OPERATOR CALCULUS FOR $\widetilde{\mathrm{Q}}$-POLYNOMIALS AND SCHUBERT POLYNOMIALS

Alain Lascoux and Piotr Pragacz

dédié à Maria

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Introduction

The goal of the present paper is to develop convenient algebraic tools to compute in the cohomology ring of Lagrangian flag manifolds as well as with Lagrangian degeneracy loci. There is quite a long series of papers devoted to this goal overlapping among others [B-G-G], [B-H], [B], [D1,2], [F-K], [F1,2], [H-B] and [P-R1,2]. The present paper will certainly not end this list. The cohomological formulas related to a symplectic form are extremely useful for computing in the cohomology (or Chow) rings of various moduli spaces including the moduli spaces of curves and the moduli spaces of abelian varieties in characteristic p. This is, perhaps, a raison d'être for the present paper.

Our basic tool here is a certain basis which is orthonormal with respect to the scalar product defined by the top symplectic divided difference. This basis has a natural geometric explanation. Consider a vector bundle V of rank $2n$, equipped with a nondegenerate symplectic form. Then the Lagrangian flag bundle parametrizing all complete Lagrangian flags in V fibers over the Lagrangian Grassmann bundle parametrizing Lagrangian n-bundles with a fiber being the "usual" flag variety of all complete flags in an n-dimensional vector space. The basis is a natural "product-basis" associated with this fibration. It consists of products of $\widetilde{Q}$-polynomials from [P-R2] (which provide a basis for the cohomology of the base of the fibration) times Schubert polynomials (which provide a basis for the cohomology of the fiber). We extensively investigate the action of different operators (notably divided differences) on this basis. It appears that such tools as a certain vanishing property (which is much in spirit of interpolation theory) and vertex operators (which have their origin in the theory of infinite-dimensional Lie algebras) are extremely useful to achieve this goal. The main technical result of the present paper is the "key formula" (Theorem 5.1). Many formulas reduce to the computation of the image of products of the type $x_1^k\widetilde{Q}_J$ under divided differences. The main algebraical upshot of this work is a study of the structure of a polynomial ring as a free module over the ring of polynomials symmetrical in the squares of the variables.

Even if the elements of the basis in question do not represent the Schubert classes in the Lagrangian flag manifold, they are very useful in the study of such classes. Recall that the polynomial representatives for those classes were defined already in [B-G-G] and [D1,2]. Recently, some algebro-combinatorial constructions of such representatives were studied in [B-H] and [F-K] and some other appeared implicitly in [F1,2]. Following [P-R2], we study in this paper still other polynomial representatives of Lagrangian Schubert classes in Appendix A. They seem to be well suited for the purposes of algebraic geometry. They are defined with the help of divided differences starting from some very simple "top" polynomial, and for maximal Grassmannian elements they are equal to certain $\widetilde{Q}$-polynomials - exactly those predicted for them by the Giambelli-type formula from [P, Sect.6]. Also, as shown in the present paper, these polynomials behave well w.r.t. different operators which geometrically often correspond to Gysin maps. Symplectic Schubert polynomials à la polonaise are not positive in general (though involve much smaller quantities of negative monomials than the original polynomials of [B-G-G] and [D1,2]). In general, these polynomials involve much fewer monomials than the other known families of Lagrangian Schubert classes representatives; hence they are also better suited for numerical computations.

This paper contains also some simple proofs of previously known results: a new operator proof of the Giambelli-type formula for maximal Lagrangian Schubert classes from [P, Sect.6], and a purely algebraic proof of the congruence modulo the corresponding ideal of two (competing) expressions for diagonals from [F1,2] and [P-R2].

The article is organized as follows.

In Section 1 we collect needed information on the hyperoctahedral groups and divided differences associated with them. Moreover, we recall two families of polynomials crucial for this paper: Schubert polynomials and $\widetilde{Q}$-polynomials. Several scalar products on polynomial rings are also discussed together with their orthogonality properties.

In Section 2 we study two reproducing kernels for a certain scalar product defined on the ring of symmetric polynomials with values in the ring of symmetric polynomials in the squares of the variables. These kernels stem from [F1,2] and [P-R2] and are motivated by geometry. More precisely, they are polynomial representatives of the classes of relative diagonals in Lagrangian Grassmann bundles. For a recent geometric development of this idea, we refer the reader to [G]. We prove that these kernels satisfy a certain vanishing property that characterizes them almost completely (see [L-S] for the type A_n).

In Section 3 we derive in an elementary way (invoking only simple properties of divided differences and the linearity formula for $\widetilde{Q}$-polynomials) how a certain key operator ∇ acts on our product-basis.

In Section 4 we achieve the samy goal using the vanishing property. The proof is much shorter, but it relies on the operator characterization of $\widetilde{Q}$-polynomials.

Section 5 is the heart of the paper. The key formula is proven there with the help of vertex operators associated with Schur's P-functions and Q'-functions from [L-L-T2]. The starting point of our approach is a presentation of the "exceptional" divided difference ∂_0 as an (infinite) differential operator. This observation is, in fact, the main motivation of using the vertex operators in various calculations in this section.

Appendix A contains the proofs of two crucial properties of symplectic Schubert polynomials à la polonaise i.e. stability and the maximal Grassmannian property. Using the key formula, we establish in a purely algebraic way the above mentioned operator characterization of $\widetilde{Q}$-polynomials. The table of the polynomials for $n = 3$ is also included.

In Appendix B we apply the above calculations to give a new operator proof of the Lagrangian Giambelli-type formula, to derive a new Gysin formula as well as to provide a new formula for Lagrangian flagged degeneracy loci.

In the present paper we prove, among others, the results announced in [P-R2, Appendix B]. We plan to present the orthogonal variants of this theory in a separate publication.

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1. Divided differences associated with the hyperoctahedral groups.

Recall first the definition of the hyperoctahedral group i.e. the Weyl group of the symplectic group. Let n be a positive integer. Let S_n be the symmetric group (of all bijections of $\{1,\ldots,n\}$) generated by the simple transpositions $s_i = [1,\ldots,i-1,i+1,i,i+2,\ldots,n]$, $i=1,\ldots,n-1$. The hyperoctahedral group $W=W_n$ is an extension of S_n by an element s_0 such that

$$(1)\qquad s_0^2=1, \quad s_0\cdot s_1\cdot s_0\cdot s_1 = s_1\cdot s_0\cdot s_1\cdot s_0, \quad s_0\cdot s_i = s_i\cdot s_0 \quad \text{for} \quad i\geq 2.$$

More precisely, $W = S_n \ltimes \mathbb{Z}_2^n$ where S_n acts on $\mathbb{Z}_2^n$ by permutations. Writing a typical element of W as $w=(\sigma,\tau)$ with $\sigma\in S_n$ and $\tau\in\mathbb{Z}_2^n$, we have for $w'=(\sigma',\tau')$,

$$w\cdot w' = (\sigma\sigma',\delta),$$

where $\delta_i = \tau_{\sigma'(i)}\cdot\tau'_i$. To represent elements of W we use the standard "barred-permutation" notation, writing them as permutations with bars in those places (numbered with "i") where $\tau_i=-1$. We have $s_0=[\overline{1},2,\ldots,n]$, and the right multiplication by s_i is given by

$$[w_1,w_2,\ldots,w_n]\cdot s_0 = [\overline{w}_1,w_2,\ldots,w_n]$$

$$\text{and}\quad [w_1,w_2,\ldots,w_n]\cdot s_i = [w_1,\ldots,w_{i-1},w_{i+1},w_i,w_{i+2},\ldots,w_n].$$

Let $\omega=\omega^{(n)}, w_0=w_0^{(n)}$ denote respectively the elements of maximal length of S_n and W. In addition, we set

$$v_0 = v_0^{(n)} = w_0\cdot\omega = \omega\cdot w_0.$$

In the above barred permutation notation,

$$\omega=[n,n-1,\ldots,1], \quad w_0=[\overline{1},\ldots,\overline{n}], \quad v_0=[\overline{n},\ldots,\overline{1}].$$

Let $X=X_n=(x_1,\ldots,x_n)$ be a sequence of commuting independent variables. The group algebra of W acts on the ring $\mathbb{Z}[X]$ of polynomials in X: s_i exchanges x_i and x_{i+1}, $1\leq i\leq n-1$, and s_0 replaces x_1 by $-x_1$ (all remaining variables being unchanged).

In the present paper we shall make an extensive use of the algebra of divided differences, with generators $\partial_0,\partial_1,\ldots,\partial_{n-1}$ acting on the ring $\mathbb{Z}[X]$ according to

$$\partial_i(f) = (f-s_i f)/(x_i-x_{i+1}), \quad i\geq 1$$

$$\text{and} \quad \partial_0 = (f - s_0 f)/2x_1.$$

These generators satisfy the standard braid relations for the symmetric group S_n and the braid relations (1). Consequently, for every $w \in W$, there exists a well-defined divided difference $\partial_w = \partial_{i_1} \circ \ldots \circ \partial_{i_l}$ which can be obtained from any reduced decomposition $s_{i_1} \cdot \ldots \cdot s_{i_l}$ of w (see [B-G-G] and [D1,2]). In particular,

$$\partial_{v_0} = \partial_0(\partial_1\partial_0)\ldots(\partial_{n-1}\ldots\partial_1\partial_0) = (\partial_0\partial_1\ldots\partial_{n-1})\ldots(\partial_0\partial_1)\partial_0 \tag{2}$$

and

$$\partial_{w_0} = \partial_{v_0} \circ \partial_\omega = \partial_\omega \circ \partial_{v_0}. \tag{3}$$

Recall (see e.g. [Mcd2]) that the operator ∂_ω coincides with the Jacobi symmetrizer

$$f \mapsto \sum_{\mu \in S_n} \mu[f/\vartriangle(X)],$$

where $\vartriangle(X) = \prod_{i<j\le n}(x_i - x_j)$.

On the other hand, the operator ∂_{w_0} (that appears in the Weyl character formula for type C_n) coincides with

$$f \mapsto (-1)^{n(n-1)/2} \sum_{w \in W} w[f/\square(X)],$$

where

$$\square(X) = \vartriangle(X) \prod_{i \le j \le n}(x_i + x_j) = 2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j\le n}(x_i^2 - x_j^2).$$

This follows, for example, from [P-R2, Proposition 5.5] by noting that s_0 appears n times in the reduced decomposition

$$w_0 = s_0 \cdot (s_1 \cdot s_0 \cdot s_1) \cdot \ldots \cdot (s_{n-1} \cdot \ldots \cdot s_2 \cdot s_1 \cdot s_0 \cdot s_1 \cdot s_2 \cdot \ldots \cdot s_{n-1}).$$

Observe that these expressions of ∂_ω and ∂_{w_0} imply that for every symmetric (in X) polynomial f, the operator ∂_{v_0} acts as

$$f \mapsto (-1)^{n(n-1)/2} \sum_{w \in W/S_n} w\Big[f \Big/ \prod_{i \le j \le n}(x_i + x_j)\Big]. \tag{4}$$

(To see this, recall that the Jacobi symmetrizer is a surjection onto the ring of symmetric polynomials in X.)

Let now T_i, $i = 1, \ldots, n$, be the operator on $\mathbb{Z}[X]$ acting only on x_i by $T_i(x_i) = -x_i$. Note that T_i is conjugate to s_0:

$$T_i = s_{i-1} \circ \ldots \circ s_1 \circ s_0 \circ s_1 \circ \ldots \circ s_{i-1}.$$

With the help of the T_i's, the operator ∂_{v_0} has a third expression which respects the symmetry between the variables.

Lemma 1.1. *One has the following equalities of operators on $\mathbb{Z}[X]$:*

$$\sum_{w\in W}(-1)^{l(w)}w = \sum_{\mu\in S_n}(-1)^{l(\mu)}\mu\ (1-T_n)\dots(1-T_1)$$
$$= (1-T_n)\dots(1-T_1)\sum_{\mu\in S_n}(-1)^{l(\mu)}\mu.$$

Indeed, these equalities follow immediately from $W = S_n \ltimes \mathbb{Z}_2^n$ invoking the following formula for the length $l(w)$ of $w=(\sigma,\tau)\in W$:

$$l(w) = \sum_{i=1}^{n} a_i + \sum_{j,\tau_j=-1}(2b_j+1), \tag{5}$$

where $a_i = \operatorname{card}\{p \mid p > i \ \&\ \sigma_p < \sigma_i\}$ and $b_j = \operatorname{card}\{p \mid p < j \ \&\ \sigma_p < \sigma_j\}$. The equality (5) can be proved by an easy induction on $l(w)$.

To simplify signs in many formulas of the present paper, we shall also use $\partial_i' = -\partial_i$ $(1 \le i \le n-1)$. For $w \in W$, by ∂_w' we will denote the divided difference operator where we use the ∂_i''s instead of the ∂_i's. Finally, instead of ∂_{v_0}, we will use

(6)
$$\nabla := \partial_0(\partial_1'\partial_0)\dots(\partial_{n-1}'\dots\partial_1'\partial_0) = (\partial_0\partial_1'\dots\partial_{n-1}')\dots(\partial_0\partial_1')\partial_0 = (-1)^{n(n-1)/2}\partial_{v_0}.$$

With the help of the T_i's, the restriction of ∂_{v_0} (or ∇) to symmetric polynomials has a simple expression:

Lemma 1.2. *For any symmetric polynomial f in X,*

$$\partial_{v_0}(f) = \frac{1}{\square(X)}(1-T_n)\dots(1-T_1)(f\cdot\triangle(X))$$
$$= \frac{1}{2x_n}(1-T_n)\dots\frac{1}{2x_1}(1-T_1)\big(f\big/\prod_{i<j}(x_i+x_j)\big).$$

Indeed, let us first record

$$\Big(\sum_{\mu\in S_n}(-1)^{l(\mu)}\mu\Big)\triangle(X) = n!\triangle(X). \tag{7}$$

Since f is symmetric, one has by (7)

$$f = \frac{1}{n!}\partial_\omega(\triangle(X)\cdot f),$$

and thus

$$\partial_{v_0}(f) = \frac{1}{n!}\partial_{w_0}(\triangle(X)\cdot f)$$
$$= \frac{1}{\square(X)}(1-T_n)\dots(1-T_1)(f\cdot\triangle(X))$$

by Lemma 1.1 and the equality (7) again.

The second equality is a restatement of the first one because x_i^2 are scalars w.r.t. T_j.

Let $\mathcal{SP}(X)$ denote the ring of symmetric polynomials in X. It follows from Lemma 1.2 (e.g. by remarking that $(1-T_i)^2 = 0$) that the operator ∇ sends $\mathcal{SP}(X)$ to $\mathcal{SP}(X^2)$ where $X^2 := (x_1^2, \dots, x_n^2)$. More explicitly, the image of the $\mathbb{Z}$-linear basis of Schur polynomials is given by the next proposition (see [P-R, Theorem 5.13]).

We shall use the following notation. Given a sequence $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will denote by x^α the monomial $x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Given two sequences $\alpha, \beta \in \mathbb{N}^n$, we shall denote by $\alpha + \beta$ the sequence $(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$. Moreover, set $\rho_k = (k, k-1, \dots, 2, 1)$.

