A FORMULA FOR THE EULER CHARACTERISTIC OF SINGULAR HYPERSURFACES[1]

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Introduction

Formulas expressing topological invariants of varieties using different characteristic classes are among the most useful applications/tools of algebraic geometry. A particularly rich work on this subject has been devoted to the dependency loci od vector bundle sections and, more generally, degeneracy loci of vector bundle morphisms (i.e. sets of points where a morphism drops the rank) - this theory is surveyed in [P-P1].

In [P-P2] the authors gave a formula for the (topological) Euler characteristic of the degeneracy locus of a general morphism $\varphi: F \to E$ of holomorphic vector bundles on a variety X (that is such that the induced section $s_\varphi: X \to \underline{Hom}(F,E)$ is transverse to all tautological degeneracy loci). It is natural to ask whether this formula can be generalized to the case of a broader family of morphisms. For instance, the Giambelli-Thom-Porteous formula describing the fundamental class of a degeneracy locus is valid under the weaker assumption that the codimension of the degeneracy locus is "expected".

It turns out that the problem is highly nontrivial even in the simplest possible case, that is when the morphism is a nontrivial section s of a line bundle L and the degeneracy locus is its zero set Z.

As it is well-known (see, for instance, [Di] and [Pa1]) if Z has only isolated singular points, the difference between the Euler characteristic $\chi(Z)$ of Z and the number expected for the Euler characteristic of the zero set of a general section of L is (up to sign) the sum of Milnor numbers of the singular points of Z. Therefore the difference in question may be thought as a "generalized Milnor number".

In [Pa1] the first named author gave the following characterization of the generalized Minor number (in fact this property served for the definition of this number in loc.cit.). Fix a Hermitian metrix on L and consider a standard decomposition of the associated metric connection $D = D' + D''$. Then the generalized Milnor number equals the intersection index of $D's$ and the zero section computed near Z.

The main purpose of this paper is to give a new formula for the generalized Milnor number which is completely different in the spirit and is worked out in the framework of algebraic and analytic geometry (Theorem 4). The present approach involves three major tools: a Whitney stratification $\mathcal{Z}$ of Z, local invariants of the singularities of Z coming from the Milnor fibration and Chern-MacPherson classes of the strata closures. Our formula for the generalized Milnor number has a clear "algebro-geometric" form. It is a sum of the following expressions indexed by the strata S of $\mathcal{Z}$

$$\int_{\bar{S}} \alpha(S)\big(c(L|_{\overline{S}})^{-1} \cap c_*(\bar{S})\big)\,,$$

where $\alpha(S)$ is a certain number determined by the Milnor fibration attached to Z and $c_*(\bar{S})$ is the Chern-MacPherson class of the closure of S (see [McP] and also [S],[B-S]).

The proof of Theorem 4 combines the techniques from differential geometry and topology with that of projective algebraic geometry. The first ones allow us to find a simple expression for the generalized Milnor number in the situation when there exists a smooth divisor Z' linearly equivalent to Z and transverse to $\mathcal{Z}$ (Proposition 7). Using the Bertini theorem in the version of Verdier ([V]) and a property of Chern-MacPherson classes (Lemma 8), this allows us to prove our formula for very ample L. Now, given an arbitrary line bundle L we pick a very ample M such that $L \otimes M$ is also very ample. Let H be a nonsingular and transverse to $\mathcal{Z}$ zero set of a section of M. Then, knowing that the formula is true for $Z \cup H$ we show that it holds also for Z, by induction on $dim Z$. The induction step depends heavily on subtle calculations of the Euler characteristics of some Milnor fibers (Lemma 3) and a specialization of the Hirzebruch functional equation for the virtual T_y-genus (cf.(6)). The main formula is proven under the assumption of projectivity of the ambient space X. We conjecture (see the end of this article) that this assumption can be dropped.

Some of the results presented here were announced in [P-P1].

Conventions:

Let X be a complex manifold and let L be a holomorphic line bundle on X. Having chosen a Hermitian metric on L, the norm of a vector in L will be denoted by $\|v\|$. By $\mathring{\mathbf{B}}_\varepsilon \subset \mathbf{C}^n$ (resp. $\mathbf{S}^{2n-1}_\varepsilon \subset \mathbf{C}^n$) we will denote the open ball (resp. the sphere of real dimension $2n-1$) with center at the origin and the radius ε. Finally, a complex number c will be called small if $|c|$ (the absolute value of c) is a small positive real number.

Let X be a compact n-dimensional complex manifold and let L be a holomorphic line bundle over X. Take $s \in H^0(X, L)$ a holomorphic section of L such that the zero set Z of s is a (nowhere dense) hypersurface in X.

We define the number $\mu(Z, X)$ as follows

$$\mu(Z, X) := (-1)^n(\chi(Z) - \chi(X|L)),$$

where for a vector bundle E over X, we define

$$\chi(X|E) := \int_X c(E)^{-1} c_{rankE}(E) c(X)\,.$$

Recall that $\chi(X|E)$ equals the Euler characteristic of the zero set of a general section of E, i.e. a section transverse to the zero section (see e.g. [P-P]). Usually, we will write $\mu(Z)$ instead of $\mu(Z, X)$.

