota\colon U \hookrightarrow \mathbb{R}^p$ is the inclusion.)

Now observe that the above expression for $\mathcal{G}H(u)$ is not handy. Yet there are charts in each Grassmannian.

The chart in a typical fibre G_p over a point in the base $\mathbb{R}^{p+t}$, good for $\mathcal{G}H(u)$ consists of all the entries in the bottommost rows (indexed by numbers $p+1,\ p+2,\ \ldots,\ p+t$) in the $(p+t)\times p$ matrices

$$\left[\begin{array}{c|c|c|c} v_1 & v_2 & \ldots & v_p \end{array}\right]$$

with non-zero upper $p\times p$ minor, after multiplying the matrix on the right by the inverse of that upper $p\times p$ submatrix

That is to say, taking as the local coordinates all the entries in the rows $p+1,\ldots,\ p+t$ of the following matrix

$$\left[\begin{array}{cccc} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \\ \hline \multicolumn{4}{|c|}{\mathbf{v_j^i}, \quad i = p+1, \ldots, p+t} \\ \multicolumn{4}{|c|}{j = 1, \ldots, p} \end{array}\right] \left[\begin{array}{cccc} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \end{array}\right]^{-1}.$$

Or, more explicitly, these coordinates are the entries of

$$\boxed{\begin{array}{c} \mathbf{v_j^i}, \quad i = p+1, \ldots, p+t \\ j = 1, \ldots, p \end{array}} \left[\begin{array}{cccc} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \end{array}\right]^{-1}.$$

We get a handy expression

$$\mathcal{G}H(u) = \Big(u,\ h(u)\,;\ \frac{\partial h}{\partial u}(u)\Big),$$

where under the symbol $\frac{\partial h}{\partial u}(u)$ understood are all the entries of this *jacobian* $(t \times p)$-matrix written in row and separated by commas.

We come back to Theorem 2. We assume without loss of generality that both M and $\widetilde{M}$ are, in the vicinities of x^0, just graphs of C^r mappings, and the parametrisations q and $\tilde{q}$ are the graphs of those mappings. That is,

$q(u) = (u, f(u)),\ f\colon U \to \mathbb{R}^{m-p}\big(y_{p+1}, \ldots, y_m\big)$ and similarly
$\tilde{q}(u) = (u, \tilde{f}(u)),\ \tilde{f}\colon U \to \mathbb{R}^{m-p}\big(y_{p+1}, \ldots, y_m\big)$,
$x^0 = \big(u^0, f(u^0)\big) = \big(u^0, \tilde{f}(u^0)\big)$.

We are going to show that Proposition implies Theorem 2.

Lemma 2

For $1 \leq l \leq k$ there exists such a local chart on the Grassmannian $G_p\big(M^{(l-1)}\big)$ in which the mapping $\mathcal{G}^{(l)}q$ evaluated at u has the form

$$\left(u,\ f(u);\ \binom{l}{1} \times f_{[1]}(u),\ \binom{l}{2} \times f_{[2]}(u),\ \ldots,\ \binom{l}{l} \times f_{[l]}(u)\right),$$

where $f_{[\nu]}(u)$ is the aggregate of all the partials of the ν-th order at u, of all the components of f, which are in the number $p \times (m-p)^{\nu}$, and $N \times (*)$ stands for the N copies going in row and separated by commas, of an object $(*)$.

Attention. In this lemma we systematically ignore the symmetricity of the partial derivatives of smooth mappings.

Proof. $l = 1$. We note that

$$\mathcal{G}^{(1)}q\,(u) = \Big(u,\ f(u);\ \mathrm{span}\{\partial_j + f_j(u)\colon j = 1,\,2,\ldots,\,p\}\Big),$$

in the relevant Grassmannian chart, is but

$$\big(u,\ f(u);\ f_{[1]}(u)\big)\ =\ \left(u,\ f(u);\ \binom{I}{1}\times f_{[1]}(u)\right).$$

The beginning of induction is done.

$l \Rightarrow l+1$, $l < k$. The mapping $\mathcal{G}^{(l)}q\colon U \to M^{(l)}$, evaluated at u, is already written down, in appropriate local chart assumed to exist in $M^{(l)}$, as

$$\left(u,\ f(u),\ \binom{l}{1} \times f_{[1]}(u),\ \binom{l}{2} \times f_{[2]}(u), \ldots,\ \binom{l}{l} \times f_{[l]}(u) \right).$$

We work with $\mathcal{G}^{(l+1)}q = \mathcal{G}\left(\mathcal{G}^{(l)}q\right)$. Now, the last expression being clearly of the form $H(u) = \big(u,\ h(u)\big)$ in the previously introduced notation, the mapping h reads

$$h(u) \ = \ \left(f(u),\ \binom{l}{1} \times f_{[1]}(u),\ \binom{l}{2} \times f_{[2]}(u), \ldots,\ \binom{l}{l} \times f_{[l]}(u) \right).$$

In order to have $\mathcal{G}H(u)$ written down, in view of

$$\mathcal{G}H(u) = \left(u,\ h(u)\,;\ \frac{\partial h}{\partial u}(u)\right),$$

one ought to write in row: u, then $h(u)$, and then all the entries of the jacobian matrix $\frac{\partial h}{\partial u}(u)$, also written in row and separated by commas:

$$\binom{l}{0} \times f_{[1]}(u),\ \binom{l}{1} \times f_{[2]}(u),\ \binom{l}{2} \times f_{[3]}(u), \ldots,\ \binom{l}{l} \times f_{[l+1]}(u).$$

These entries on the right hand side are to be juxtaposed with the former entries $\big(u,\ h(u)\big)$.