Proposition 1.3. *The image of an S-polynomial under ∇, $\nabla\big(s_I(X)\big)$, is nonzero only if the partition I is of the form $\rho_n + 2J$ for some partition J. In this case,*

$$\nabla\big(s_{\rho_n+2J}(X)\big) = \psi_2\big(s_J(X)\big),$$

where $\psi_2 : \mathbb{Z}[X] \to \mathbb{Z}[X^2]$ is the ring homomorphism defined by $\psi_2(x_i) = x_i^2$, $i = 1, \dots, n$.

Proof. Since

$$s_I(X) = \partial_\omega(x^{I+\rho_{n-1}}),$$

we have

$$\nabla(s_I(X)) = \partial_{w_0}(x^{I+\rho_{n-1}}).$$

Using the first equality in Lemma 1.1, we see that $\nabla(s_I(X)) \neq 0$ only if all the parts of $I + \rho_{n-1}$ are odd. This holds iff $l(I) = n$ and I is strict, or equivalently, iff $I = I' + \rho_n$ for some partition I'. But $I' + \rho_n + \rho_{n-1}$ has all the parts odd iff $I' = 2J$ for some partition J.

Since we have

$$(1-T_n)\dots(1-T_1)(x^{2J+\rho_n+\rho_{n-1}}) = 2^n x^{2J+\rho_n+\rho_{n-1}},$$

we infer finally that

$$\begin{aligned}\nabla(s_{2J+\rho_n}(X)) &= \partial_{w_0}(x^{2J+\rho_n+\rho_{n-1}}) \\ &= 2^n \sum_{\mu\in S_n} \mu[x^{2J+\rho_n+\rho_{n-1}}/\square(X)] \\ &= \sum_{\mu\in S_n} \mu[x^{2J+2\rho_{n-1}}/\vartriangle(X^2)] \\ &= \psi_2(s_J(X)),\end{aligned}$$

as asserted. $\square$

It is, perhaps, in order to mention that D.E. Littlewood used in [Li2] the notions of 2-quotient, 2-core and 2-sign (whose definitions can be found e.g. in loc.cit.,

[Mcd1] and [D-L-T]) to evaluate $\psi_2\big(s_J(X)\big)$. In fact, he computed more generally $\psi_p\big(s_J(X)\big)$, where, for any integer $p \geq 2$, $\psi_p(x_i) = x_i^p$, $i = 1, 2, \dots$ For another computation of $\psi_p\big(s_J(X)\big)$ as a quotient of two S-polynomials see [P-R2, Proposition 5.11].

The expression of ∇ as a sum, with rational-function coefficients, of elements of W, is not totally straightforward, but we shall only need the leading term described by the following lemma.

Lemma 1.4. *Write $\nabla = \sum_{w \in W} a_w w$, where a_w are rational functions in X. Then there is a unique $w = v_0$ of maximal length such that $a_w \neq 0$. Moreover, $a_{v_0} = (-1)^{n(n+1)/2} \prod_{i \leq j \leq n} (x_i + x_j)^{-1}$.*

Indeed, we have, using (6), the following presentation of ∇:

$$\nabla = \frac{1}{2x_1}(1-s_0)\frac{1}{x_2 - x_1}(1-s_1)\dots\frac{1}{x_n - x_{n-1}}(1-s_{n-1})\dots$$
$$\dots\frac{1}{2x_1}(1-s_0)\frac{1}{x_2-x_1}(1-s_1)\frac{1}{2x_1}(1-s_0)$$

which implies, by an easy induction on n, that its term of maximal length equals

$$(-1)^{n(n+1)/2}\frac{1}{2x_1}s_0\frac{1}{x_2-x_1}s_1\dots\frac{1}{x_n-x_{n-1}}s_{n-1}\dots$$
$$\dots\frac{1}{2x_1}s_0\frac{1}{x_2-x_1}s_1\frac{1}{2x_1}s_0$$
$$= (-1)^{n(n+1)/2}\frac{1}{2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j\leq n}(x_i + x_j)}\ v_0,$$

as asserted.

Corollary 1.5. *With the notation of Lemma 1.4, the unique $w = (\sigma, \tau)$ for which* $\operatorname{card}\{j \mid \tau_j = -1\} = n$ *and $a_w \neq 0$ is $w = v_0$.*

Indeed, this follows directly from the previous lemma once we observe that, by (5), v_0 is the element of the smallest length among the barred permutations with n bars.

The following "commutation property" will be frequently used in this article.

Lemma 1.6. *For any $\mu \in S_n$,*

$$\nabla \circ \partial_\mu = \partial_{\omega\mu\omega} \circ \nabla \,.$$

Indeed, recalling that the symmetric group S_n acts on the set of barred permutations from the <u>right</u> by permuting the places, and, from the <u>left</u> by permuting the values, we get the equality

$$[\overline{n}, \overline{n-1}, \dots, \overline{2}, \overline{1}] \cdot s_i = s_{n-i} \cdot [\overline{n}, \overline{n-1}, \dots, \overline{2}, \overline{1}] \,,$$

whence

$$\nabla \circ \partial_i = \partial_{n-i} \circ \nabla$$

which immediately implies the assertion.

All the operators introduced above will be used to describe the ring $\mathbb{Z}[X]$ as an $\mathcal{SP}(X^2)$-module. First we consider $\mathbb{Z}[X]$ as an $\mathcal{SP}(X)$-module.

The ring $\mathbb{Z}[X]$ is a free $\mathcal{SP}(X)$-module with a basis provided by Schubert polynomials $\mathfrak{S}_\mu$, $\mu \in S_n$, defined in the following way. For a permutation $\mu \in S_n$ we set

$$\mathfrak{S}_\mu := \mathfrak{S}_\mu(X) := \partial_{\mu^{-1}\omega}(x_1^{n-1} x_2^{n-2} \ldots x_{n-1})$$

- the *Schubert polynomial* associated with μ. Observe that, equivalently, the polynomials $\mathfrak{S}_\mu$ are defined inductively by the equation

$$\partial_i \mathfrak{S}_\mu = \mathfrak{S}_{\mu s_i}$$

if $\mu(i) > \mu(i+1)$, the "top" polynomial being $\mathfrak{S}_\omega = x^{\rho_{n-1}}$. Recall that the Schubert polynomials have the following stability property. Let $S_n \hookrightarrow S_m$ be the embedding of the symmetric groups via the first n components ($n < m$). Then we have for $\mu \in S_n \subset S_m$,

$$\mathfrak{S}_\mu(X_m)|_{x_{n+1}=\ldots=x_m=0} = \mathfrak{S}_\mu(X_n)\,.$$

It follows from this equality that $\mathfrak{S}_\mu$ is a well-defined polynomial for each permutation $\mu \in S_\infty = \bigcup_n S_n$.

On $\mathbb{Z}[X]$ there is a scalar product

$$(\ ,\ ) : \mathbb{Z}[X] \times \mathbb{Z}[X] \to \mathcal{SP}(X),$$

defined for $f, g \in \mathbb{Z}[X]$ by

$$(f, g) = \partial_\omega(f \cdot g)\,.$$

Schubert polynomials are almost orthonormal basis w.r.t. $(\ ,\ )$. Given $w \in W$, we set $X^w = (w(x_1), \ldots, w(x_n))$ and $\overline{X} = X^{w_0} = (-x_1, \ldots, -x_n)$. In particular, $X^\omega = (x_n, \ldots, x_2, x_1)$ and $\overline{X}^\omega = (-x_n, \ldots, -x_1)$.

One has for any $\mu, \nu \in S_n$,

$$\big(\mathfrak{S}_\mu(X), \mathfrak{S}_{\nu\omega}(\overline{X}^\omega)\big) = \delta_{\mu,\nu}$$

(the Kronecker delta). For more on Schubert polynomials see [Mcd2] and the references therein.

Now we will describe the ring $\mathcal{SP}(X)$ as an $\mathcal{SP}(X^2)$–module. As is well-known, $\mathcal{SP}(X) \otimes \mathbb{Q}$ is a free $\mathcal{SP}(X^2) \otimes \mathbb{Q}$–module with a basis provided by Schur's Q-polynomials $\{Q_I(X)\}$, where strict $I \subset \rho_n$. In this article, we want a basis over $\mathbb{Z}$, and moreover geometry imposes the need to take a basis better adapted to the scalar product $\langle\ ,\ \rangle$ defined below. Therefore, we shall use the following basis of

$\widetilde{Q}$-polynomials modelled on Schur's Q-polynomials. The reader can find in [P-R2], where these polynomials were extensively studied, another geometric motivation of their choice. We set $\widetilde{Q}_i := \widetilde{Q}_i(X) := e_i(X)$ - the i-th elementary symmetric polynomial in X. Given two nonnegative integers i, j, we define

$$\widetilde{Q}_{i,j} := \widetilde{Q}_{i,j}(X) = \widetilde{Q}_i \cdot \widetilde{Q}_j + 2\sum_{p=1}^{j}(-1)^p \widetilde{Q}_{i+p} \cdot \widetilde{Q}_{j-p}\,.$$

Finally, for any partition $I = (i_1 \geq i_2 \geq \ldots \geq i_{l(I)} > 0)$, the polynomial $\widetilde{Q}_I = \widetilde{Q}_I(X)$ is defined recurrently on $l = l(I)$ by putting for odd l,

$$\widetilde{Q}_I := \widetilde{Q}_I(X) = \sum_{j=1}^{l}(-1)^{j-1}\widetilde{Q}_{i_j} \cdot \widetilde{Q}_{(i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_l)}$$

and for even l,

$$\widetilde{Q}_I := \widetilde{Q}_I(X) = \sum_{j=2}^{l}(-1)^{j}\widetilde{Q}_{i_1,i_j} \cdot \widetilde{Q}_{(i_2,\ldots,i_{j-1},i_{j+1},\ldots,i_l)}\,.$$

The ring $\mathcal{SP}(X)$ is a free $\mathcal{SP}(X^2)$-module with a basis provided by $\widetilde{Q}$-polynomials $\widetilde{Q}_I(X)$, where strict $I \subset \rho_n$. The polynomial $\widetilde{Q}_I(X_n)$ can be defined in another recursive way as follows. For any strict partition I one has

$$\widetilde{Q}_I(X_n) = \sum_{j=0}^{l(I)} x_n^j \Big(\sum_{|I|-|J|=j} \widetilde{Q}_J(X_{n-1})\Big), \tag{8}$$

where the sum is over all (i.e. not necessary strict) partitions $J \subset I$ such that I/J has at most one box in every row (see [P-R2, Proposition 4.1]). Moreover, given a strict partition $I' = (i_1, \ldots, j, j, \ldots, i_k)$ and denoting $I = (i_1, \ldots, i_k)$, one has the factorization property

$$\widetilde{Q}_{I'} = \widetilde{Q}_{j,j} \cdot \widetilde{Q}_I. \tag{9}$$

The following scalar product $\langle\ ,\ \rangle : \mathcal{SP}(X) \times \mathcal{SP}(X) \to \mathcal{SP}(X^2)$ defined by $\langle f, g\rangle = \nabla(f \cdot g)$, will be of particular importance for the present paper. We have, for strict $I, J \subset \rho_n$,

$$\langle \widetilde{Q}_I(X), \widetilde{Q}_{\rho_n \smallsetminus J}(X)\rangle = \delta_{I,J}\,, \tag{10}$$

where $\rho_n \smallsetminus J$ denotes the strict partition whose parts complement the parts of J in $\{n, n-1, \ldots, 2, 1\}$ (see [P-R, Theorem 5.23]).

Denote by

$$\langle\!(\ ,\ )\!\rangle : \mathbb{Z}[X] \times \mathbb{Z}[X] \to \mathcal{SP}(X^2)$$

the scalar product defined for $f,g\in\mathbb{Z}[X]$ by

$$\langle\langle f,g\rangle\rangle=(-1)^{n(n-1)/2}\partial_{w_0}(f\cdot g)\,.$$

The $\mathcal{SP}(X^2)$-basis $\{\mathfrak{S}_\mu\cdot\widetilde{Q}_I\}$ of $\mathbb{Z}[X]$, where $\mu\in S_n$ and $I\subset\rho_n$ are strict, is especially convenient with regard to the scalar product $\langle\langle\ ,\ \rangle\rangle$ because obviously

$$\langle\langle \mathfrak{S}_\mu(X)\cdot\widetilde{Q}_I\ ,\ \mathfrak{S}_{\nu\omega}(\overline{X}^{\omega})\cdot\widetilde{Q}_{\rho_n\smallsetminus J}\rangle\rangle=\delta_{\mu,\nu}\delta_{I,J}\,.$$

Note, however, that the elements of this basis are not representatives of Schubert cycles in the manifold of complete Lagrangian flags, but form a natural "product-basis" associated with a fibration of this variety over the Lagrangian Grassmannian of maximal isotropic subspaces, with the "usual" complete flag variety as its fiber (for more about that, see [P-R2, Sect.5]).

The main theme of the present paper is to study the action of different operators on this basis. Among the operators in question the following divided difference operators will be of particular importance:

$$\nabla_k=(\partial'_{n-k}\dots\partial'_1\partial_0)\dots(\partial'_{n-2}\dots\partial'_1\partial_0)(\partial'_{n-1}\dots\partial'_1\partial_0)\tag{11}$$

(thus $\nabla=\nabla_n$).

As a consequence of the main results of the present paper, we will obtain a purely algebraic proof of the next proposition not invoking the characteristic map and the result from [P] quoted below. For details, see Corollary A.8 in Appendix A. Since, however, we will need this result in Section 4, we record it now with a proof which uses proprieties available in already published literature.

Proposition 1.7. *For each strict partition $I\subset\rho_n$ there exists a divided difference operator (in the ∂'_i's and ∂_0) $\partial^I:\mathbb{Z}[X]\to\mathbb{Z}[X]$ such that $\partial^I(\widetilde{Q}_I)=1$ and such that, for any strict $J\neq I$ with $|J|=|I|$, $\partial^I(\widetilde{Q}_J)=0$.*

(Note that from the first assertion it follows that $\partial^I=\partial'_w\circ\partial_0$ for some w.)

Proof. This proof uses the Borel characteristic map [B] and the Giambelli-type formula for Lagrangian Grassmannians from [P]. Recall (see [P]) that Schubert classes $\sigma(I)$ in the Lagrangian Grassmannian G of isotropic n-subspaces in $\mathbb{C}^{2n}$ equipped with a nondegenerate symplectic form, are indexed by strict partitions $I\subset\rho_n$. There exists a ring homomorphism (the characteristic map)

$$c:\mathcal{SP}(X)\longrightarrow A(G)$$

such that for a homogeneous symmetric polynomial f,

$$c(f)=\sum_{I,|I|=\deg f}\partial'_{w_I}(f)\sigma(I),$$

where $w_I\in W$ is the barred permutation

$$w_I=[\overline{i_1}>\ldots>\overline{i_l},j_1<\ldots<j_{n-l}]\tag{12}$$

(for details, see [B], [B-G-G] and [D1,2]). Now [P, Theorem 6.17] says that

$$c(\widetilde{Q}_I)=\sigma(I)\ .$$

This implies that $\partial^I:=\partial'_{w_I}$ fulfills the assertion of the proposition. $\square$

2. Reproducing kernels and a vanishing property.

Let $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n)$ be two sequences of independent variables (we fix n).

