Multiplying Schubert classes

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Abstract

We show how to compute the structure constants for cohomological multiplication of Schubert classes by exploiting the action of the Weyl group and that of BGG-operators, on the cohomology ring of a flag variety. We illustrate this method with simple proofs of the Chevalley and Pieri formulas.

1 Introduction

One of the main problems of Schubert calculus on flag varieties (or generalized flag varieties G/B) is to give expressions for the structure constants for the cohomological multiplication of Schubert classes. The *main problem* is to describe these structure constants as *the cardinalities of some sets*, but also "closed" formulas for the structure constants are of some interest. This is a classical topic starting with the work of Schubert, Giambelli, Pieri, Lesieur, Hodge-Pedoe, Littlewood-Richardson [1], Borel, Chevalley, Monk, and Horrocks, and continuing in recent years with the work of Bernstein-Gelfand-Gelfand, Demazure, Koch, Lascoux-Schützenberger, Kleiman-Laksov, Stoll, Carrell, Kostant-Kumar, Hiller-Boe, Stembridge, Akyildiz, Sertöz, Fulton, Pragacz-Ratajski, Bergeron-Sottile, Knutson, Vakil, Buch-Kresch-Tamvakis, Duan, and Gatto – to mention a few. We do not attempt to survey this activity here, but in the bibliographical references the reader may find a vast discussion of the structure constants.

The purpose of this note is to give a closed formula (Theorem 1) for these constants. This formula evolved from a sequence of papers [24], [25], and [26] (see also [23] and [9]). The first main tool that we use is the action of the Weyl group on $H^\bullet(G/B, \mathbf{Q})$ expressed in terms of Schubert classes. The second main tool is the theory of BGG-operators acting as skew derivations on $H^\bullet(G/B, \mathbf{Q})$.

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[1] The famous combinatorial Littlewood-Richardson rule governing the multiplication of Schubert classes on Grassmannians, was found, in fact, in a parallel context of representation theory. The same remark applies to the contribution of Stembridge.

These fundamental tools were developed mainly by Bernstein-Gelfand-Gelfand [1] and Demazure [6], [7] in the 70's, as a continuation of the work of Borel [2].

We illustrate our method with short proof the Chevalley formula (Theorem 2), and transparent, purely algebro-combinatorial proof of the classical Pieri formula (Theorem 3). (These proofs were mentioned in [11], p.122 and [26], p.50, respectively.)

Background. The content of this note was obtained in the 90's and has not been written up until now.[2] Following an encouragement of Michel Brion, we have decided to publish it now because of an increasing interest in the structure constants. The story told here is closely related to the lecture notes by Brion [3], Buch [4], Duan [8], and Tamvakis [27] in the present volume. This is, in fact, the main reason for the appearance of this note here.

2 Characteristic map and BGG-operators

A general reference for group-theoretic notions used in this note is [17].

Let G be a semisimple algebraic group and $B \subset G$ a Borel subgroup.

Let X be a variety on which B acts freely (from the right). Suppose that the quotient X/B exists so that $p : X \to X/B$ is a principal B-bundle. On the other hand, let $\mu : B \to GL(V)$ be a linear representation. We denote by $\mathcal{L}_\mu$ the vector bundle $X \times^B V$ that is the quotient of $X \times V$ by the equivalence relation

$$(x, v) \sim \big(xb, \mu(b)^{-1}v\big),$$

where $x \in X$, $b \in B$, and $v \in V$. Equivalently, if U is an open subset of X/B, then $\Gamma(U, \mathcal{L}_\mu)$ is the set of morphisms $\varphi : p^{-1}(U) \to V$ such that $\varphi(xb) = \mu(b)^{-1}\varphi(x)$.

In particular, with any character χ of B (that is, a homomorphism of B into the multiplicative group) there is associated a line bundle $\mathcal{L}_\chi$; this induces a homomorphism of groups $X^*(B) \to \mathrm{Pic}(X/B)$, where $X^*(B)$ denotes the group of characters of B.