Example 1. Assume that Z has only isolated singular points. Pick one such $x \in Z$. In local coordinates $z = (z_1, \ldots, z_n)$ around $x \in Z$ the hypersurface Z is defined by a holomorphic function f. We may assume that x is the origin in $\mathbf{C}^n$ and f is defined in a neighbourhood of x. For small positive ε and δ (and $0 < \delta \ll \varepsilon$) the intersection $f^{-1}(\delta) \cap \mathring{\mathbf{B}}_\varepsilon$ has the homotopy type of a bouquet $\mathbf{S} \vee \ldots \vee \mathbf{S}$, where $\mathbf{S}$ is a sphere of real dimension $n-1$ (see [Mi]). The number of spheres μ_x, say, is called the Milnor number of Z at x. Since $Z \cap \mathring{\mathbf{B}}_\varepsilon$ is contractible

$$\chi(Z \cap \mathring{\mathbf{B}}_\varepsilon) - \chi(f^{-1}(\delta) \cap \mathring{\mathbf{B}}_\varepsilon) = (-1)^n \mu_x\,.$$

It is not difficult to see (cf. [Pa1] for instance) that if Z has only isolated singularities, then

$$\mu(Z) = \chi(Z) - \chi(X|L) = (-1)^n \sum_{x \in Sing(Z)} \mu_x\,.$$

Recall that an alternative algebraic expression for the Milnor number is given by the formula

$$\mu_x = dim_{\mathbf{C}}\, \mathbf{C}\{z\}/(\partial f/\partial z_1, \cdots, \partial f/\partial z_n)\,.$$

For other interpretations of the Milnor number we refer the reader to [M] and [O].

In virtue of the example $\mu(Z)$ may be thought as a "generalized Milnor number".

Let x be an arbitrary point of $Sing(Z)$ and assume that in local coordinates around x the hypersuface Z is the zero set of (the germ of) an analytic function $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. Choose an $\varepsilon > 0$ small enough and $\delta > 0$ such that $0 < \delta \ll \varepsilon$. Let $D^*_\delta = \{z \in \mathbf{C}; 0 < |z| < \delta\}$ be a small open punctured disc in $\mathbf{C}$. Then, f restricted to $\mathring{\mathbf{B}}_\varepsilon \cap f^{-1}(D^*_\delta)$ is a smooth locally trivial fibration for any $0 < \delta \ll \varepsilon$ small enough (this variant of the Milnor fibration theorem [Mi] stems from [Lê] and [Ha]). We call this fibre *the Milnor fibre attached to x* and will denote it by F_x.

Definition. We define $\mu(Z,x) := (-1)^{n-1}(\chi(F_x)-1)$.

Example 2. If x is a nonsingular point of Z, then $\mu(Z,x)=0$. If x is an isolated singularity, then $\mu(Z,x) = (-1)^{n-1}\Big(1+(-1)^{n-1}dim_{\mathbf{Q}}H^{n-1}(F_x,\mathbf{Q})-1\Big) = \mu_x$, the usual Milnor number of Z at x.

Let $\mathcal{Z}$ be a Whitney stratification (see e.g. [G-M]) of Z. Then by a recent result obtained independently in [Pa2] and [B-M-M], $\mathcal{Z}$ is a "good" stratification of Z. Recall that a stratification $\mathcal{Z}$ of Z is traditionally called "good" if it satisfies the following local condition (which is independent of the choice of local coordinates). Assume that as above Z is described locally as the zero set of $f\colon \mathbf{C}^n \to \mathbf{C}$. We say that $\mathcal{Z}$ is a "good" stratification of Z if for each sequence $x_k \in \mathbf{C}^n \setminus Z$ converging to $x \in Z$ and such that the sequence $T_{x_k}\big(f^{-1}(f(x_k))\big)$ of the tangent spaces to the fibres of f has a limit T (in $\mathbf{P}^{n-1}$), T contains the tangent space to the stratum containing x. Now we pass to a characterization of "good" stratifications using local coordinates. Assume that x is the origin and the stratum containing x is given locally by the equations $z_1 = \ldots = z_j = 0$. Then the above condition can be described as follows:

$$\frac{\|(\partial f/\partial z_{j+1}(z),\ldots,\partial f/\partial z_n(z))\|}{\|(\partial f/\partial z_1(z),\ldots,\partial f/\partial z_j(z))\|} \xrightarrow{z\to 0} 0\,. \tag{1}$$

If $\mathcal{Z}$ is a Whitney stratification of Z then by Thom's Second Isotopy Lemma (see e.g. [G, Theorem 5.8]) the topological type of the Milnor fibres F_x at x is constant along the strata of $\mathcal{Z}$.

Notation. For $S \in \mathcal{Z}$ we denote by μ_S the value of $x \mapsto \mu(Z,x)$ on S.

The following result, which is a particular case of Theorem A of [N], gives an example of a more elaborate calculation of $\mu(Z,x)$, which will be important in the inductive step of the proof of Theorem 4. We present a short, independent proof for the reader's convenience.