In the following, by $\equiv$ we will understand the congruence in the ring $\mathcal{SP}(X) \otimes \mathcal{SP}(Y)$ modulo the ideal generated by all $f(X^2) - f(Y^2)$, where f symmetrical and X^2 stands for $(x_1^2, \dots, x_n^2)$. We define the following polynomials in $\mathcal{SP}(X) \otimes \mathcal{SP}(Y)$ (the definition of $F(X,Y)$ is inspired by [F1,2] and the one of $\widetilde{Q}(X,Y)$ - by [P-R2]).

$$F(X,Y) := \ |e_{n+1+j-2i}X + e_{n+1+j-2i}Y|_{1\le i,j\le n}$$

$$= \begin{vmatrix} e_nX + e_nY & 0 & \dots & \\ e_{n-2}X + e_{n-2}Y & e_{n-1}X + e_{n-1}Y & \dots & \\ \vdots & \vdots & \ddots & \\ & & 2 & e_1X + e_1Y \end{vmatrix},$$

and

$$\widetilde{Q}(X,Y) := \sum \widetilde{Q}_I X \cdot \widetilde{Q}_{\rho_n \smallsetminus I} Y, \tag{13}$$

where the summation is over all strict partitions $I \subset \rho_n$.

The following results will give us some reproducing kernels for the scalar product

$$\langle \ , \ \rangle : \mathbb{Z}[X] \times \mathbb{Z}[X] \longrightarrow \mathcal{SP}(X^2)$$

defined in the previous section.

Proposition 2.1. *For any $f \in \mathcal{SP}(X)$,*

$$\langle f(X), \widetilde{Q}(X,Y) \rangle \equiv f(Y)\,.$$

Proof. Since we are working modulo $\equiv$ which allow us to exchange $\mathcal{SP}(X^2)$-coefficients to the corresponding $\mathcal{SP}(Y^2)$-coefficients, it suffices to check the equality:

$$\langle \widetilde{Q}_I(X), \widetilde{Q}(X,Y) \rangle = \widetilde{Q}_I(Y)\,,$$

where $I \subset \rho_n$ is a strict partition. This equality follows immediately from the orthogonality of the $\widetilde{Q}_I$'s w.r.t. $\langle \ , \ \rangle$ recalled in (10). $\square$

To prove a similar property for $F(X,Y)$, we need the following result.

Proposition 2.2. *(i) For $w \in S_n \subset W_n$,*

$$F(X, X^w) = F(X,X) = 2^n s_{\rho_n}(X)\,.$$

(ii) For $w \in W_n \setminus S_n$,

$$F(X, X^w) = 0\,.$$

Proof. (i) This is obvious from a familiar expression for s_{ρ_n} as a determinant in elementary symmetric functions. (Note that we use that the degree-zero-entries in the matrix defining F are equal to 2.)
(ii) Let $w \in W_n \setminus S_n$. If w has odd number of bars, then the (1,1)-entry $e_n X + e_n Y$ becomes zero for $Y = X^w$ and thus $F(X, X^w) = 0$. Hence we can assume that the number of bars is $2k$ where $k > 0$. Since, without loss of generality, we can renumber the variables, we may assume that

$$w = [1, 2, \ldots, n-2k, \overline{n-2k+1}, \ldots, \overline{n}].$$

Let $X' = (x_1, \ldots, x_{n-2k})$ and $X'' = (x_{n-2k+1}, x_{n-2k+2}, \ldots, x_n)$. Then specializing $Y = X^w$, $e_i X + e_i Y$ becomes

$$2 \sum_{j \geq 0} e_{i-2j}(X') \cdot e_{2j}(X'')\,.$$

We claim that if we replace the entries of the matrix of $F(X,Y)$ by putting $\sum_{j\geq 0} e_{i-2j}(X')e_{2j}(X'')$ instead of $e_i X + e_i Y$, then the determinant of the so-obtained matrix M is zero. This can be seen, e.g., as follows. Consider the $n \times n$-matrix $[e_{n+1+j-2i}]$, where e_i are indeterminates. If we substitute $e_i := e_i(X')$, then the determinant of this specialized matrix N vanishes; indeed, $s_{\rho_n}(X') = 0$ because $\mathrm{card}(X') < n$. But using some elementary operations on rows of N, we arrive at M . More precisely, this is done as follows:

- add $e_{2l-2}(X'')$ times the l-th row to the top row (all $l \geq 1$);
  this changes the top row of N to the top row of M .
- add $e_{2l-4}(X'')$ times the l-th row to the second row (all $l \geq 2$);
  this changes the second row of N to the second row of M .

Continuing down the rows, N is changed to M . The assertion follows. □

We will refer to the assertions of the lemma as a "vanishing property".

Proposition 2.3. *For any $f \in \mathcal{SP}(X)$,*

$$\langle f(X), F(X,Y) \rangle \equiv f(Y)\,.$$

Proof. Observe that $\langle f(X), F(X,Y)\rangle$ is a combination of symmetric polynomials in Y with coefficients from $\mathcal{SP}(X^2)$:

$$\langle f(X), F(X,Y)\rangle = \sum_I c_I(X^2) \cdot s_I(Y),$$

where I runs over partitions and $c_I(X^2) \in \mathcal{SP}(X^2)$. By (4) we have

$$\langle f(X), F(X,Y)\rangle = \sum_{w\in W_n/S_n} f(X^w)\cdot F(X^w,Y)/2^n s_{\rho_n}(X^w)\,.$$

The assertion to be proved says: by replacing in $\langle f(X), F(X,Y)\rangle$ the coefficients $c_I(X^2)$ by the polynomials $c_I(Y^2)$ from $\mathcal{SP}(Y^2)$, we get $f(Y)$. Hence, it suffices to check that the following specialized polynomial:

$$\langle f(X), F(X,Y)\rangle|_{Y=X} = \sum_{w\in W_n/S_n} f(X^w)\cdot F(X^w,X)/2^n s_{\rho_n}(X^w)$$

is equal to $f(X)$. But we know by Proposition 2.2 that only one summand on the right-hand side survives, giving us $f(X)$, as desired. $\square$

The content of the next proposition provides an algebraic link between the formulas of [F1,2] and [P-R2].

Proposition 2.4. *:* $F(X,Y) \equiv \widetilde{Q}(X,Y)\,.$

Proof. We can write

$$F(X,Y) \equiv \sum f_I(Y)\widetilde{Q}_I(X)\,,$$

where the sum is over strict partitions $I \subset \rho_n$ and $f_I(Y) \in \mathcal{SP}(Y)$. By Proposition 2.3 and the orthogonality (10) of the $\widetilde{Q}_I(X)$'s, we get for strict $I \subset \rho_n$

$$\begin{aligned}\widetilde{Q}_I(Y) &\equiv \langle \widetilde{Q}_I(X), F(X,Y)\rangle\\ &\equiv \langle \widetilde{Q}_I(X), \sum f_J(Y)\widetilde{Q}_J(X)\rangle\\ &= f_{\rho_n \smallsetminus J}(Y).\end{aligned}$$

This implies $f_I(Y) \equiv \widetilde{Q}_{\rho_n\smallsetminus I}(Y)$. Consequently,

$$F(X,Y) \equiv \sum_{\text{strict } I\subset\rho_n} \widetilde{Q}_{\rho_n\smallsetminus I}(Y)\widetilde{Q}_I(X) = \widetilde{Q}(X,Y)\,,$$

as asserted. $\square$

Remark 2.5. This result was known earlier from comparison of geometric formulas of [F1,2] and [P-R2]. Here, we have given a purely algebraic proof of it.

Example 2.6. For $n=2,3$ $F(X,Y) = \widetilde{Q}(X,Y)$. For $n=4$,
$\widetilde{Q}(X,Y) - F(X,Y) = (x_1^2+x_2^2+x_3^2+x_4^2-y_1^2-y_2^2-y_3^2-y_4^2)(x_1^2x_2^2x_3^2x_4^2-y_1^2y_2^2y_3^2y_4^2)\,.$

Finally, observe that Proposition 2.4 (combined with Proposition 2.2) implies that $\widetilde{Q}(X,Y)$ has also the vanishing property. The following variant of it will be used in Section 4.

Lemma 2.7. *(i) For any* $w \in S_n$ *(and* $w_0 = [\overline{1},\overline{2},\dots,\overline{n}]$ *),*

$$\widetilde{Q}(X^{w_0 w},\overline{X}) = 2^n s_{\rho_n}(\overline{X})\,.$$

(ii) For $w \in W_n \smallsetminus w_0 S_n$ *,*

$$\widetilde{Q}(X^w,\overline{X}) = 0\,.$$

3. Action of ∇ on the basis $\{\mathfrak{S}_\mu \cdot \widetilde{Q}_I\}$ - an inductive approach.

In this section we give an elementary inductive proof of the following

Theorem 3.1. *(i) For $\mu \in S_n$, writing $X = X_n$ and $\omega = \omega^{(n)}$, one has*

$$\nabla\big(\mathfrak{S}_\mu(X)\cdot \widetilde{Q}_{\rho_n}(X)\big) = (-1)^{l(\mu)}\mathfrak{S}_\mu(X^\omega) = \mathfrak{S}_\mu(\overline{X}^\omega).$$

(ii) For strict $I \subsetneq \rho_n$, $\mu \in S_n$, $\nabla\big(\mathfrak{S}_\mu(X)\cdot \widetilde{Q}_I(X)\big) = 0$.

We give first an outline of the proof. For the need of induction we write

$$\nabla^{(k)} = (\partial_0\partial'_1 \dots \partial'_{k-1})(\partial_0\partial'_1 \dots \partial'_{k-2}) \ \dots \ (\partial_0\partial'_1)\partial_0 .$$

We have

$$\nabla = \nabla^{(n)} = \partial_0\partial'_1 \dots \partial'_{n-1}\nabla^{(n-1)}. \tag{14}$$

We first show that it suffices to prove assertions (i) and (ii) for $\mu = \omega$. Then, we prove (i) and (ii) for $\mu = \omega$ by a simultaneous induction on n using (14). In the course of the proof of (i), we use the linearity formula (8) for $\widetilde{Q}_I$ and

$$\partial_1\partial_2 \dots \partial_{n-1}(x_1x_2^2 \dots x_{n-1}^{n-1}) = x_3x_4^2 \dots x_n^{n-2}.$$

In the course of the proof of (ii), we use the linearity formula for $\widetilde{Q}_I$ and the vanishing property:

$$\partial_0\partial_1 \dots \partial_{n-1}(x_n^j x_{n-1}^{n-2}x_{n-2}^{n-3} \ \dots \ x_2) = 0$$

for $0 \le j \le n-1$.

We now pass to a detailed proof.

Lemma 3.2. *It suffices to verify assertions (i) and (ii) of the theorem for $\mu = \omega$.*

Proof. Denote $v = v_0$; so that $\nabla = \partial'_v$ and $\mathfrak{S}_\mu = \mathfrak{S}_\mu(X_n)$, $\widetilde{Q}_I = \widetilde{Q}_I(X_n)$ for brevity. Since $\widetilde{Q}_I$ is symmetric in X, we have

$$\begin{aligned}
\nabla(\mathfrak{S}_\mu\widetilde{Q}_I) &= (-1)^{n(n-1)/2}\partial_v(\mathfrak{S}_\mu\widetilde{Q}_I) \\
&= \pm\partial_{v\mu^{-1}\omega}(x^{\rho_{n-1}}\widetilde{Q}_I) \\
&= \pm\partial_{v\omega\nu^{-1}\omega\omega}(x^{\rho_{n-1}}\widetilde{Q}_I) && \text{(we substitute} \quad \mu = \omega\nu\omega) \\
&= \pm\partial_{\nu^{-1}\omega v}(x^{\rho_{n-1}}\widetilde{Q}_I) && (v\omega = \omega v \text{ is the element of biggest length} \\
& && \text{in } W \text{ and commutes with any element of it)} \\
&= \partial_{\nu^{-1}\omega}\nabla(x^{\rho_{n-1}}\widetilde{Q}_I)
\end{aligned}$$

$$= \begin{cases} (-1)^{n(n-1)/2}\partial_{\nu^{-1}\omega}\big(\omega(x^{\rho_{n-1}})\big) & \text{for } I=\rho_n \text{ by the case } \mu=\omega \\ 0 & \text{for } I\neq\rho_n \text{ by the case } \mu=\omega \end{cases}$$

$$= \begin{cases} (-1)^{l(\mu)}\mathfrak{S}_\mu(X^\omega) & \text{for } I=\rho_n \\ 0 & \text{for } I\neq\rho_n. \end{cases}$$

□

In the proof of the next proposition we will need the following lemma.

Lemma 3.3. *:* $\partial_1\partial_2\ldots\partial_{n-1}(x_1x_2^2\ldots x_{n-1}^{n-1}) = x_3x_4^2\ldots x_n^{n-2}$.

Proof. We show by descending induction on i ($i=n,n-1,\ldots,1$) that after applying ∂_i and before applying ∂_{i-1} the resulting sum has the properties:
1) it contains a unique summand without x_i, namely

$$x_1x_2^2\ldots x_{i-1}^{i-1}x_{i+1}^{i-1}x_{i+2}^{i}\ldots x_n^{n-2}$$

2) all but one summands are monomials of the form

$$x_1x_2^2\ldots x_{i-1}^{i-1}x_i^a x_{i+1}^{\bullet}\ \ldots\ ,$$

where $0<a\le i-1$.

The beginning of the induction is trivial: before applying ∂_{n-1}, we have the unique summand $x_1x_2^2\ldots x_{n-1}^{n-1}$. All monomials in the image of 2) under ∂_{i-1} contain x_{i-1}, so the unique summand without x_{i-1} in the image of ∂_{i-1} stems by applying ∂_{i-1} to 1) and equals

$$x_1x_2^2\ldots x_{i-2}^{i-2}x_i^{i-2}x_{i+1}^{i-1}\ldots x_n^{n-2}.$$

All other monomials in the image of 1) are of the form:

$$x_1x_2^2\ldots x_{i-2}^{i-2}x_{i-1}^a x_i^{\bullet}\ \ldots\ ,$$

where $0<a\le i-2$. This is also the form of all monomials in the image of 2) under ∂_{i-1}, and the inductive step follows. Consequently, after applying ∂_2 and before applying ∂_1, all but one summands are of the form: $x_1x_2x_3^{\bullet}\ldots$, and they go to zero under ∂_1. The unique summand not containing x_2 is

$$x_1x_3x_4^2\ \ldots\ x_n^{n-2},$$

and it goes under ∂_1 to $x_3x_4^2\ldots x_n^{n-2}$. This proves the lemma. □

Proposition 3.4. : $\nabla\big(x^{\rho_{n-1}}\cdot\widetilde{Q}_{\rho_n}(X_n)\big)=(-1)^{n(n-1)/2}\omega(x^{\rho_{n-1}})$.