We put together the groups of same partials. In view of the elementary identities $\binom{l}{\nu-1}+\binom{l}{\nu}=\binom{l+1}{\nu}$, we get

$$\left(u,\ f(u), \binom{l+1}{1}\times f_{[1]}(u), \binom{l+1}{2}\times f_{[2]}(u), \dots, \binom{l+1}{l+1}\times f_{[l+1]}(u)\right)$$

Lemma 2 is now proved by induction.

We now take $l=k$ in Lemma 2 and get, for arbitrary $u\in U$, two similar expressions for $\mathcal{G}^{(k)}q\,(u)$ and $\mathcal{G}^{(k)}\tilde{q}\,(u)$. Suppose that Proposition – without Φ – holds for $u=u^0$. As a consequence, Theorem 2 now follows. Conversely, assuming Theorem 2, we get that the partial derivatives of q and $\tilde{q}$ at u^0 are mutually equal. This gives the Proposition.

1. What about singular algebraic varieties – branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness?

2. Working in real category, there is close link with contact geometry in dimension 3. (See recent book by Geiges.)

3. There exists a variant – Lagrangian tangency order. It produces interesting symplectic invariants to be compared with index of isotropy (Domitrz, Trębska).

Let M be a finite-dimensional real analytic manifold, d be a distance function on M induced by a Riemannian metric on M, and let $X, Y \subset M$ be closed subanalytic sets. The following important fact says that X and Y are regularly separated at any x_0:

Theorem (Łojasiewicz) For any $x_0 \in X \cap Y$ there exist $\nu > 0$ and $C > 0$ such for some neighbourhood $\Omega \subset M$ of x_0

$$d(x, X) + d(x, Y) \geq Cd(x, X \cap Y)^{\nu},$$

where $x \in \Omega$.

The exponent ν is called a *regular separation exponent* of X and Y at x_0. The infimum of all regular separation exponents of X and Y at x_0 is called the *Łojasiewicz exponent* of X and Y at x_0 and denoted $\mathcal{L}_{x_0}(X, Y)$.

Example Let $C = \{(x, y) \colon (y - x^2)^2 = x^5\}$,

The two branches of C issuing from the point $(0, 0)$,

$$C_- = \{y = x^2 - x^{5/2}\,,\ x \geq 0\} \text{ and } C_+ = \{y = x^2 + x^{5/2}\,,\ x \geq 0\}\,,$$

could be naturally extended to one-dimensional manifolds D_- and D_+, both of class C^2 – the graphs of functions

$$y_-(x) = x^2 - |x|^{5/2} \quad \text{and} \quad y_+(x) = x^2 + |x|^{5/2}\,,$$

respectively. The Taylor polynomials of degree 2 about $x = 0$ of y_- and y_+ coincide. Hence D_- and D_+ have at $(0, 0)$ the order of tangency at least 2 (and clearly *not* at least 3).

This example suggests that, in the real algebraic geometry category, it would be suitable to use non-integer measures of closeness. For instance, for the above sets $y_-(x)$ and $y_+(x)$, we may take

$$\sup\{\alpha > 0 : \; y_+(x) - y_-(x) = o(|x|^\alpha) \text{ when } x \to 0\}\,.$$

This generalised order of tangency would be $5/2$ in the above example. This is the minimal regular separation exponent of the semialgebraic sets C_- and C_+. That quantity is also the Łojasiewicz exponent $\mathcal{L}_{(0,0)}(C_-, \, C_+)$.

This example generalises, for $(y - x^N)^2 = x^{2N+1}$, to a pair of C^N manifolds having the order of tangency at least N, not at least $N + 1$, and the minimal separation exponent $\nu = N + \frac{1}{2}$.

Example Consider two curves N and Z in $\mathbb{R}^2(x, y)$ intersecting at $(0, 0)$:

$$N = \{y = 0\} \quad \text{and} \quad Z = \{y^d + yx^{d-1} + x^s = 0\},$$

where $1 < d < s$, and assume that d is odd. What is their minimal regular separation exponent at $(0, 0)$? We want to present Z as the graph of some function $y(x)$.

Lemma 3

There is a locally unique function

$$y(x) = x^{s-d+1}z(x) - \ x^{s-d+1}$$

whose graph is Z, with a C^∞ function $z(x)$, $z(0) = 0$.

Since $z(0) = 0$, this is the last summand of $y(x)$ which dominates the computation.

Using $y(x)$, we compute the minimal reqular separation exponent. Here is a sketch.

We discuss the inequality defining the regular separation exponent at $(0,0)$. Let $A=(x,0)$ be the points on N, $B=(x,y(x))$ be the points on Z, and let O be the point $(0,0)$. Using the above function $y(x)$, the length AB is of order $|x|^{s-d+1}$. Since AO and BO are of order $|x|$, the triangle inequality:

$$AB \leq AO + BO$$

implies the inequality from the Łojasiewicz theorem. We get that the exponent is equal to $s-d+1$.

HAPPY BIRTHDAY TO ADAM!