Lemma 3. *Let Z be a hypersurface in X and let H be a nonsingular hypersurface of X transverse to a Whitney stratification of Z. Then for each $x \in Z \cap H$*

$$\mu(Z\cup H, x) = (-1)^n$$

(in the other words the Euler characteristic of the Milnor fibre of $Z \cup H$ attached to x is zero).

Proof. Assume that x is the origin in $\mathbf{C}^n$ and let Z and H be the zero sets of f and z_n respectively, so $Z \cup H$ is the zero set of $g(z) = z_n f(z)$. For $z \in \mathbf{C}^n$ we write $z = (z', z_n)$, where $z' \in \mathbf{C}^{n-1}$. We assume that the line $l = \{(0, z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}\}$

is contained in the stratum of a Whitney stratification $\mathcal{Z}$ of Z which contains x. Denote by G the Milnor fibre of $Z \cup H$ at the origin. We shall show that, up to homotopy equivalence, G fibres over a circle and therefore $\chi(G) = 0$, as desired. In the proof we use standard facts of real analytic geometry which can be found in [Ł] and [G].

First, consider the family of hypersurfaces $Z(z_n)$ in $\mathbf{C}^{n-1}$ given by the equations $f(*, z_n) = 0$ with fixed z_n. Since $\mathcal{Z}$ is "good", the Milnor fibres of $Z(z_n)$ at $0 \in \mathbf{C}^{n-1}$ are homeomorphic to the Milnor fibre F' of $Z \cap H$ at the origin, provided z_n is sufficiently small. Moreover, by the regular separation [Ł, §18], one may find $\varepsilon_0 > 0$ and $m \in \mathbf{N}$ such that for every $|z_n| \leq \varepsilon_0$ and $0 < \varepsilon \leq \varepsilon_0$, $c \in \mathbf{C}$ with $0 < |c| \leq \varepsilon^m$, the space

$$\{z' \in \mathbf{C}^{n-1}; \|z'\| \leq \varepsilon, f(z', z_n) = c\},$$

depending on z_n and c, is homeomorphic to F'. Instead of the ordinary representatives of G

$$G_{\varepsilon,c} = g^{-1}(c) \cap \mathring{\mathbf{B}}_\varepsilon \qquad c \in \mathbf{C}\,, 0 < |c| \ll \varepsilon \ll 1$$

we consider

$$\tilde{G}_{\varepsilon,c} = g^{-1}(c) \cap U_\varepsilon \qquad c \in \mathbf{C}\,, 0 < |c| \ll \varepsilon \ll 1\,,$$

where

$$U_\varepsilon = \{(z', z_n) \in \mathbf{C}^n; \|z'\| < \varepsilon, |z_n| < \varepsilon, |f(z', z_n)| < \varepsilon^m\}$$

(we will show below that they are homotopically equivalent). Note that the image of $\tilde{G}_{\varepsilon,c}$ by the projection π_n defined by $\pi_n(z) = z_n$, is the annulus $\{z \in \mathbf{C}; \frac{c}{\varepsilon^m} < z < \varepsilon\}$ and the fibres are homeomorphic to F'. Since π_n restricted to $\tilde{G}_{\varepsilon,c}$ is semi-analytic and can be stratified, this already gives $\chi(\tilde{G}_{\varepsilon,c}) = 0$. But, in fact, using (1) one may easily prove that $\pi_{n|\tilde{G}_{\varepsilon,c}}$ is a locally trivial smooth fibration.

To complete the proof we will show that $\tilde{G}_{\varepsilon,c}$ are homotopically equivalent to $G_{\varepsilon,c}$ for $0 < |c| \ll \varepsilon \ll 1$.

Consider a one-parameter family of neighbourhoods of the origin in $\mathbf{C}^n$

$$V_\varepsilon = \varphi^{-1}([0, \varepsilon))\,,$$

where $\varphi \colon \mathbf{C}^n \to \mathbf{R}$ is a semi-analytic continuous function (the same argument is valid if φ is subanalytic) and $\varphi^{-1}(0) = 0$. Fix $\varepsilon'_0 > 0$ and $c \in \mathbf{C} \setminus \{0\}$ and let

$$\Phi \colon V_{\varepsilon'_0} \cap g^{-1}(\{tc; 0 \leq t \leq 1\}) \to \mathbf{R}_+ \times [0, 1]$$

be given by $\Phi(z) = (\|z\|, g(z)/c)$. Since Φ is semi-analytic and continuous, it can be stratified and by the properties of semi-analytic sets there exist ε_0 and $m \in \mathbf{N}$ such that all

$$G'_{\varepsilon,t} = g^{-1}(tc) \cap V_\varepsilon\,.$$

are homeomorphic if only $(\varepsilon, t) \in \{(\varepsilon, t); 0 < \varepsilon \leq \varepsilon_0, 0 < t \leq \varepsilon^m\} =: A$, say. Moreover, the homeomorphisms can be obtained by the integration of vector fields