Proof. We have

$$\nabla\big(x^{\rho_{n-1}}\widetilde{Q}_{\rho_n}(X_n)\big)=\nabla^{(n)}\big(x^{\rho_{n-2}}\widetilde{Q}_{\rho_{n-1}}(X_n)x_1^2\dots x_{n-1}^2x_n\big)$$

$$=\partial_0\partial'_1\dots\partial'_{n-1}\nabla^{(n-1)}\big(x^{\rho_{n-2}}\widetilde{Q}_{\rho_{n-1}}(X_{n-1}+x_n)x_1^2\dots x_{n-1}^2x_n\big)\,.$$

We apply the linearity formula (8) to $\widetilde{Q}_{\rho_{n-1}}(X_{n-1}+x_n)$. All summands different from $\widetilde{Q}_{\rho_{n-1}}(X_{n-1})$ are of the form $\widetilde{Q}_J(X_{n-1})x_n^\bullet$ where $J\subsetneq\rho_{n-1}$. After applying to J the factorization property (9) (i.e. factoring out $\widetilde{Q}_{pp}(X_{n-1})$ corresponding to all pairs of equal rows), as well as factoring out $x_n^\bullet$, we apply $\nabla^{(n-1)}$ to $x^{\rho_{n-2}}\widetilde{Q}_{J'}(X_{n-1})$ where $J'\subsetneq\rho_{n-1}$ is the resulting strict partition. We get the vanishing by the induction assumption (ii) for $n-1$. Thus the above expression is equal to

$$\begin{aligned}&\partial_0\partial'_1\dots\partial'_{n-1}\nabla^{(n-1)}\big(x^{\rho_{n-2}}\widetilde{Q}_{\rho_{n-1}}(X_{n-1})x_1^2\dots x_{n-1}^2x_n\big)\\&\quad=\partial_0\partial'_1\dots\partial'_{n-1}\Big(x_1^2\dots x_{n-1}^2x_n\nabla^{(n-1)}\big(x^{\rho_{n-2}}\widetilde{Q}_{\rho_{n-1}}(X_{n-1})\big)\Big)\\&\quad=\partial_0\partial'_1\dots\partial'_{n-1}\big(x_1^2\dots x_{n-1}^2x_n(x_2x_3^2\dots x_{n-1}^{n-2}\cdot x_1\dots x_{n-1})\big)\\&\quad=(-1)^{(n-1)(n-2)/2}\partial_0\big(x_1\dots x_n\ \partial'_1\dots\partial'_{n-1}(x_1x_2^2\dots x_{n-1}^{n-1})\big)\end{aligned}$$

by the induction assumption (i) for $n-1$.

Now, using Lemma 3.3, the above expression is rewritten as $(-1)^{n(n-1)/2}$ times

$$\partial_0(x_1x_2x_3^2\dots x_n^{n-1})=x_2x_3^2\dots x_n^{n-1}=\omega(x^{\rho_{n-1}}),$$

and the proof of the proposition is complete. □

Proposition 3.5. : $\nabla\big(x^{\rho_{n-1}}\cdot\widetilde{Q}_I(X_n)\big)=0$ *for strict* $I\subset\rho_n$.

Proof. We apply the linearity formula (8) to $\widetilde{Q}_I(X_n)=\widetilde{Q}_I(X_{n-1}+x_n)$. Consider a typical summand $\widetilde{Q}_J(X_{n-1})\cdot x_n^\bullet$. Recall that J is a partition (not necessary strict) obtained from I by subtracting boxes from I, but no more then two from one row. Moreover, all parts of J are less than or equal to $n-1$. By dividing $x^{\rho_{n-1}}$ by $x_1\cdot\ldots\cdot x_{n-1}$ and multiplying $\widetilde{Q}_J(X_{n-1})\cdot x_n^\bullet$ by $x_1\cdot\ldots\cdot x_{n-1}$, we get the summand

$$x^{\rho_{n-2}}\widetilde{Q}_{(n-1)J}(X_{n-1})\cdot x_n^\bullet\,,$$

whose factor depending on X_{n-1} is an element of the basis associated with X_{n-1}. If $J=(n-1)J'$ then $J'\subsetneq\rho_{n-1}$ ($J'=\rho_{n-1}$ implies $I=\rho_n$, which is impossible). Then using the factorization property (9) to $\widetilde{Q}_{(n-1)J}(X_{n-1})$ and factoring out all possible $\widetilde{Q}_{p,p}(X_{n-1})$ for $p=n-1,\dots,1$, we apply $\nabla^{(n-1)}$ to $x^{\rho_{n-2}}\widetilde{Q}_{J''}(X_{n-1})$ where $J''\subsetneq\rho_{n-1}$ is strict, thus getting the vanishing by the induction assumption (ii) for $n-1$.
Hence we can assume $j_1\le n-2$. Observe that in J no three consecutive rows are equal. Therefore it is impossible to subtract from J a pair of equal rows to get

ρ_{n-2}. Hence the only J which after a possible subtraction of pairs of equal parts and adding the part $(n-1)$, gives us ρ_{n-1} is the partition ρ_{n-2} itself. The unique possibilities for the initial partition I are then

$$I = (n-1, n-2, \ldots, j+2, j, j-1, \ldots, 1) \qquad j = 1, \ldots, n-1.$$

(Pictorially:

$$\begin{array}{ccccccc}
\bullet & & & & & & \\
\bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & * & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & * & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & *
\end{array}$$

where the total number of rows is $n-2$ and the number of rows without "*" equals j. In the above figure, we have $n = 8$ and $j = 3$.) This brings us to the summand

$$x^{\rho_{n-2}} \widetilde{Q}_{\rho_{n-1}}(X_{n-1}) x_n^j \qquad 1 \le j \le n-1.$$

We have, by the induction assumption (i) for $n-1$,

$$\begin{aligned}
\nabla\left(x^{\rho_{n-1}} \widetilde{Q}_{\rho_{n-1}}(X_{n-1}) x_n^j\right) &= \partial_0 \partial_1' \ldots \partial_{n-1}' \left(x_n^j \nabla^{(n-1)}\left(x^{\rho_{n-2}} \widetilde{Q}_{\rho_{n-1}}(X_{n-1})\right)\right) \\
&= \pm \partial_0 \partial_1 \ldots \partial_{n-1}\left(x_n^j x_{n-1}^{n-2} x_{n-2}^{n-3} \ldots x_2\right).
\end{aligned}$$

We show that this expression is zero for $1 \le j \le n-1$ by induction on n. If $j = n$ or $j = n-1$, then

$$\partial_0 \partial_1 \ldots \partial_n \left(x_{n+1}^j x_n^{n-1} x_{n-1}^{n-2} \ldots x_2\right) = 0$$

by direct calculation. Assume $1 \le j \le n-1$. Then ∂_n sends $x_{n+1}^j x_n^{n-1} x_{n-1}^{n-2} \ldots$ to

$$x_{n+1}^j x_n^j (x_n^{n-1-j-1} + x_n^{n-1-j-2} x_{n+1} + \ldots) x_{n-1}^{n-2} \ldots$$

(for $j = n$, the third factor is empty). In every monomial, the variable x_n appears with exponent $\le j + n - 1 - j - 1 = n - 2$. Hence, by the induction assumption, $\partial_0 \partial_1 \ldots \partial_{n-1}$ annihilates this expression. The proposition has been proved. $\square$

This completes the proof of Theorem 3.1.

Corollary 3.6. $\mathrm{Im}(\nabla)$ *is a free* $\mathcal{SP}(X^2)$*-module with a basis* $\{\mathfrak{S}_\mu(X^\omega)\}_{\mu \in S_n}$.

We end this section with the following, related formula.

Proposition 3.7. *For any $\mu \in S_n$, $\nabla\big(\mathfrak{S}_\mu \cdot s_I(X)\big) \neq 0$ iff I is of the form $\rho_n + 2J$. In this case,*

$$\nabla\big(\mathfrak{S}_\mu(X) \cdot s_I(X)\big) = (-1)^{l(\mu)} \mathfrak{S}_\mu(X^\omega) \cdot \psi_2\big(s_J(X)\big)$$

(where ψ_2 is the ring homomorphism sending x_i to x_i^2, $i = 1, \dots, n$).

Proof. It follows from Theorem 3.1 (e.g. by developing $s_I(X)$ in the $\mathcal{SP}(X^2)$-basis of $\widetilde{Q}$-polynomials) that

$$\nabla\big(\mathfrak{S}_\mu(X) \cdot s_I(X)\big) = c \cdot \mathfrak{S}_\mu(X^\omega)$$

for some $c \in \mathcal{SP}(X^2)$. The coefficient c is computed in the following way, using Lemma 1.6:

$$\begin{aligned} c &= \partial'_{\omega\mu\omega} \nabla\big(\mathfrak{S}_\mu(X) \cdot s_I(X)\big) = \nabla \partial'_\mu \big(\mathfrak{S}_\mu(X) \cdot s_I(X)\big) \\ &= (-1)^{l(\mu)} \nabla\big(s_I(X)\big), \end{aligned}$$

and the assertion follows from the formula recalled in Proposition 1.3. □

4. Action of ∇ on the basis $\{\mathfrak{S}_\mu \cdot \widetilde{Q}_I\}$ via the vanishing property.

The goal of this section is to derive the formula of Theorem 3.1 via the vanishing property of Section 2. The starting point of the approach of this section is the determination of the image of $\nabla : \mathbb{Z}[X] \to \mathbb{Z}[X]$.

Proposition 4.1. $\mathrm{Im}(\nabla)$ *is a free $\mathcal{SP}(X^2)$-module with a basis $\{\mathfrak{S}_\mu(X^\omega)\}_{\mu \in S_n}$.*

Proof. We write for brevity: $\widetilde{Q}_I = \widetilde{Q}_I(X)$, $\mathcal{SP} = \mathcal{SP}(X)$ and $\mathcal{SP}^2 = \mathcal{SP}(X^2)$. Knowing that $\mathbb{Z}[X]$ is an $\mathcal{SP}$-module with a basis $\{\mathfrak{S}_\mu(X^\omega)\}_{\mu \in S_n}$ and $\mathcal{SP}$ is an $\mathcal{SP}^2$-module with a basis $\{\widetilde{Q}_I\}_{\text{strict } I \subset \rho_n}$, we can write any element in $\mathrm{Im}(\nabla)$ as a unique expression

$$\sum c_{\mu,I} \cdot \mathfrak{S}_\mu(X^\omega) \cdot \widetilde{Q}_I\,, \tag{15}$$

where $c_{\mu,I} \in \mathcal{SP}^2$ and μ, I are as above. Suppose that there exist ν, J with $|J| \geq 1$ such that $c_{\nu,J} \neq 0$. Let μ be a permutation of biggest length among such ν's. The operator $\partial'_{\omega\mu\omega}$ sends $\mathfrak{S}_\mu(X^\omega)$ to 1 and all $\mathfrak{S}_\nu(X^\omega)$, where $\nu \neq \mu$ and $l(\nu) \leq l(\mu)$, to 0. Applying this operator to (15), we get

$$\sum c_{\mu,J} \cdot \widetilde{Q}_J + \big(\mathcal{SP}^2\text{-combination of the } \mathfrak{S}_\nu(X^\omega)\text{'s}\big). \tag{16}$$

Let I be a partition of biggest weight among those J such that $c_{\mu,J} \neq 0$. Let ∂^I be the divided difference operator (in the ∂'_i's and ∂_0) s.t. $\partial^I(\widetilde{Q}_I) = 1$ (hence $\partial^I = \partial'_w \circ \partial_0$ for some w), and $\partial^I(\widetilde{Q}_J) = 0$ for all strict $J \neq I$, $|J| \leq |I|$ (see

Proposition 1.7). Then the image of (16) under ∂^I is $c_{\mu,I} \neq 0$. (Observe that $\partial^I(\mathfrak{S}_\nu(X^\omega) = 0$ because the $\mathfrak{S}_\nu(X^\omega)$'s do not depend on x_1.)

On the other hand, using $\nabla\partial'_{\omega\mu\omega} = \partial'_\mu\nabla$ (Lemma 1.6), we infer that (16) belongs to $\mathrm{Im}(\nabla)$. Since $\partial_0\nabla = 0$, we have $\partial^I\nabla = 0$, and we get a contradiction with the above calculation.

The proposition has been proved. $\square$

We now give

Another proof of Theorem 3.1.

By the preceding proposition we know that

$$\nabla\big(\mathfrak{S}_\mu(X)\cdot\widetilde{Q}_{\rho_n\smallsetminus I}(X)\big) = \sum c_{I,\nu}\cdot\mathfrak{S}_\nu(X^\omega)\,,$$

where $c_{I,\nu} \in \mathcal{SP}(X^2)$. Invoking the definition (13) of $\widetilde{Q}(X,Y)$, we thus have, by linearity,

$$\begin{aligned}\nabla\big(\mathfrak{S}_\mu(X)\cdot\widetilde{Q}(X,\overline{Y})\big) &= \nabla\big(\mathfrak{S}_\mu(X)\cdot\sum_{\text{strict } I\subset\rho_n}(-1)^{|I|}\widetilde{Q}_{\rho_n\smallsetminus I}(X)\cdot\widetilde{Q}_I(Y)\big)\\ &= \sum_{I,\nu}(-1)^{|I|}c_{I,\nu}\cdot\mathfrak{S}_\nu(X^\omega)\cdot\widetilde{Q}_I(Y).\end{aligned}$$

Specialize now Y in X (that is, we put $y_1 = x_1,\ldots,y_n = x_n$). Since the family $\{\mathfrak{S}_\mu(X^\omega)\cdot\widetilde{Q}_I(X)\}$ is an $\mathcal{SP}(X^2)$-basis, the value of $\nabla\big(\mathfrak{S}_\mu(X)\cdot\widetilde{Q}(X,\overline{Y})\big)$ under this specialization still determines the coefficients $c_{I,\nu}$.

Write $\nabla = \sum_{w\in W} a_w w$, where a_w are rational functions in X. Then

$$\nabla\big(\mathfrak{S}_\mu(X)\cdot\widetilde{Q}(X,\overline{Y})\big) = \sum_{w\in W} a_w\mathfrak{S}_\mu(X^w)\cdot\widetilde{Q}(X^w,\overline{Y})\,,$$

where W acts only on X-variables. It follows, from Corollary 1.5 and Lemma 2.7, that among those w, for which $a_w \neq 0$ there is only one $w = v_0 = [\overline{n},\overline{n-1},\ldots,\overline{1}] = w_0\omega$, for which $\widetilde{Q}(X^w,\overline{X}) \neq 0$; in fact, we have

$$\widetilde{Q}(X^{v_0},\overline{X}) = 2^n s_{\rho_n}(\overline{X}).$$

Since $a_{v_0} = \big(2^n s_{\rho_n}(\overline{X})\big)^{-1}$ by Lemma 1.4, we get finally

$$\begin{aligned}\nabla\big(\mathfrak{S}_\mu(X)\cdot\widetilde{Q}(X,\overline{Y})\big)\,|_{X=Y} &= \mathfrak{S}_\mu(X^{v_0})\\ &= (-1)^{l(\mu)}\mathfrak{S}_\mu(X^\omega).\end{aligned}$$

This implies that the coefficients $c_{I,\nu}$ are as asserted, and the proof is complete. $\square$

5. Key formula and vertex operators

In this section we prove the following identity.