(Thom's First Isotopy Lemma [G, Theorem 5.2]) and therefore for all $(\varepsilon, t), (\varepsilon', t) \in A$, $\varepsilon \le \varepsilon'$, $G'_{\varepsilon,tc}$ is a deformation retract of $G'_{\varepsilon',tc}$. Take ε_0 good for both families $\mathring{\mathbf{B}}_\varepsilon$ and U_ε and choose ε_i $(i = 1, 2, 3)$ such that

$$\mathring{\mathbf{B}}_{\varepsilon_0} \supset U_{\varepsilon_1} \supset \mathring{\mathbf{B}}_{\varepsilon_2} \supset U_{\varepsilon_3} \, .$$

By the above, for sufficiently small $t > 0$, $G_{\varepsilon_0,tc}$ is a deformation retract of $G_{\varepsilon_2,tc}$ and $\tilde{G}_{\varepsilon_1,tc}$ is a deformation retract of $\tilde{G}_{\varepsilon_3,tc}$. Therefore $G_{\varepsilon_2,tc}$ and $\tilde{G}_{\varepsilon_1,tc}$ are homotopically equivalent. This ends the proof. □

The next theorem, which is the main result of this paper, gives a formula for the generalized Milnor number $\mu(Z)$ in terms of a Whitney stratification of Z, local invariants of the singularities of Z coming from the Milnor fibration and Chern-MacPherson classes of the strata closures.

Theorem 4. *Let X be a nonsingular subvariety of $\mathbf{P}^N$ and let Z be the zero set of a holomorphic section of a holomorphic line bundle L over X. Let $\mathcal{Z}$ be a Whitney stratification of Z. Then*

$$\mu(Z) = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \int_{\bar{S}} \left(c(L|_{\bar{S}})^{-1} \cap c_*(\bar{S}) \right) ,$$

where $\alpha(S) = \mu_S - \sum_{S' \neq S, \bar{S}' \supset S} \alpha(S')$, and $c_*(\bar{S})$ denotes the Chern-MacPherson class of $\bar{S}$ (-the closure of the stratum S).

For a definition of Chern-MacPherson classes the reader can consult [McP]; classes essentially equivalent to the above were defined independently by M. H. Schwartz [S], [B-S]. In [P-P2, Section 1] we recall a more algebraic approach to Chern-MacPherson classes, given by González-Sprinberg, Verdier and others and provide some auxiliary properties of Chern-MacPherson classes needed in the sequel of the present paper.

Before we start the proof we give some examples illustrating the theorem.

Example 5. Assume that $Y = Sing(Z)$ is nonsingular and that the pair $(Z \setminus Y, Y)$ satisfies Whitney Conditions. Then, for $x \in Y$

$$\mu_Y = \mu(Z, x) = (-1)^m \mu^{(n-m)}(Z, x) \, ,$$

where $m = dim\, Y$ and $\mu^{(n-m)}(Z, x)$ is the $(n-m)$-th Teissier number of Z at x (see [T]). Recall that $\mu^{(n-m)}(Z, x)$ is the Milnor number at x of $Z \cap L$ where L is a sufficiently general $(n-m)$-dimensional linear space containing x. Now the theorem asserts

$$\mu(Z) = \mu_Y \int_Y c(Y) \cdot c(L|_Y)^{-1} \, .$$

As it was proved in [Pa1], using the index characterization of $\mu(Z)$ recalled in the Introduction, the above formula holds without the assumption of projectivity of X.

Example 6. Let $Y = Sing(Z)$ be nonsingular and let $\mathcal{Z}$ consists of 3 strata: $S_1 = Z - Y, S_2 = Y - S_3$ and S_3. Then the theorem gives

$$\mu(Z) = \mu_{S_2} \int_Y c(Y) \cdot c(L|_Y)^{-1} + (\mu_{S_3} - \mu_{S_2}) \int_{S_3} c(S_3) \cdot c(L|_{S_3})^{-1} .$$

In order to prove Theorem 4, let us first prove the following proposition (in which we do not assume the projectivity of X).

Proposition 7. *Let L be a holomorphic line bundle over a compact n-dimensional manifold X. Assume that a hypersurface Z in X is the zero set of $s \in H^0(X, L)$ and let s' be a holomorphic section of L such that the zero set Z' of s' is nonsingular and transverse to a Whitney stratification $\mathcal{Z}$ of Z. Then*

$$\mu(Z) = \sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z') = \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') .$$

Proof. We will approximate Z by the zero sets Z_t of the sections $s_t = s - ts'$ ($t \in \mathbf{C}$). First, we note that for small $t \neq 0$ all the Z_t are nonsingular and transverse to $\mathcal{Z}$. In fact, by Bertini's Theorem (e.g [G-H, p.137]) the singularities of generic Z_t are contained in $Z \cap Z'$. Fix $x_0 \in Z \cap Z'$ and investigate Z_t around x_0. Denote the sections s and s' by f and g respectively and consider them as functions. By the transversality of Z' to $\mathcal{Z}$ and the fact that $\mathcal{Z}$ is a "good" stratification, the levels of f and g are transverse with the angle bounded from below by a nonzero positive constant. In particular, $d(f - tg) = df - t \cdot dg$ nowhere vanishes on Z_t (for $t \neq 0$) and Z_t is transverse to $\mathcal{Z}$.

Let us fix a Hermitian metric on L. For t small enough it is easy to see that $Z \cap Z'$ is a strong deformation retract of

$$Z_{t,\epsilon} = \{x \in Z_t;\ \|s'(x)\| \ < \epsilon\}$$

provided ϵ is sufficiently small. (For $t = 0$, this follows from the transversality assumption.)

Step 1. We claim that for a sufficiently small t we can find an universal $\epsilon > 0$ such that $Z \cap Z'$ is a strong deformation retract of $Z_{t,\epsilon}$.

Proof of the assertion of Step1. Take small $t \neq 0$. We shall show that $Z_{t,\epsilon}$ can be retracted onto $Z \cap Z'$ using the flow generated by the orthogonal projection on $Z_{t,\epsilon}$ of $grad\,\|s'\|^2$. To prove it, it suffices to show that the projected vector field does not vanish on $Z_{t,\epsilon} \setminus Z \cap Z'$. We proceed locally in a neighbourhood of some $x_0 \in Z \cap Z'$. So assume that x_0 is the origin in $\mathbf{C}^n$ and $s = f \cdot \mathbf{e}$, $s' = g \cdot \mathbf{e}$, where $\mathbf{e}$ is a nonvanishing holomorphic section of L defined in a neighbourhood of x_0 and f, g are holomorphic functions. Let D be the associated connection and θ-the connection form with respect to $\mathbf{e}$. Then

$$
\begin{aligned}
d\,(\|s'\|^2) &= \langle g\cdot\mathbf{e}, Dg\cdot\mathbf{e}\rangle + \langle Dg\cdot\mathbf{e}, g\cdot\mathbf{e}\rangle \\
&= (d\,|g|^2 + |g|^2(\theta+\bar\theta))\|\mathbf{e}\|^2 \qquad (2)\\
&= ((\bar g\,dg + |g|^2\theta) + \overline{(\bar g\,dg + |g|^2\theta)})\|\mathbf{e}\|^2 .
\end{aligned}
$$

As x_0 is a regular point of g, we may choose such local coordinates with the origin in x_0, that $g(z) \equiv z_n$ and in a neighbourhood of x_0 a given stratum of $\mathcal{Z}$ (that is transverse to $H = \{z; z_n = 0\}$ by the assumption) is given locally by the equations $z_1 = \ldots = z_j = 0$ where $j < n$. Then, writing $df = d'f + (\partial f/\partial z_n)dz_n$, we have (near x_0) by (1)

$$
\|d'f\| \geq C\cdot|\partial f/\partial z_n| , \qquad (3)
$$

for some universal $C > 0$.

Take $z \notin Z \cup Z'$ and near x_0 (these properties of z will be assumed up to the end of Step 1). We show that the levels of $\|s'\|^2$ and f/g are transverse at z. For this purpose we consider the conormal vectors to them. Let $t(z) := f(z)/g(z)$. The holomorphic part of the conormal vector to the former, given by (2), equals

$$
\bar g(z)\,dg(z) + |g(z)|^2\theta(z) = \bar z_n dz_n + |z_n|^2\theta(z) ,
$$

and the holomorphic conormal vector to the latter is

$$
g(z)^{-1}((df(z) - t(z)\cdot dg(z)) = z_n^{-1}(df(z) - (f(z)/z_n)dz_n) .
$$

To prove the statement it is enough to show that the above vectors are independent for z sufficiently close to x_0.

Suppose that they are linearly dependent. Then this implies the linear dependence of $l_1(z) = dz_n + z_n\theta(z)$ and $l_2(z) = dz_n + (\partial f/\partial z_n(z) - f(z)/z_n)^{-1}d'f(z)$ for z sufficiently close to x_0. We use the following inequality due to Łojasiewicz [Ł, §18 Proposition 1]:

$$
\|(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)\| \geq |f|^\alpha ,
$$

for some $0 < \alpha < 1$, which holds in some neighbourhood of x_0. Since θ is bounded in a neighbourhood of x_0, the second summand of $l_1(z)$ is bounded near x_0. On the other side, it follows from the Łojasiewicz inequality combined with (3) that the second summand of $l_2(z)$ is unbounded in a neghbourhood of x_0. This leads to a contradiction.

The proof of the assertion of Step 1 is complete. □

Fix ϵ given by Step 1 and let Y be a compact manifold (with boundary) defined as $X \setminus \{x \in X; \|s'(x)\| < \epsilon\}$. Note that the stratification $\mathcal{Z}$ is transverse to ∂Y.

One of the main properties of Whitney stratification is a topological equisingularity. It says that if $(Z, \mathcal{Z})$ is a set with Whitney stratification, then the topological type of Z at $x \in S \in \mathcal{Z}$, does not depend on the choice of the point x on a given stratum S. This follows from Thom's First Isotopy Lemma, whose proof is based on the technique of extending vector fields (the reader can consult [G, Chapter II]) and requires a construction of the system of tubular neighbourhoods of the strata (loc.cit. Chapter II §2).