Theorem 5.1 (Key formula). *Suppose $n \geq k > 0$. Let IkJ be a strict partition and H a strict partition not containing k. Then*

$$(-1)^{n-1}\partial_{n-1}\partial_{n-2}\ldots\partial_1\partial_0\big(x_1^{n-k}\widetilde{Q}_{IkJ}(X_n)\big) = (-1)^{l(I)}\widetilde{Q}_{IJ}(X_n)$$

and

$$\partial_{n-1}\partial_{n-2}\ldots\partial_1\partial_0\big(x_1^{n-k}\widetilde{Q}_{H}(X_n)\big) = 0.$$

Our proof of the theorem will use some differential operators (called usually "vertex operators"). The vertex operators that we shall use appeared already in the literature (see e.g. [D-K-M], [D-J-K-M1] and [D-J-K-M2]), but in the context different from the one in the present paper. Our notation and conventions for vertex operators follow rather the exposition of [C-T].

Besides $\widetilde{Q}$-polynomials we will need Schur's P-functions and Q'-functions from [L-L-T2]. (We use the term "function" in the sense of [Mcd1] i.e., a function depends on an infinite sequence of variables $(x_1, x_2, \ldots)$.) Given a symmetric function f, we denote

$$f(X_n) = f_{|x_{n+1}=x_{n+2}=\ldots=0},$$

the corresponding symmetric polynomial in the first n variables.

For the purposes of this paper it suffices to work with Schur P-functions indexed by sequences of positive integers (like in Schur's original paper [S]). Recall that for $J \in \mathbb{N}^{*m}$ with odd m,

$$P_J = P_{j_1} \cdot P_{j_2,\ldots,j_m} - P_{j_2} \cdot P_{j_1,j_3,\ldots,j_m} + \ldots + P_{j_m} \cdot P_{j_1,\ldots,j_{m-1}}$$

and with even m,

$$P_J = P_{j_1,j_2} \cdot P_{j_3,\ldots,j_m} - P_{j_1,j_3} \cdot P_{j_2,j_4,\ldots,j_m} + \ldots + P_{j_1,j_m} \cdot P_{j_2,\ldots,j_{m-1}}\,.$$

Here, $P_j = \sum s_L$, sum over all hook partitions L of j, and

$$P_{i,j} = P_i \cdot P_j + 2\sum_{k=1}^{j-1}(-1)^k P_{i+k}P_{j-k} + (-1)^j P_{i+j}.$$

For another expression of P_J in the form of a quadratic polynomial in the S-functions, see [L-L-T1].

If we restrict ourselves to a finite set of indeterminates X_n, where $n > |J|$, then $P_J(X_n)$ is given by the following symmetrization:

$$P_J(X_n) = \pi\Big(x^J \prod_{\substack{i<j\leq n\\ i\leq m}} \big(1 + \frac{x_j}{x_i}\big)\Big), \tag{17}$$

where π is the symmetrizer defined for $f \in \mathbb{Z}[X_n]$ by

$$\pi(f) = \sum_{w \in S_n} w\big[f\, x^{\rho_{n-1}}/\triangle(X)\big] = \partial_\omega(f \cdot x^{\rho_{n-1}}).$$

Recall that P_J is an antisymmetric function of the components of J; in particular, $P_J = 0$ if two components of J are equal.

Given a symmetric function f, let D_f be the adjoint operator to the multiplication by f w.r.t. the standard scalar product on the ring of symmetric functions (see [Mcd1]).

Proposition 5.2. *Let $k > 0$. Then*

$$P_{kJ} = \sum_{j \geq 0} (-1)^j P_{k+j} \cdot D_{e_j}(P_J).$$

Proof. Suppose that $n >> 0$ (in fact, $n \geq k + |J|$ will do the job). Using (17) and $\pi^2 = \pi$, we get

$$P_{kJ}(X_n) = \pi\Big(x_1^k(1 + \frac{x_2}{x_1}) \dots (1 + \frac{x_n}{x_1}) P_J(x_2, \dots, x_n)\Big).$$

Passing to the λ-ring setup, we have

$$\begin{aligned} P_{kJ}(X_n) &= \frac{1}{2}\pi\Big(x_1^k(1 + \frac{x_1}{x_1})(1 + \frac{x_2}{x_1}) \dots (1 + \frac{x_n}{x_1}) P_J(X_n - x_1)\Big) \\ &= \frac{1}{2}\pi\Big(\big(\sum_{i=0}^{n} x_1^{k-i} e_i(X_n)\big)\big(\sum_{j \geq 0}(-x_1)^j \cdot (D_{e_j}P_J)(X_n)\big)\Big) \\ &= \frac{1}{2}\sum_i \sum_j (-1)^j s_{k+j-i}(X_n) \cdot e_i(X_n) \cdot (D_{e_j}P_J)(X_n) \\ &= \sum_{j \geq 0}(-1)^j P_{k+j}(X_n) \cdot (D_{e_j}P_J)(X_n). \end{aligned}$$

Above, we have used the equalities $\pi(x_1^i) = s_i(X_n)$ for $i \in \mathbb{N}$, and

$$\big(\sum e_i\big) \cdot \big(\sum s_j\big) = 1 + 2\sum P_i.$$

The proposition has been proved. $\square$

By introducing the following differential operator on the ring of symmetric functions

$$V_k^e := P_k - P_{k+1}D_{e_1} + P_{k+2}D_{e_2} - \dots\ ,$$

we can restate the formula in Proposition 5.2 as the equation

$$V_k^e(P_J) = P_{kJ}\ .$$

We also need Q'-functions: for a partition I, Q'_I is the image of $\widetilde{Q}_I$ under the involution which exchanges e_i and s_i, $i \in \mathbb{N}$. In the following remark, for the reader's convenience, we collect some properties of these functions.

Remark 5.3. (i) Let $Q'_I(q)$ be the Hall-Littlewood polynomial where the initial alphabet X is replaced by $X/(1-q)$ (in the sense of λ-rings). Then Q'_I equals $Q'_I(q)$ when q is specialized to -1.

(ii) Equivalently, using raising operators R_{ij} (see e.g. [Mcd1]), one has

$$Q'_I = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}} h_I \ ,$$

where h_I is the product of complete homogeneous symmetric functions associated with the parts of I.

(iii) Q'-functions can be defined also using differential operators. In fact, this approach extends their definition to all finite sequences of integers. For $k \in \mathbb{Z}$ one has

$$Q'_{kJ} = (s_k - 2s_{k+1}D_{P_1} + 2s_{k+2}D_{P_2} - \ldots)(Q'_J) \ .$$

(iv) Littlewood [Li1] showed that the P and Q-Hall-Littlewood bases are adjoint w.r.t. a certain scalar product. Going back to the standard scalar product on the ring of symmetric functions, his result says that $\{P_I(q)\}$ is adjoint to $\{Q'_I(q)\}$, where I runs over all partitions. Specializing q to -1, one gets that $\{Q'_J\}$ and $\{P_J\}$, where J runs over the set of strict partitions, are two adjoint bases of some free $\mathbb{Z}$-modules A and B of symmetric functions w.r.t. the standard scalar product.

We refer the reader to [Li1], [Mcd1], [L-L-T2] and [D-L-T] for details and precise references to the results described in this remark.

Introduce now two other differential operators on the ring of symmetric functions:

$$U^s_k := D_{P_k} - s_1 D_{P_{k+1}} + s_2 D_{P_{k+2}} - \ldots$$

and

$$U^e_k := D_{P_k} - e_1 D_{P_{k+1}} + e_2 D_{P_{k+2}} - \ldots \ .$$

In particular, by Remark 5.3(iii), we have for $k > 0$

$$U^s_k(Q'_J) = (-1)^k \frac{1}{2} Q'_{(-k)J} \ .$$

Our goal now is to investigate the action of the operator U^s_k on $\widetilde{Q}$-functions. We compute first the action of U^e_k on Q'-functions.

Proposition 5.4. *Suppose $k > 0$. Let IkJ be a strict partition and H- a strict partition not containing k. Then*

$$U^e_k(Q'_{IkJ}) = (-1)^{l(I)} Q'_{IJ}$$

and

$$U_k^e(Q'_H) = 0 \ .$$

Proof. It follows from Proposition 5.2 that $V_k^e(B) \subset B$, and hence $U_k^e(A) \subset A$. Denote by $[\ ,\ ]$ the standard scalar product on the ring of symmetric functions. For strict partitions K, L we must evaluate

$$\big[U_k^e(Q'_K), P_L\big] = \big[Q'_K, V_k^e(P_L)\big] = \big[Q'_K, P_{kL}\big] \,.$$

This immediately yields both assertions. □

Applying the involution on the ring of symmetric polynomials, that exchanges e_i with s_i for $i \in \mathbb{N}$, we get

Proposition 5.5. *Suppose $k > 0$. Let IkJ be a strict partition and H- a strict partition not containing k . Then*

$$U_k^s(\widetilde{Q}_{IkJ}) = (-1)^{l(I)}\widetilde{Q}_{IJ}$$

and

$$U_k^s(\widetilde{Q}_H) = 0 \,.$$

We now want to link the above differential operators with divided difference operators. To this end we give first the following presentation of the operator induced by s_0 on the ring of symmetric functions.

Lemma 5.6. *For every symmetric function f,*

$$s_0(f) = f - 2x_1 D_{P_1}(f) + 2x_1^2 D_{P_2}(f) - 2x_1^3 D_{P_3}(f) + \dots \,.$$

Proof. We will use the λ-ring setup. For any symmetric function f and three alphabets X, B, C, one has

$$f(X - B + C) = \sum s_I(-B + C) D_{s_I}(f)(X),$$

where $C - B$ denotes the formal difference of the alphabets C and B.

Take now $B = x_1$, $C = \overline{x}_1$; then $X - B + C$ is obtained from X by changing x_1 to $\overline{x}_1$. On the other hand, the only nonzero $s_I(-B + C)$ for those special B and C, are those for which I is a hook $I = (j + 1, 1^i)$.

In that case, for $I \neq 0$, one has the explicit value

$$s_I(-B + C) = (\overline{x}_1 - x_1)\overline{x}_1^j(-x_1)^i$$

which specializes to $2(-x_1)^{|I|}$ under $\overline{x}_1 \mapsto -x_1$. Since the sum of all hook Schur S-functions of weight k is equal to P_k, collecting all hooks of the same weight, one finally gets

$$f\big((-x_1), x_2, x_3, \dots\big) = f - 2x_1 D_{P_1}(f) + 2x_1^2 D_{P_2}(f) - \dots \,,$$

the right-hand side being evaluated in $x_1, x_2, \dots$. □

Corollary 5.7. *For every symmetric function f ,*

$$\partial_0(f) = D_{P_1}(f) - x_1 D_{P_2}(f) + x_1^2 D_{P_3}(f) - \dots .$$

Let us define the following operator $D_h^{(n)}$ on the ring of symmetric functions by

$$D_h^{(n)}(f) := \partial_{n-1} \dots \partial_1 \partial_0 (x_1^h \cdot f) .$$

Proposition 5.8. *Fix $n \in \mathbb{N}$. Let $0 \le h < n$. If f is a symmetric function, then one has the equality*

$$\Big((-1)^{n-1} D_h^{(n)} f\Big)(X_n) = (U_{n-h}^s f)(X_n) ;$$

in other words,

$$\Big((-1)^{n-1} D_h^{(n)} f\Big)(X_n) = \Big(D_{P_{n-h}} f - s_1 \cdot D_{P_{n-h+1}} f + s_2 \cdot D_{P_{n-h+2}} f - \dots\Big)(X_n).$$

Proof. We have

$$\partial_0(x_1^{2j+1} \cdot f) = (-x_1)^{2j+1} \partial_0(f) + x_1^{2j} \cdot f$$

and

$$\partial_0(x_1^{2j} \cdot f) = x_1^{2j} \partial_0(f).$$

If $2j+1 < n$, then $\partial_{n-1} \dots \partial_1 (x_1^{2j} f) = 0$. The assertion follows from Corollary 5.7 by using the well-known formula $\partial_{n-1} \dots \partial_1 (x_1^p) = s_{p-n+1}(X_n)$ for $p \in \mathbb{N}$. $\square$

This completes the proof of Theorem 5.1. (Note that the case $k = n$ was proved earlier in [P-R2] by another method.)

We end this section by showing how the key formula explicates the action of the operators ∇_k defined by (11) on the functions specified in the following

Proposition 5.9. *Let $0 < k \le n$ and $n-1 \ge \alpha_1 > \alpha_2 > \dots > \alpha_k \ge 0$. Then the image under ∇_k of*

$$\partial'_{\omega^{(k)}}\big(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k} \cdot \widetilde{Q}_I(X_n)\big) = \partial'_{\omega^{(k)}}\big(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}\big) \cdot \widetilde{Q}_I(X_n)$$

is nonzero only if $n-\alpha_1, n-\alpha_2, \dots, n-\alpha_k$ are parts of I . In this case, the image is

$$(-1)^{\sum p_j - k} \widetilde{Q}_J(X_n) ,$$

where J is the strict partition with parts $\{i_1, i_2, \dots\} \smallsetminus \{n-\alpha_1, \dots, n-\alpha_k\}$ and p_j is such that $i_{p_j} = n - \alpha_j$ for $j = 1, \dots, k$.

We need some properties of the powers of ∇_1.

Lemma 5.10. *Let $\alpha_i \le n-i$ for $i=1,\dots,k$. Then for a symmetric function f,*

$$\nabla_1^k(x_1^{\alpha_1}\cdot\ldots\cdot x_k^{\alpha_k}\cdot f)=\nabla_1\Big(x_1^{\alpha_k}\cdot\ldots\cdot\nabla_1(x_1^{\alpha_2}\cdot\nabla_1(x_1^{\alpha_1}\cdot f))\ldots\Big).$$

In other words, by identifying a monomial with the operator of multiplication by this monomial, the lemma asserts the following equality of operators on the ring of symmetric functions:

$$\nabla_1^k\circ x_1^{\alpha_1}\ldots x_k^{\alpha_k}=\nabla_1\circ x_1^{\alpha_k}\circ\ldots\circ\nabla_1\circ x_1^{\alpha_2}\circ\nabla_1\circ x_1^{\alpha_1}.$$

Proof. The proof is by induction on k. For simplicity, we only prove the case $k=2$ and show how the case $k=2$ implies the case $k=3$. This is sufficient to see the general pattern of the inductive proof. For $k=2$ we have, using the Leibniz rule,

$$\begin{aligned}
&\nabla_1^2(x_1^\alpha x_2^\beta\cdot f)=\\
&\quad=\nabla_1\partial'_{n-1}\ldots\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)\\
&\quad=\nabla_1\partial'_{n-1}\ldots\partial'_1\big[\partial_0(x_1^\alpha\cdot f)\cdot x_2^\beta\big]\\
&\quad=\nabla_1\partial'_{n-1}\ldots\partial'_2\big[\partial'_1\partial_0(x_1^\alpha\cdot f)\cdot x_1^\beta+\partial_0(x_1^\alpha\cdot f)\cdot\partial'_1(x_2^\beta)\big].
\end{aligned}$$

Observe that

$$\partial'_{n-1}\ldots\partial'_2\big(\partial_0(x_1^\alpha\cdot f)\cdot\partial'_1(x_2^\beta)\big)=0$$

because $\partial_0(x_1^\alpha\cdot f)$ is a scalar w.r.t. $\partial'_{n-1}\ldots\partial'_3\partial'_2$ and we have $\beta\le n-2$. We conclude that

$$\nabla_1^2(x_1^\alpha x_2^\beta\cdot f)=\nabla_1\big(x_1^\beta\cdot\nabla_1(x_1^\alpha\cdot f)\big),$$

as asserted because x_1^β is a scalar w.r.t. $\partial'_{n-1}\ldots\partial'_2$.