Step 2 (A construction of a system of tubular neighbourhoods Γ_S of $S \cap Y$ in Y)

For $S \in \mathcal{Z}$ we define Γ_S inductively on $dimS$ as follows:

$$\Gamma_S = \{x \in Y; dist(x, S) \leq \delta_S\} \cup \bigcup_{S' \subset \bar{S} \setminus S} \Gamma_{S'} \,,$$

where δ_S is a sufficiently small number such that:

(a) $G_S = \Gamma_S \setminus \bigcup_{S' \subset \bar{S} \setminus S} Int(\Gamma_{S'})$ is a manifold with corners which (as a stratified set) is transverse to $\mathcal{Z}$.

(b) G_S is a locally trivial topological fibration over $\tilde{S} := S \cap G_S$ (by Thom's First Isotopy Lemma) We denote this fibration by π_S.

(c) $\tilde{S}$ is a manifold with corners with the same homotopy type as $S \cap Y$ (which can be shown by gluing the vector fields given by (b)).

(A more complicated system of tubular neighbourhoods satisfying the above properties was constructed by Dubson [Du, Proposition I 1.4.2.B])

Claim: The map $\pi_S|_{Z_t \cap G_S} \colon Z_t \cap G_S \to \tilde{S}$ is a locally trivial topological fibration and its fibre $\tilde{F}_x$, $x \in \tilde{S}$, is homotopically equivalent to the Milnor fibre F_x.

Indeed, since $\mathcal{Z}$ is a "good" stratification, Z_t (for sufficiently small $t \neq 0$) is transverse to the fibres of π_S. In particular $\pi_S|_{Z_t \cap G_S} \colon Z_t \cap G_S \to \tilde{S}$ is a locally trivial fibration. Its fibre $\tilde{F}_x$ at $x \in \tilde{S}$ is homotopically equivalent to the Milnor fibre F_x by Thom's First Isotopy Lemma.

Finally we have

$$\begin{aligned}
\chi(Z) - \chi(Z_t) &= \chi(Z \cap Y) - \chi(Z_t \cap Y) \\
&= \sum_{S \in \mathcal{Z}} (\chi(\tilde{S}) - \chi(Z_t \cap G_S)) \\
&= \sum_{S \in \mathcal{Z}} \big(\chi(\tilde{S}) - \chi(\tilde{S})\chi(\tilde{F}_x)\big) \qquad \text{(by Claim; here, } x \in \tilde{S}) \\
&= (-1)^n \sum_{S \in \mathcal{Z}} \chi(\tilde{S})\mu_S \,.
\end{aligned}$$

If ϵ is sufficiently small, then $\chi(\tilde{S}) = \chi(S \cap Y) = \chi(S \setminus Z')$ for every $S \in \mathcal{Z}$. Therefore, by the above we get

$$\begin{aligned}
\mu(Z) = (-1)^n[\chi(Z) - \chi(Z_t)] &= \sum_{S \in \mathcal{Z}} \mu_S \cdot \chi(S \setminus Z') \\
&= \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') \,,
\end{aligned}$$

where the latter expression is just a simple rewriting of the former.

This completes the proof of Proposition 7. $\square$

We record the following formula for a proof of which we refer to [P-P2, Sect.1].

Lemma 8. *Let s be a general section of a vector bundle E over a possibly singular variety X. Denoting by Z the zero set of s, one has*

$$\chi(Z) = \int_X c(E)^{-1} \cdot c_{rankE}(E) \cap c_*(X) . \quad \square$$

Proof of Theorem 4.

Step 1 We claim that the assertion is true if L is very ample.

By Bertini theorem in the version of Verdier (see [V] and [G-H p.137]) there exists a section s' of L whose zero set Z' is nonsingular and transverse to $\mathcal{Z}$. Then we have

$$\begin{aligned}
\mu(Z) &= \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \chi(\bar{S} \setminus Z') \\
&= \sum_{S \in \mathcal{Z}} \alpha(S) \cdot [\chi(\bar{S}) - \chi(\bar{S} \cap Z')] \\
&= \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \Big[\int_{\bar{S}} c_*(\bar{S}) - \int_{\bar{S}} c(L|_{\overline{S}})^{-1} c_1(L|_{\overline{S}}) \cap c_*(\bar{S}) \Big] \\
&= \sum_{S \in \mathcal{Z}} \alpha(S) \cdot \int_{\bar{S}} (c(L|_{\overline{S}})^{-1} \cap c_*(\bar{S})) .
\end{aligned}$$

Here, the first equality follows from Proposition 7 and the third one - from a standard property of the Chern-MacPherson classes and Lemma 8.

Now consider the general case. We proceed by induction on $n = dimX$.

Let M be a very ample line bundle on X such that $L \otimes M$ is also very ample (such a bundle exists since X is projective). Let H be the zero set of a section of M such that H is nonsingular and transverse to $\mathcal{Z}$. Then the family $S \setminus H$ (for $S \in \mathcal{Z}$), $S \cap H$ (for $S \in \mathcal{Z}$) and $H \setminus Z$ defines a Whitney stratification of $Z \cup H$.

Let T be the zero set of a general section of $L \otimes M$ such that T is nonsingular and transverse to the above stratification of $Z \cup H$.

Step 2 We claim that

$$\begin{aligned}
\mu(Z \cup H) = &\sum_{S \in \mathcal{Z}} \alpha(S) \Big[\chi(\bar{S}) - \chi(\bar{S}|M) - \chi(\bar{S}|L \otimes M)) - \chi(\bar{S}|M \oplus (L \otimes M)) \Big] \\
&- \mu(Z \cap H) - \mu(Z \cap H \cap T) \\
&+ (-1)^n (\chi(X|L \oplus M) - \chi(X|L \oplus M \oplus (L \otimes M)) ,
\end{aligned} \tag{4}$$

where $\mu(Z\cap H)=\mu(Z\cap H,H)$ and $\mu(Z\cap H\cap T)=\mu(Z\cap H\cap T,H\cap T)$.

Indeed, by considering the above stratification of $Z\cup H$, we have

$$\begin{aligned}\mu(Z\cup H)=&\sum_{S\in\mathcal{Z}}\mu_{S\setminus H}(Z\cup H)\cdot\chi(S\setminus H\setminus T)+\mu_{H\setminus Z}(Z\cup H)\cdot\chi(H\setminus Z)\\&+\sum_{S\in\mathcal{Z}}\mu_{S\cap H}(Z\cup H)\cdot\chi(S\cap H\setminus T)\qquad\text{(by Proposition 7)}\\=&\sum_{S\in\mathcal{Z}}\mu_S\cdot\chi(S\setminus H\setminus T)+(-1)^n\sum_{S\in\mathcal{Z}}\chi(S\cap H\setminus T)\end{aligned}$$

because obviously $\mu_{S\setminus H}(Z\cup H)=\mu_S(Z)=\mu_S$, $\mu_{H\setminus Z}=0$, and $\mu_{S\cap H}(Z\cup H)=(-1)^n$ by Lemma 3. Thus

$$\mu(Z\cup H)=\sum_{S\in\mathcal{Z}}\alpha_S\cdot\chi(\bar{S}\setminus H\setminus T)+(-1)^n[\chi(Z\cap H)-\chi(Z\cap H\cap T)]\,.$$

But we have

$$\chi(\bar{S}\setminus H\setminus T)=\chi(\bar{S})-\chi(\bar{S}|M)-\chi(\bar{S}|L\otimes M)+\chi(\bar{S}|M\oplus(L\otimes M))$$

and

$$\begin{aligned}(-1)^n\chi(Z\cap H)&=-\mu(Z\cap H)+(-1)^n\chi(H|L)\\&=-\mu(Z\cap H)+(-1)^n\chi(X|L\oplus M)\,,\end{aligned}$$

$$\begin{aligned}(-1)^{n-1}\chi(Z\cap H\cap T)&=-\mu(Z\cap H\cap T)+(-1)^{n-1}\chi(H\cap T|L)\\&=-\mu(Z\cap H\cap T)+(-1)^{n-1}\chi(X|L\oplus M\oplus(L\otimes M))\,.\end{aligned}$$

All these equalities give (4).

Step 3 We claim that

$$\begin{aligned}\mu(Z\cup H)=&\mu(Z)+\mu(Z\cap H)\\&+(-1)^n\big[\chi(X|L)+\chi(X|M)-\chi(X|L\oplus M)-\chi(X|L\otimes M)\big]\,.\end{aligned}\tag{5}$$

Indeed, by the definition of $\mu(*)$ and the additivity of Euler characteristic, we have

$$\begin{aligned}\mu(Z\cup H)=&(-1)^n[\chi(Z\cup H)-\chi(X|L\otimes M)]\\=&(-1)^n[\chi(Z)+\chi(H)-\chi(Z\cap H)-\chi(X|L\otimes M)]\\=&\mu(Z)+(-1)^n[\chi(H)-\chi(Z\cap H)+\chi(X|L)-\chi(X|L\otimes M)]\,.\end{aligned}$$

But $\chi(H)=\chi(X|M)$ and $(-1)^{n-1}[\chi(Z\cap H)-\chi(X|L\oplus M)]=\mu(Z\cap H)$. Thus the above equation gives (5).

Step 4 For arbitrary line bundles L and M on any compact analytic variety Y, the following equality holds

$$(6)\quad 2\chi(Y|L\oplus M)+\chi(Y|L\otimes M)=\chi(Y|L)+\chi(Y|M)+\chi(Y|L\oplus M\oplus(L\otimes M))\,.$$

This equation was proved in a more general framework in [H, Theorem 11.3.1]. We leave to the reader a verification of the following equality

$$\begin{aligned}&2ab(1+a)^{-1}(1+b)^{-1}+(a+b)(1+a+b)^{-1}\\ =&a(1+a)^{-1}+b(1+b)^{-1}+ab(a+b)(1+a)^{-1}(1+b)^{-1}(1+a+b)^{-1}\,.\end{aligned}$$

This equality (with $a=c_1(L)$ and $b=c_1(M)$) and the definition of $\chi(X|E)$ implies (6).