For $k=3$ we have by the Leibniz rule

$$\begin{aligned}
&\nabla_1^3(x_1^\alpha x_2^\beta x_3^\gamma\cdot f)=\\
&\quad=\nabla_1^2\partial'_{n-1}\ldots\partial'_1\big[\partial_0(x_1^\alpha x_2^\beta\cdot f)\cdot x_3^\gamma\big]\\
&\quad=\nabla_1^2\partial'_{n-1}\ldots\partial'_2\big[\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)\cdot x_3^\gamma\big]\\
&\quad=\nabla_1^2\partial'_{n-1}\ldots\partial'_3\big[\partial'_2\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)\cdot x_2^\gamma+\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)\cdot\partial'_2(x_3^\gamma)\big].
\end{aligned}$$

Observe that

$$\partial'_{n-1}\ldots\partial'_3\big[\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)\cdot\partial'_2(x_3^\gamma)\big]=0$$

because $\partial'_1\partial_0(x_1^\alpha x_2^\beta\cdot f)$ is a scalar w.r.t. $\partial'_{n-1}\ldots\partial'_3$ and $\gamma\le n-3$. Since x_2^γ is a scalar w.r.t. $\partial'_{n-1}\ldots\partial'_3$, the expression in question can be rewritten as

$$\begin{aligned}
&\nabla_1^2\big(x_2^\gamma\cdot\nabla_1(x_1^\alpha x_2^\beta\cdot f)\big)\\
&\quad=\nabla_1\partial'_{n-1}\ldots\partial'_1\big(x_2^\gamma\cdot\partial_0\nabla_1(x_1^\alpha x_2^\beta\cdot f)\big)\\
&\quad=\nabla_1\partial'_{n-1}\ldots\partial'_2\Big(\partial'_1\partial_0\nabla_1(x_1^\alpha x_2^\beta\cdot f)\cdot x_1^\gamma+\partial_0\nabla_1(x_1^\alpha x_2^\beta\cdot f)\cdot\partial'_1(x_2^\gamma)\Big)
\end{aligned}$$

by the Leibniz rule. But $\partial_0\nabla_1(x_1^\alpha x_2^\beta\cdot f)=\partial_0\big(x_1^\beta\cdot\nabla_1(x_1^\alpha\cdot f)\big)$ is a scalar w.r.t. $\partial'_{n-1}\dots\partial'_2$ and $\gamma\le n-3$; this implies

$$\partial'_{n-1}\dots\partial'_2\big[\partial_0\nabla_1(x_1^\alpha x_2^\beta\cdot f)\cdot\partial'_1(x_2^\gamma)\big]=0.$$

Since x_1^γ is a scalar w.r.t. $\partial'_{n-1}\dots\partial'_2$, the expression in question equals

$$\nabla_1\big(x_1^\gamma\cdot\nabla_1^2(x_1^\alpha x_2^\beta\cdot f)\big)=\nabla_1\Big(x_1^\gamma\cdot\nabla_1\big(x_1^\beta\cdot\nabla_1(x_1^\alpha\cdot f)\big)\Big)$$

by the case $k=2$ proved above. $\square$

Proof of Proposition 5.9: The assertion follows directly from Lemma 5.10, the equality $\nabla_k\circ\partial'_{\omega^{(k)}}=\nabla_1^k$ and the key formula (Theorem 5.1). $\square$

Example 5.11. $n=6,\ k=3,\ \widetilde{Q}_I=\widetilde{Q}_I(X_6)$.

$$\begin{aligned}\nabla_3\big(\partial'_{\omega^{(3)}}(x_1^4x_2^1x_3^0)\cdot\widetilde{Q}_{654321}\big)&\\ &=\pm\nabla_1\Big(x_1^0\cdot\nabla_1\big(x_1^1\cdot\nabla_1(x_1^4\cdot\widetilde{Q}_{654321})\big)\Big)\\ &=\pm\nabla_1\big(x_1^0\cdot\nabla_1(x_1^1\cdot\widetilde{Q}_{65431})\big)\\ &=\pm\nabla_1(x_1^0\cdot\widetilde{Q}_{6431})\\ &=\widetilde{Q}_{431}\,.\end{aligned}$$

We give now other useful consequences of the key formula. Given two barred permutations w and v, we shall say that " u transforms w to v " (or " we pass from w to v using u ") if $w\cdot u^{-1}=v$ and $l(w)-l(u)=l(v)$.

Proposition 5.12. *Let $I=(i_1,\dots,i_l)\subset\rho_n$ be a strict partition. Then*

$$\partial'_{i_1-1}\dots\partial'_1\partial_0\big(\widetilde{Q}_I(X_n)\big)=\widetilde{Q}_{(i_2,\dots,i_l)}(X_n)\,.$$

Proof. Suppose that $\{i_1,\dots,i_l,j_1<j_2<\dots<j_k\}=\{1,\dots,n\}$. Let w, v, u and μ be elements of W_n, which transform the barred permutations as indicated in the following square

$$\begin{array}{ccc}[\overline{j_1},\dots,\overline{j_k},\overline{i_1},\overline{i_2},\dots,\overline{i_l}] & \xleftarrow{\ \mu\ } & [\overline{j_1}<\dots<\overline{j_{p-1}}<\overline{i_1}<\overline{j_p}<\dots<\overline{j_k},\overline{i_2},\dots,\overline{i_l}]\\ \Big\downarrow v & & \Big\downarrow w\\ [\overline{i_1},\overline{i_2},\dots,\overline{i_l},j_1,\dots,j_k] & \xrightarrow{\ u\ } & [\overline{i_2},\dots,\overline{i_l},j_1<\dots<j_{p-1}<i_1<j_p<\dots<j_k]\end{array}$$

for some p. Note that $\mu=s_k\cdot\ldots\cdot s_{p+1}\cdot s_p$.

We have $w=u\cdot v\cdot\mu$ and $l(w)=l(u)+l(v)+l(\mu)$. Hence $\partial'_w=\partial'_u\circ\partial'_v\circ\partial'_\mu$. Observe that

$$v=(s_l\dots s_1s_0)\dots(s_{n-2}\dots s_1s_0)(s_{n-1}\dots s_1s_0)\cdot\omega^{(k)}\,.$$

Therefore by Proposition 5.9, we have

$$\partial'_v\big(x_1^{n-j_1}x_2^{n-j_2}\cdot\ldots\cdot x_k^{n-j_k}\widetilde{Q}_{\rho_n}(X_n)\big) = \\ = (-1)^{n-j_1+n-j_2+\ldots+n-j_k}\widetilde{Q}_{(i_1,\ldots,i_l)}(X_n)\,.$$

In a similar way, we infer

$$\partial'_w\big(x_1^{n-j_1}\cdot\ldots\cdot x_p^{n-i_1}\cdot\ldots\cdot x_{k+1}^{n-j_k}\widetilde{Q}_{\rho_n}(X_n)\big) = \\ = (-1)^{n-j_1+\ldots+n-i_1+\ldots+n-j_k}\widetilde{Q}_{(i_2,\ldots,i_l)}(X_n)\,.$$

Since $\partial'_{i_1-1}\ldots\partial'_1\partial_0 = \partial'_u$, to prove the assertion it suffices to show that

$$\partial'_\mu\big(x_1^{n-j_1}\cdot\ldots\cdot x_p^{n-i_1}\cdot\ldots\cdot x_{k+1}^{n-j_k}\big) = (-1)^{n-i_1}x_1^{n-j_1}\cdot\ldots\cdot x_k^{n-j_k}\,.$$

But this is an obvious equality for *dominant* Schubert polynomials:

$$\partial_k\ldots\partial_{p+1}\partial_p\mathfrak{S}_{[n+1-j_1,\ldots,n+1-i_1,\ldots,n+1-j_k,n+1-i_2,\ldots,n+1-i_l]} = \\ = \mathfrak{S}_{[n+1-j_1,\ldots,n+1-j_k,n+1-i_1,\ldots,n+1-i_l]}\,,$$

where $n+1-i_1$ is placed in the p-th place, in the first expression.

Thus the proposition has been proved. $\square$

Example 5.13. $n=9$, $I=(6,4,3,2)$. The elements w, v, u and μ of W_9 are defined by the following diagram:

$$\begin{tikzcd}
{[\overline{1},\overline{5},\overline{7},\overline{8},\overline{9},\overline{6},\overline{4},\overline{3},\overline{2}]} \arrow[d, "v"] & {[\overline{1},\overline{5},\overline{6},\overline{7},\overline{8},\overline{9},\overline{4},\overline{3},\overline{2}]} \arrow[l, "\mu"'] \arrow[d, "w"] \\
{[\overline{6},\overline{4},\overline{3},\overline{2},1,5,7,8,9]} \arrow[r, "u"] & {[\overline{4},\overline{3},\overline{2},1,5,6,7,8,9]}
\end{tikzcd}$$

We have, with $\widetilde{Q}_I = \widetilde{Q}_I(X_9)$,

$$\partial'_v(x_1^8x_2^4x_3^2x_4^1x_5^0\cdot\widetilde{Q}_{\rho_9}) = \pm\widetilde{Q}_{6432}\,,$$
$$\partial'_w(x_1^8x_2^4x_3^3x_4^2x_5^1x_6^0\cdot\widetilde{Q}_{\rho_9}) = \pm\widetilde{Q}_{432} \text{ and}$$
$$\partial_5\partial_4\partial_3(x_1^8x_2^4x_3^3x_4^2x_5^1x_6^0) = x_1^8x_2^4x_3^2x_4^1x_5^0\,.$$

Corollary 5.14. *Let $I=(i_1,\ldots,i_k,\ldots,i_l)$ be a strict partition. Then*

$$\partial'_{i_k-1}\ldots\partial'_1\partial_0\ldots\partial'_{i_1-1}\ldots\partial'_1\partial_0\big(\widetilde{Q}_I(X_n)\big) = \widetilde{Q}_{(i_{k+1},i_{k+2},\ldots,i_l)}(X_n)\,.$$

In particular (invoking the notation of (12)), we get

$$\partial'_{w_I}\big(\widetilde{Q}_I(X_n)\big) = 1\,.$$

Corollary 5.15. *Let $I=(i_1,\dots,i_l)$ be a strict partition. Then for $j>i_1$,*

$$\partial_{j-1}\dots\partial_1\partial_0\big(\widetilde{Q}_I(X_n)\big)=0\,.$$

Indeed, we have

$$\partial_{j-1}\dots\partial_{i_1}\partial_{i_1-1}\dots\partial_1\partial_0\big(\widetilde{Q}_I(X_n)\big)=\partial_{j-1}\dots\partial_{i_1}\widetilde{Q}_{(i_2,\dots)}(X_n)=0\,.$$

Observe that this implies that for any strict partition $J\neq I$ which precedes I in the reverse lexicographic order (see [Mcd1]),

$$\partial_{w_J}\big(\widetilde{Q}_I(X_n)\big)=0.$$

Remark 5.16. Some of the above results for "trapezoidal" partitions $I=(n,n-1,\dots,k+1)$ were originally proved in [P-R2] by different methods.

Appendix A (written in collaboration with J. Ratajski): Symplectic Schubert polynomials à la polonaise.

Billey and Haiman [B-H] and Fomin and Kirillov [F-K] have defined various Schubert polynomials for type C_n. Following [P-R2, Appendix B], we propose here other candidates to represent the Schubert classes in the variety of complete Lagrangian flags.

Let $(x_1,x_2,\dots)$ be a sequence of independent variables. Let $w_0=w_0^{(n)}$ be the longest element in the Weyl group W_n of type C_n. Define

$$\mathcal{C}_{w_0}=\mathcal{C}_{w_0}(X_n):=(-1)^{n(n-1)/2}x^{\rho_{n-1}}\widetilde{Q}_{\rho_n}(X_n)\,,$$

and for an arbitrary $w\in W_n$,

$$\mathcal{C}_w:=\mathcal{C}_w(X_n)=\partial'_{w^{-1}w_0}(\mathcal{C}_{w_0})$$

- the *symplectic Schubert polynomial* associated with w.

Example A.1. Symplectic Schubert polynomials for $n=2$ are:

$$\begin{gathered}
\mathcal{C}_{[\bar1,\bar2]}=-\,x_1^3x_2-x_1^2x_2^2,\\
\mathcal{C}_{[1,\bar2]}=-x_1^2x_2\qquad,\qquad \mathcal{C}_{[\bar2,\bar1]}=x_1^2x_2+x_1x_2^2,\\
\mathcal{C}_{[\bar2,1]}=x_1x_2\qquad,\qquad \mathcal{C}_{[2,\bar1]}=x_2^2,\\
\mathcal{C}_{[2,1]}=x_2\qquad,\qquad \mathcal{C}_{[\bar1,2]}=x_1+x_2,\\
\mathcal{C}_{[1,2]}=1\,.
\end{gathered}$$

Their expression in terms of the $\mathfrak{S}\widetilde{Q}$-basis is:

$$\mathcal{C}_{[\overline{1},\overline{2}]} = -\,\mathfrak{S}_{[2,1]}\widetilde{Q}_{21}\ ,$$
$$\mathcal{C}_{[1,\overline{2}]} = -\mathfrak{S}_{[2,1]}\widetilde{Q}_2 \quad , \quad \mathcal{C}_{[\overline{2},\overline{1}]} = \widetilde{Q}_{21}\ ,$$
$$\mathcal{C}_{[\overline{2},1]} = \widetilde{Q}_2 \quad , \quad \mathcal{C}_{[2,\overline{1}]} = -\mathfrak{S}_{[2,1]}\widetilde{Q}_1 + (x_1^2+x_2^2) + \widetilde{Q}_2\ ,$$
$$\mathcal{C}_{[2,1]} = -\mathfrak{S}_{[2,1]} + \widetilde{Q}_1 \quad , \quad \mathcal{C}_{[\overline{1},2]} = \widetilde{Q}_1\ ,$$
$$\mathcal{C}_{[1,2]} = 1\,.$$

Theorem A.2. (Stability) *Suppose that $m > n$. Let $W_n \hookrightarrow W_m$ be the embedding via the first n components. Then for any $w \in W_n \subset W_m$, the following equality holds:*

$$\mathcal{C}_w(X_m)|_{x_{n+1}=\ldots=x_m=0} = \mathcal{C}_w(X_n)\,.$$

Proof. To show the assertion we must prove that the above definition is consistent with the divided difference recurrences. Taking into account the equality

$$w_0^{(n)} = w_0^{(n-1)} s_{n-1}s_{n-2}\ldots s_1 s_0 s_1 \ldots s_{n-2}s_{n-1}\,,$$

and our sign conventions, the wanted assertion is an immediate consequence of the equality:

$$\partial'_{n-1}\ldots\partial'_2\partial'_1\partial_0\partial_1\partial_2\ldots\partial_{n-1}(f_n)|_{x_n=0} = f_{n-1}\,,$$

where $f_n = x^{\rho_{n-1}}\widetilde{Q}_{\rho_n}(X_n)$.

We need a simple

Lemma A.3. *For any polynomial $f = f(X_n)$,*

$$\partial_{n-1}\ldots\partial_1\partial_0(x_1^2x_2\ldots x_n\cdot f)|_{x_n=0} = 0\,.$$

Indeed, after applying ∂_0 we get a (possibly zero) sum of monomials divisible by $x_1\cdot\ldots\cdot x_n$. Since $x_1\cdot\ldots\cdot x_n$ is a scalar w.r.t. $\partial_{n-1}\ldots\partial_1$, the final substitution $x_n = 0$ gives the vanishing.

Suppose now that

$$\partial_1\partial_2\ldots\partial_{n-1}(x^{\rho_{n-1}}) = \sum_J a_J x^J\,, \tag{18}$$

where $a_J \in \mathbb{Z}$, so that

$$\partial_1\partial_2\ldots\partial_{n-1}(f_n) = \Big(\sum a_J x^J\Big)\widetilde{Q}_{\rho_n}(X_n)\,.$$

We claim that if for $J = (j_1,\ldots)$ with $a_J \neq 0$ we have $j_1 \geq 1$, then

$$\partial_{n-1}\ldots\partial_1\partial_0\big(x^J\widetilde{Q}_{\rho_n}(X_n)\big)\,|_{x_n=0} = 0\,.$$

This follows from the lemma since $\widetilde{Q}_{\rho_n}(X_n) = x_1\cdot\ldots\cdot x_n\cdot\widetilde{Q}_{\rho_{n-1}}(X_n)$. We next observe

Lemma A.4. *With the notation of (18), if $a_J \neq 0$ and $j_1 = 0$, then $J = (0, n-2, n-3, \ldots, 2, 1, 0)$.*

Indeed, let M be a monomial appearing in the decomposition of

$$\partial_2 \ldots \partial_{n-1}(x_2^{n-2} x_3^{n-3} \ldots x_{n-1})$$

as a sum of monomials. To get in $\partial_1(x_1^{n-1} M)$ a monomial without x_1, the monomial M must be independent of x_2 ($x_1 x_2$ is a scalar w.r.t. ∂_1). By induction we can assume $M = x_3^{n-3} \ldots x_{n-1}$. But then, the unique monomial in $\partial_1(x_1^{n-1} x_3^{n-3} \ldots x_{n-1})$ that does not depend on x_1, is $x_2^{n-2} x_3^{n-3} \ldots x_{n-1}$, as claimed.

Hence we must compute:

$$\begin{aligned}
&\partial'_{n-1} \ldots \partial'_1 \partial_0 \big(x_2^{n-2} \cdot x_3^{n-3} \cdot \ldots \cdot x_{n-1} \cdot \widetilde{Q}_{\rho_n}(X_n)\big) \,|_{\, x_n = 0} \\
&\qquad = \partial'_{n-1} \ldots \partial'_1 \partial_0 \big(x_2^{n-2} \cdot \ldots \cdot x_{n-1} \cdot x_1 \cdot \ldots \cdot x_n \cdot \widetilde{Q}_{\rho_{n-1}}(X_n)\big) \,|_{\, x_n = 0} \\
&\qquad = \partial'_{n-1} \ldots \partial'_1 \partial_0 \big(x_2^{n-2} \cdot \ldots \cdot x_{n-1} \cdot x_1 \cdot \ldots \cdot x_n \cdot \widetilde{Q}_{\rho_{n-1}}(x_2, \ldots, x_n)\big) \,|_{\, x_n = 0}
\end{aligned}$$

by Lemma A.3. The last expression can be rewritten as

$$\partial'_{n-1} \ldots \partial'_1 \big(x_2^{n-1} \ldots x_{n-1}^2 x_n \cdot \widetilde{Q}_{\rho_{n-1}}(x_2, \ldots, x_n)\big) \,|_{\, x_n = 0}. \tag{19}$$

Now we need

Lemma A.5. *Let x, y be indeterminates and $f = f(x,y)$ a polynomial function depending on x, y and perhaps on some other variables. Then denoting by ∂_{xy} the divided difference $f \mapsto \frac{f(x,y) - f(y,x)}{x-y}$, we have*

$$\partial_{xy}\big(y \cdot f(x,y)\big) \,|_{\, y=0} = \partial_{xy}\big(y \cdot f(0,y)\big) \,|_{\, y=0} \,.$$

Indeed, $y \cdot f(x,y) - y \cdot f(0,y) = xy \cdot g$ for some polynomial g. Then

$$\partial_{xy}\big(y \cdot f(x,y) - y f(0,y)\big) = \partial_{xy}(xy \cdot g) = xy \partial_{xy}(g)$$

vanishes after the substitution $y = 0$.

Coming back to our expression (19), we see that before applying ∂'_{n-1} we can substitute $x_{n-1} = 0$, so before applying ∂'_{n-2} we can substitute $x_{n-2} = 0$ etc. In other words, in the course of computing (19) we can after applying ∂'_1 substitute $x_2 = 0$, after applying ∂'_2 substitute $x_3 = 0, \ldots$, after applying ∂'_{n-2} substitute $x_{n-1} = 0$, and at the end substitute $x_n = 0$.

Using the Leibniz rule, we get

$$\begin{aligned}
&\partial'_1 \big(x_2^{n-1} \ldots x_{n-1}^2 x_n \cdot \widetilde{Q}_{\rho_{n-1}}(x_2, \ldots, x_n)\big) \,|_{\, x_2 = 0} \\
&= \Big[\partial'_1 (x_2^{n-1} \ldots x_n) \cdot s_1 \widetilde{Q}_{\rho_{n-1}}(x_2, \ldots, x_n)\Big] \,|_{\, x_2 = 0} \\
&\qquad\qquad + \Big[x_2^{n-1} \ldots x_n \cdot \partial'_1 \big(\widetilde{Q}_{\rho_{n-1}}(x_2, \ldots, x_n)\big)\Big] \,|_{\, x_2 = 0} \\
&= x_1^{n-2} x_3^{n-2} \ldots x_n \cdot \widetilde{Q}_{\rho_{n-1}}(x_1, x_3, \ldots, x_n) + 0.
\end{aligned}$$

Similarly, applying ∂_2' to the obtained polynomial and substituting $x_3 = 0$, we get

$$x_1^{n-2}x_2^{n-3}x_4^{n-3}x_5^{n-4}\dots x_n \cdot \widetilde{Q}_{\rho_{n-1}}(x_1, x_2, x_4, \dots, x_n)\,.$$

Continuing in this way, we get after applying ∂_{n-1}'

$$x_1^{n-2}x_2^{n-3}\dots x_{n-2}\cdot \widetilde{Q}_{\rho_{n-1}}(x_1,\dots,x_{n-1}) = f_{n-1}\,.$$

This completes the proof of the stability property. □

Theorem A.6. (the maximal Grassmannian property) *Let $I = (i_1 > \dots > i_l > 0)$ be a strict partition contained in ρ_n. Then, invoking the notation of (12), one has the equality*

$$\mathcal{C}_{w_I}(X_n) = \widetilde{Q}_I(X_n)\,.$$

Proof. The proof is very similar to the one of Proposition 5.12, but we give all details for the reader's convenience. We first explain how, in the course of the proof, we pass from w_0 to w_I.[1] Write $k = n - l$. First, by using an appropriate permutation σ, we pass from $w_0 = [\overline{1}, \overline{2}, \dots, \overline{n}]$ to

$$[\overline{j_1}, \overline{j_2}, \dots, \overline{j_k}, \overline{i_1}, \dots, \overline{i_l}]\,,$$

where $\{i_1, \dots, i_l, j_1, \dots, j_k\} = \{1, \dots, n\}$ and $j_1 < \dots < j_k$. Secondly, using $\omega^{(k)}$, we pass to

$$[\overline{j_k}, \overline{j_{k-1}}, \dots, \overline{j_1}, \overline{i_1}, \dots, \overline{i_l}]\,.$$

Thirdly, using

$$(s_{n-k}\dots s_1 s_0)\dots(s_{n-2}\dots s_1 s_0)(s_{n-1}\dots s_1 s_0)\,,$$

we pass to

$$[\overline{i_1}, \dots, \overline{i_l}, j_1, j_2, \dots, j_k] = w_I\,.$$

Observe that after applying to $\mathcal{C}_{w_0}(X_n)$ the divided difference corresponding to the first permutation, we get

$$\begin{aligned}&\mathcal{C}_{[\overline{j_1}, \overline{j_2}, \dots, \overline{j_k}, \overline{i_1}, \dots, \overline{i_l}]}\\ &\quad = (-1)^{n(n-1)/2 + l(\sigma)}\mathfrak{S}_{[n+1-j_1, \dots, n+1-j_k, n+1-i_1, \dots, n+1-i_l]}\cdot \widetilde{Q}_{\rho_n}(X_n)\end{aligned}$$

because σ transforms $\omega^{(n)}$ to

$$[n+1-j_1, \dots, n+1-j_k, n+1-i_1, \dots, n+1-i_l].$$

We have

$$\mathfrak{S}_{[n+1-j_1, \dots, n+1-j_k, n+1-i_1, \dots, n+1-i_l]} = x_1^{n-j_1}x_2^{n-j_2}\dots x_k^{n-j_k}\,,$$

[1] Recall that we pass from w to v using u (or u transforms w to v) if $w\cdot u^{-1} = v$ and $l(w) - l(u) = l(v)$. We then have $\partial_u \mathcal{C}_w = \mathcal{C}_v$. Similarly, if for $\mu, \nu, \sigma \in S_n$, σ transforms μ to ν, then $\partial_\sigma \mathfrak{S}_\mu = \mathfrak{S}_\nu$.

see [Mcd2]. The operator corresponding to the composition of the second permutation and the third barred permutation is $\nabla_k \circ \partial'_{\omega^{(k)}}$, and using Proposition 5.9 we get

$$\nabla_k \circ \partial'_{\omega^{(k)}} \left(x_1^{n-j_1} x_2^{n-j_2} \dots x_k^{n-j_k} \widetilde{Q}_{\rho_n}(X_n) \right) = (-1)^{n-j_1+n-j_2+\dots+n-j_k} \widetilde{Q}_{(i_1,\dots,i_l)}(X_n).$$

Since $n-j_1+\dots+n-j_k$ is the length of the permutation passing from $[n, n-1, \dots, 2, 1]$ to $[j_1, j_2, \dots, j_k, i_1, \dots, i_l]$ and σ passes from $[1, 2, \dots, n]$ to $[j_1, j_2, \dots, j_k, i_1, \dots, i_l]$, we see that the above signs fit to give us

$$\left(\nabla_k \circ \partial'_{\omega^{(k)}} \circ \partial'_\sigma \right) \left((-1)^{n(n-1)/2} x^{\rho_{n-1}} \widetilde{Q}_{\rho_n}(X_n) \right) = \widetilde{Q}_I(X_n),$$

as asserted. $\square$

Example A.7. $n = 6$, $I = (4, 3, 1)$. We compute, with $\widetilde{Q}_I = \widetilde{Q}_I(X_6)$,

$$\begin{aligned} \mathcal{C}_{[\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6}]} &\xrightarrow{\partial_\sigma} \mathcal{C}_{[\bar{2},\bar{5},\bar{6},\bar{4},\bar{3},\bar{1}]} \\ &= \pm \mathfrak{S}_{[5,2,1,3,4,6]} \cdot \widetilde{Q}_{\rho_6} = \pm x_1^4 x_2^1 x_3^0 \cdot \widetilde{Q}_{\rho_6}. \end{aligned}$$

Then to compute the image under $\nabla_3 \circ \partial'_{\omega^{(3)}}$ of the last element, we follow Example 5.11 to get finally $\widetilde{Q}_{(4,3,1)}$.

From the maximal Grassmannian property it follows the following operator characterization of $\widetilde{Q}$-polynomials.

Corollary A.8. *For any strict partition I, $w = w_I$ is the unique element of W such that $l(w) = |I|$ and $\partial_w(\widetilde{Q}_I(X)) \neq 0$; in fact, $\partial_{w_I}(\widetilde{Q}_I(X)) = 1$.*

We end this Appendix with the table of symplectic Schubert polynomials for $n = 3$ computed with the help of *ACE* (see [V1,2]). In this table $\mathfrak{S}_{i,j,k} := \mathfrak{S}_{[i,j,k]}(x_1, x_2, x_3)$ and $E_i := e_i(x_1^2, x_2^2, x_3^2)$.