Step 5 (Inductive step) In order to prove the formula we use the induction on $n=dimX$. Assume that the formula holds for $\mu(Z\cap H)$ and $\mu(Z\cap H\cap T)$

$$(7)\qquad \mu(Z\cap H)=\sum_{S\in\mathcal{Z}}\alpha(S)[\chi(\bar{S}|M)-\chi(\bar{S}|L\oplus M)]\,,$$

$$(8)\qquad \mu(Z\cap H\cap T)=\sum_{S\in\mathcal{Z}}\alpha(S)[\chi(\bar{S}|M\oplus(L\otimes M))-\chi(\bar{S}|L\oplus M\oplus(L\otimes M))]\,.$$

It follows from (4) and (5) that

$$\begin{aligned}\mu(Z)=&\sum_S\alpha(S)\cdot\left[\chi(\bar{S})-\chi(\bar{S}|M)-\chi(\bar{S}|L\otimes M)+\chi(\bar{S}|M\oplus(L\otimes M))\right]\\ &-2\mu(Z\cap H)-\mu(Z\cap H\cap T)\\ &+(-1)^{n-1}\Big[\chi(X|L)+\chi(X|M)-\chi(X|L\otimes M)\\ &\qquad -2\chi(X|L\oplus M)-\chi(X|L\oplus M\oplus(L\otimes M))\Big]\,,\end{aligned}$$

the latter summand being zero by (6). Using (7) and (8) we thus obtain:

$$\begin{aligned}\mu(Z)=&\sum_S\alpha(S)\Big[\chi(\bar{S})-\chi(\bar{S}|M)-\chi(\bar{S}|L\otimes M)+\chi(\bar{S}|M\oplus(L\otimes M))\\ &\qquad +2\chi(\bar{S}|M)-2\chi(\bar{S}|L\oplus M)\\ &\qquad -\chi(\bar{S}|M\oplus(L\otimes M))-\chi(\bar{S}|L\oplus M\oplus(L\otimes M))\Big]\\ =&\sum_S\alpha(S)[\chi(\bar{S})-\chi(\bar{S}|L)]\,,\end{aligned}$$

by applying (6) once again. This gives the desired assertion. □

Theorem 4 has been proven under the assumption of the projectivity of the ambient space X, which allows one to use the Bertini-Verdier theorem. Nevertheless, it seems reasonable to state the following:

Conjecture:

The formula of Theorem 4 holds for any compact complex manifold X. In other words, the assumption of projectivity of X can be dropped.
(See Example 5 for some evidence for this conjecture.)

References

[B-S] J.P. Brasselet, M.H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque **82–83** (1981), 93–147.

[B-M-M] J.Briancon, Ph.Maisonobe, M.Merle, *Localisation de systemes differentiels, stratifications de Whitney et condition de Thom*, preprint Université de Nice No. 324, Invent. Math. - to appear.

[Di] A. Dimca, *On the homology and cohomology of complete intersections with isolated singularities*, Comp. Math. **58** (1986), 321–339.

[Du] A.S. Dubson, *Calcul des invariants numériques des singularités et des applications*, preprint S.F.B. Theoretische Mathematik Universität Bonn (1981).

[G] C.G. Gibson et al., *Topological Stability of Smooth Mappings*, Lect. Notes in Math. 552, Springer, Berlin, New York, 1976.

[G-H] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[G-M] M.Goresky, R.MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **14**,, Springer, Berlin Heidelberg, 1988.

[H] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, New York, 1966.

[Ha] H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191** (1971), 235-252.

[Ł] S. Łojasiewicz, *Ensembles semi-analytiques*, I.H.E.S, 1965.

[Lê] Lê D. T., *Some remarks on Relative Monodromy*, Real and Complex Singularities, Oslo, 1976, pp. 397-403.

[McP] R.D. MacPherson, *Chern classes for singular algebraic varieties*, Annals of Math. **100** (1974), 423-432.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton Univ. Press, Princeton, 1968.

[N] A. Némethi, *The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$*, Comp. Math. **79** (1991), 63-97.

[O] P. Orlik, *The multiplicity of a holomorphic map at an isolated critical point*, Real and Complex Singularities, Oslo, 1976, pp. 405-474.

[Pa1] A. Parusiński, *A generalization of the Milnor number*, Math. Ann. **281** (1988), 247–254.

[Pa2] ———, *Limits of tangent spaces to fibres and the w_f condition*, preprint of University of Sydney No.92-38, Duke Math. Journal - to appear.

[P-P1] A. Parusiński and P. Pragacz, *Characteristic numbers of degeneracy loci*, in Enumerative Algebraic Geometry, Proceedings of the 1989 Zeuthen Symposium (S. L. Kleiman, A. Thorup, eds.), Contemporary Mathematics, vol. 123, AMS, Providence, 1991, pp. 189–198.

[P-P2] ———, *Chern-MacPherson classes and the Euler characteristic of degeneracy loci*, submitted.

[S] M.H. Schwartz, *Cycles caractéristiques définies par une stratification d'une variété analytique complexe*, Compt. Rend. Acad. Sc. **260** (1965), 3261-3264; 3535-3537.

[T] B. Teissier, *Cycles évanescentes, sections planes et conditions de Whitney*, Astérisque **7-8** (1973), 285-362.

[V] J.L. Verdier,, *Stratifications de Whitney et théoréme de Bertini-Sard*, Inv. Math. **36** (1976), 295–312.

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