$\mathcal{C}_{[1,2,3]} = 1$
$\mathcal{C}_{[1,3,2]} = x_3 = -\mathfrak{S}_{1,3,2} + \widetilde{Q}_1$
$\mathcal{C}_{[2,1,3]} = x_2 + x_3 = -\mathfrak{S}_{2,1,3} + \widetilde{Q}_1$
$\mathcal{C}_{[\bar{1},2,3]} = x_1 + x_2 + x_3 = \widetilde{Q}_1$
$\mathcal{C}_{[3,1,2]} = x_2 x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_1 + \mathfrak{S}_{3,1,2} + \widetilde{Q}_2$
$\mathcal{C}_{[2,3,1]} = x_3^2 = \mathfrak{S}_{2,3,1} - \mathfrak{S}_{1,3,2}\widetilde{Q}_1 + E_1 + \widetilde{Q}_2$
$\mathcal{C}_{[\bar{2},1,3]} = x_1 x_2 + x_1 x_3 + x_2 x_3 = \widetilde{Q}_2$
$\mathcal{C}_{[\bar{1},3,2]} = (x_1 + x_2 + x_3) x_3 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_1 + E_1 + 2\widetilde{Q}_2$
$\mathcal{C}_{[2,\bar{1},3]} = x_2^2 + x_2 x_3 + x_3^2 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_1 + E_1 + \widetilde{Q}_2$
$\mathcal{C}_{[\bar{3},1,2]} = x_1 x_2 x_3 = \widetilde{Q}_3$
$\mathcal{C}_{[3,2,1]} = x_2 x_3^2 = -\mathfrak{S}_{3,2,1} + \mathfrak{S}_{2,3,1}\widetilde{Q}_1 - E_1 \mathfrak{S}_{2,1,3} - \mathfrak{S}_{2,1,3}\widetilde{Q}_2 + \mathfrak{S}_{3,1,2}\widetilde{Q}_1 - \mathfrak{S}_{1,3,2}\widetilde{Q}_2 + \widetilde{Q}_3 + \widetilde{Q}_{21}$
$\mathcal{C}_{[\bar{2},3,1]} = (x_1 + x_2) x_3^2 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_2 + \widetilde{Q}_3 + \widetilde{Q}_{21}$

$\mathcal{C}_{[1,\overline{2},3]} = -x_1^2(x_2+x_3) = -\mathfrak{S}_{2,1,3}\widetilde{Q}_2 + \widetilde{Q}_3$

$\mathcal{C}_{[3,\overline{1},2]} = (x_2+x_3)x_2x_3 = -E_1\mathfrak{S}_{2,1,3} - 2\mathfrak{S}_{2,1,3}\widetilde{Q}_2 + \mathfrak{S}_{3,1,2}\widetilde{Q}_1 + \widetilde{Q}_3 + \widetilde{Q}_{21}$

$\mathcal{C}_{[2,3,\overline{1}]} = x_3^3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_1 - E_1\mathfrak{S}_{1,3,2} - \mathfrak{S}_{1,3,2}\widetilde{Q}_2 + E_1\widetilde{Q}_1 + \widetilde{Q}_3$

$\mathcal{C}_{[\overline{2},\overline{1},3]} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 = \widetilde{Q}_{21}$

$\mathcal{C}_{[1,\overline{3},2]} = -x_1^2x_2x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_3$

$\mathcal{C}_{[\overline{3},2,1]} = x_1x_2x_3^2 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_3 + \widetilde{Q}_{31}$

$\mathcal{C}_{[3,\overline{2},1]} = x_2^2x_3^2 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_3 - \mathfrak{S}_{2,1,3}\widetilde{Q}_{21} + \mathfrak{S}_{3,1,2}\widetilde{Q}_2 + E_2 + \widetilde{Q}_{31}$

$\mathcal{C}_{[1,3,\overline{2}]} = -(x_1^2+x_2^2)x_3^2 = \mathfrak{S}_{2,3,1}\widetilde{Q}_2 - \mathfrak{S}_{1,3,2}\widetilde{Q}_3 - E_2$

$\mathcal{C}_{[\overline{3},\overline{1},2]} = (x_1+x_2+x_3)x_1x_2x_3 = \widetilde{Q}_{31}$

$\mathcal{C}_{[3,2,\overline{1}]} = x_2x_3^3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_1 + E_1\mathfrak{S}_{2,3,1} + 2\mathfrak{S}_{2,3,1}\widetilde{Q}_2 - E_1\mathfrak{S}_{2,1,3}\widetilde{Q}_1 - \mathfrak{S}_{2,1,3}\widetilde{Q}_3 + E_1\mathfrak{S}_{3,1,2} + \mathfrak{S}_{3,1,2}\widetilde{Q}_2 - \mathfrak{S}_{1,3,2}\widetilde{Q}_3 - \mathfrak{S}_{1,3,2}\widetilde{Q}_{21} + E_2 + E_1\widetilde{Q}_2 + \mathfrak{S}_{1,2,3}\widetilde{Q}_{31}$

$\mathcal{C}_{[\overline{2},3,\overline{1}]} = (x_1^2 + x_1x_3 + x_2^2 + x_2x_3)x_3^2 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_{21} + 2E_2 + E_1\widetilde{Q}_2 + \widetilde{Q}_{31}$

$\mathcal{C}_{[\overline{1},\overline{2},3]} = -(x_1x_2 + x_1x_3 + x_2^2 + x_3^2)x_1^2 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_{21} + \widetilde{Q}_{31}$

$\mathcal{C}_{[1,2,\overline{3}]} = x_1^2x_2^2x_3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_3$

$\mathcal{C}_{[2,\overline{3},1]} = -(x_2+x_3)x_1^2x_2x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_{31} + \mathfrak{S}_{3,1,2}\widetilde{Q}_3$

$\mathcal{C}_{[\overline{3},\overline{2},1]} = (x_1x_2 + x_1x_3 + x_2x_3)x_1x_2x_3 = \widetilde{Q}_{32}$

$\mathcal{C}_{[3,1,\overline{2}]} = -(x_1^2x_2 + x_1^2x_3 + x_2^2x_3)x_2x_3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_2 + \mathfrak{S}_{2,3,1}\widetilde{Q}_3 + \mathfrak{S}_{2,3,1}\widetilde{Q}_{21} + E_2\mathfrak{S}_{2,1,3} + \mathfrak{S}_{3,1,2}\widetilde{Q}_3 - E_2\mathfrak{S}_{1,3,2} - \mathfrak{S}_{1,3,2}\widetilde{Q}_{31}$

$\mathcal{C}_{[\overline{1},\overline{3},2]} = -(x_1+x_2+x_3)x_1^2x_2x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_{31}$

$\mathcal{C}_{[\overline{3},2,\overline{1}]} = -(x_1x_2 - x_3^2)x_1x_2x_3 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_{31} + E_1\widetilde{Q}_3 + \widetilde{Q}_{32}$

$\mathcal{C}_{[3,\overline{2},\overline{1}]} = (x_1^2 + x_2x_3)(x_2+x_3)x_2x_3 = -2E_2\mathfrak{S}_{2,1,3} - E_1\mathfrak{S}_{2,1,3}\widetilde{Q}_2 - \mathfrak{S}_{2,1,3}\widetilde{Q}_{31} + \mathfrak{S}_{3,1,2}\widetilde{Q}_{21} + E_2\widetilde{Q}_1 + E_1\widetilde{Q}_3 + \widetilde{Q}_{32}$

$\mathcal{C}_{[\overline{1},3,\overline{2}]} = -(x_1^3x_3 + 2x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_2^3x_3 + x_2^2x_3^2)x_3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_{21} - \mathfrak{S}_{1,3,2}\widetilde{Q}_{31} - E_2\widetilde{Q}_1$

$\mathcal{C}_{[2,1,\overline{3}]} = x_1^2x_2^3x_3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_3 + \mathfrak{S}_{2,3,1}\widetilde{Q}_{31} - E_3$

$\mathcal{C}_{[\overline{2},\overline{3},1]} = -(x_1x_2 + x_1x_3 + x_2x_3)x_1^2x_2x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_{32}$

$\mathcal{C}_{[\overline{3},1,\overline{2}]} = -(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3)x_1x_2x_3 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_{32} + 2E_3$

$\mathcal{C}_{[\overline{1},2,\overline{3}]} = (x_1+x_2+x_3)x_1^2x_2^2x_3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_{31}$

$\mathcal{C}_{[2,\overline{3},\overline{1}]} = -(x_2^2 + x_2x_3 + x_3^2)x_1^2x_2x_3 = -E_1\mathfrak{S}_{2,1,3}\widetilde{Q}_3 - \mathfrak{S}_{2,1,3}\widetilde{Q}_{32} + \mathfrak{S}_{3,1,2}\widetilde{Q}_{31}$

$\mathcal{C}_{[\overline{3},\overline{2},\overline{1}]} = (x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2)x_1x_2x_3 = \widetilde{Q}_{321}$

$\mathcal{C}_{[3,\overline{1},\overline{2}]} = -(2x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1^2x_3^2 + x_2^3x_3 + x_2^2x_3^2)x_2x_3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_{21} + 2E_2\mathfrak{S}_{2,3,1} + E_1\mathfrak{S}_{2,3,1}\widetilde{Q}_2 + \mathfrak{S}_{2,3,1}\widetilde{Q}_{31} + E_2\mathfrak{S}_{2,1,3}\widetilde{Q}_1 + \mathfrak{S}_{3,1,2}\widetilde{Q}_{31} - E_2\mathfrak{S}_{1,3,2}\widetilde{Q}_1 - E_1\mathfrak{S}_{1,3,2}\widetilde{Q}_3 - \mathfrak{S}_{1,3,2}\widetilde{Q}_{32}$

$\mathcal{C}_{[\overline{2},1,\overline{3}]} = (x_1x_2 - x_3^2)x_1^2x_2^2x_3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_{32} - E_3\widetilde{Q}_1$

$\mathcal{C}_{[1,\overline{3},\overline{2}]} = x_1^2x_2x_3(x_1^2 + x_2x_3)(x_2+x_3) = 2E_3\mathfrak{S}_{2,1,3} + \mathfrak{S}_{3,1,2}\widetilde{Q}_{32} + E_3\widetilde{Q}_1$

$\mathcal{C}_{[2,\overline{1},\overline{3}]} = (x_2^2 + x_2x_3 + x_3^2)x_1^2x_2^2x_3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_{31} + E_1\mathfrak{S}_{2,3,1}\widetilde{Q}_3 + \mathfrak{S}_{2,3,1}\widetilde{Q}_{32}$

$\mathcal{C}_{[\overline{2},\overline{3},\overline{1}]} = -(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2)x_1^2x_2x_3 = -\mathfrak{S}_{2,1,3}\widetilde{Q}_{321}$

$\mathcal{C}_{[\overline{3},\overline{1},\overline{2}]} = -(x_1+x_2)(x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2)x_1x_2x_3 = -\mathfrak{S}_{1,3,2}\widetilde{Q}_{321}$
$\mathcal{C}_{[1,\overline{2},\overline{3}]} = -x_1^2x_2^2x_3(x_1^2+x_2x_3)(x_2+x_3) = -\mathfrak{S}_{3,2,1}\widetilde{Q}_{32} + 2E_3\mathfrak{S}_{2,3,1} + E_3\mathfrak{S}_{2,1,3}\widetilde{Q}_1 - E_3\mathfrak{S}_{1,3,2}\widetilde{Q}_1$
$\mathcal{C}_{[\overline{2},\overline{1},\overline{3}]} = (x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2)x_1^2x_2^2x_3 = \mathfrak{S}_{2,3,1}\widetilde{Q}_{321}$
$\mathcal{C}_{[\overline{1},\overline{3},\overline{2}]} = (x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2)x_1^3x_2x_3 = \mathfrak{S}_{3,1,2}\widetilde{Q}_{321}$
$\mathcal{C}_{[\overline{1},\overline{2},\overline{3}]} = -(x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2)x_1^3x_2^2x_3 = -\mathfrak{S}_{3,2,1}\widetilde{Q}_{321}.$

Appendix B: Three geometric applications

1. In Corollary A.8 we have obtained a purely algebraic proof (based on the key formula) of the following result. For any strict partition J with $|J| = |I|$, we have $\partial_{w_J}(\widetilde{Q}_I) \neq 0$ only if $I = J$, and, in this case, we get $\partial'_{w_I}(\widetilde{Q}_I) = 1$. Apply this to the characteristic map

$$c : \mathcal{SP}(X) \longrightarrow A^*(G)$$

given for a homogeneous symmetric polynomial f by

$$c(f) = \sum_{I,|I|=\deg f} \partial'_{w_I}(f)\sigma(I)$$

(the notation as in Section 1). We infer

$$c(\widetilde{Q}_I) = \sigma(I) \tag{20}$$

which is the Giambelli-type formula for the Lagrangian Grassmannian, originally obtained in [P, Sect.6] by a different method. Note that this purely algebraic operator proof of (20) gives us, after invoking the Pieri-type formula for Hall-Littlewood functions from [Mo], a Pieri-type formula for Lagrangian Schubert classes obtained originally, by other methods, in [H-B] (see also [P-R1] for a divided difference approach).

2. The key formula admits the following interpretation in terms of Gysin maps. Let X be a scheme and V a rank $2n$ vector bundle on X equipped with a nondegenerate symplectic form. Let V_n be a rank n isotropic subbundle of V. By $p : \mathcal{F} \to X$ we denote the flag bundle parametrizing pairs $A \subset B$ of subbundles of V such that rank $A = n-1$, rank $B = n$, $A \subset V_n$ and B is Lagrangian. Let $C \subset D$ be the tautological bundles of respective ranks $n-1$ and n, on $\mathcal{F}$. Set $S = D/C$.

Suppose $n \geq k > 0$. Let IkJ be a strict partition and H a strict partition not containing k. Then the key formula says:

$$p_*\big(c_1(S^\vee)^{n-k} \cdot \widetilde{Q}_{IkJ}(D^\vee)\big) = (-1)^{l(I)}\widetilde{Q}_{IJ}(V_n^\vee) \tag{21}$$

and

$$p_*\big(c_1(S^\vee)^{n-k} \cdot \widetilde{Q}_H(D^\vee)\big) = 0\,.$$

For $k = n$, this result agrees with Proposition 3.1 in [P-R2], which was proved using geometric arguments. Does it exist a geometric proof for the key formula in general? One more comment may be here in order. Note that the arguments in both sides of (21) are different whereas in the key formula, they are the same. As a matter of fact, in the above translation of the key formula, we have used a result from [P-R2, Sect.8] allowing us to make an appropriate exchange of the sequences of the Chern roots of the bundles involved under the action of p_*.

3. The results of [P-R2] and the present paper combined with some results of [F1,2] allow us to generalize the formulas for the Schubert classes in Lagrangian Grassmann bundles from [P-R2] to the case of arbitrary Lagrangian flag bundles and flagged Lagrangian degeneracy loci. Here, for simplicity, we describe just the difference between our formulas and the ones obtained by Fulton in [F1,2]. Let $\mathcal{X}_w$ $(w \in W_n)$ be the degeneracy locus defined as in [F1, p.243] with the help of a nondegenerate symplectic form on a rank $2n$ vector bundle V which is equipped with two complete filtrations:

$$E_1 \subset E_2 \subset \ldots \subset E_n \subset V,$$

$$F_1 \subset F_2 \subset \ldots \subset F_n \subset V$$

by isotropic subbundles. Then, it follows from Proposition 2.4 of the present paper (or its geometric counterpart in [P-R2]), and the results of [F1,2], that the fundamental class of $\mathcal{X}_w$ is evaluated by the expression given in [F1, Theorem (C_n) on p.243], where instead of $F(X,Y)$ (denoted by $\triangle$ in loc.cit.) one uses $\widetilde{Q}(X,Y)$. More precisely, if the ambient variety is Cohen-Macaulay and $\mathcal{X}_w$ is of pure codimension $n^2 - l(w)$, then

$$[\mathcal{X}_w] = \partial_{w^{-1}}\Big(\prod_{i+j\le n}(x_i - y_j)\widetilde{Q}(X,Y)\Big) \cap [X],$$

where we specialize $x_i := -c_1(E_i/E_{i-1})$ and $y_i := -c_1(F_{n+1-i}/F_{n-i})$ for $i = 1, \ldots, n$.

We point out this difference between the approach of [F1,2] and ours because it appears that for numerical computations (e.g. with the help of ACE of [V1,2]) the reproducing kernel $\widetilde{Q}(X,Y)$ is more economical than $F(X,Y)$.

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A.L. : Institut Gaspard Monge, Université de Marne-la-Vallée, 2 rue de la Butte-Verte, 93166 Noisy-le-Grand cedex, France

P.P. : Mathematical Institute of Polish Academy of Sciences, Chopina 12, 87-100 Toruń, Poland

J.R. : Mathematical Institute of Polish Academy of Sciences, Śniadeckich 8, 00-950 Warszawa, Poland