Composing this homomorphism with the homomorphism of the first Chern class from $\mathrm{Pic}(X/B)$ to $H^2(X/B, \mathbf{Z})$, one gets a homomorphism from $X^*(B)$ to $H^2(X/B, \mathbf{Z})$, which extends to a homomorphism of graded rings

$$c : S^\bullet\big(X^*(B)\big) \to H^\bullet(X/B, \mathbf{Z})$$

from the symmetric algebra of the $\mathbf{Z}$-module $X^*(B)$ to the cohomology ring of X/B; this homomorphism is called the *characteristic map* of the fiber bundle $p : X \to X/B$. In this note, $S = \oplus S^k$ will denote the symmetric algebra $S^\bullet\big(X^*(B)\big) = \oplus S^k\big(X^*(B)\big)$.

[2]The material of this note was presented at various Impanga seminars, and, e.g., at the Littelmann-Mathieu seminar in Strasbourg (December, 1994), at the Summer School "Schubert Varieties" in Thurnau (June, 1995), and at the topology seminar at CAS in Beijing (June, 2002).

Choose a maximal torus $T \subset B$ with the Weyl group $W = N_G(T)/T$ of (G,T). Then W acts on the group of characters $X^*(T)$ of T, and since $X^*(B) = X^*(T)$, this induces an action of W on S.

The root system of (G,T) is denoted by R; the set R^+ of positive roots consists in the opposites of roots of (B,T). Let $\Delta \subset R^+$ be the associated basis of R. The Weyl group W is generated by simple reflections, i.e. by the reflections associated with the elements of Δ. For any root $\alpha \in R$, we denote by s_α the reflection associated with α. The reflection s_α can be realized as a linear endomorphism of the Euclidean space $X^*(T) \otimes \mathbf{R}$, equipped with a W-invariant inner product $(\ ,\ )$. We have $s_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha$, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

By a *reduced decomposition* of an element $w \in W$ we understand a presentation $w = s_{\alpha_1} \cdots s_{\alpha_l}$ where all $\alpha_p \in \Delta$, and l is the smallest number occurring in such a presentation, called the *length* of w and denoted $l(w)$.

By w_0 we denote the longest element of W, the unique element of W with length equal to the cardinality of R^+.

We shall need the following "BGG-operators" A_w, $w \in W$, acting on the ring S (cf. [1], [6], and [7]).

Definition 1 *Given a root α and $f \in S$, we set*

$$A_\alpha(f) := \frac{f - s_\alpha(f)}{\alpha}\,.$$

The operator A_α is a well defined (group) endomorphism on S lowering the degree by 1. Note that $A_\alpha(f) = (\alpha^\vee, f)$ for $f \in S^1$; this will be used in the proof of Theorem 2.

We now record (cf. [1], Theorem 3.4 and [6], Théorème 1):

Lemma 1 *If $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ are simple roots such that*

$$s_{\alpha_1} \cdots s_{\alpha_k} = s_{\beta_1} \cdots s_{\beta_k}$$

are two reduced decompositions, then

$$A_{\alpha_1} \cdots A_{\alpha_k} = A_{\beta_1} \cdots A_{\beta_k}\,.$$

Thus for $w \in W$, given its reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_k}$, the operator

$$A_w := A_{\alpha_1} \cdots A_{\alpha_k}$$

is well-defined (i.e. doesn't depend on a reduced decomposition of w).

The following result says how the BGG-operators act on products (cf., e.g., [6], Eq. (6), p.289):

Lemma 2 *We have for $f, g \in S$ and a simple root α,*

$$A_\alpha(fg) = A_\alpha(f)g + s_\alpha(f)A_\alpha(g)\,. \tag{1}$$

Geometric interpretations of BGG-operators are related to correspondences in flag bundles (cf., e.g., [11], Chap.2 and 6), and Gysin maps for *Bott-Samelson schemes*. These schemes are described in the notes by Brion [3] and Duan [8] in the present volume. This last aspect of BGG-operators is discussed in [11], Appendix C.

The reader may also consult [15] for a detailed treatment of the so-called Schubert calculus of the coinvariant algebra, that is based on BGG-operators.

3 Structure constants for Schubert classes

In the geometry of flag manifolds G/B a large role is played by the *Schubert cells* BwB/B and their closures called *Schubert varieties*. We set $X^w := \overline{Bw_0wB/B}$. The cohomology class $[X^w]$ of X^w lies in $H^{2l(w)}(G/B, \mathbf{Z})$. The Schubert cells form a cellular decomposition of G/B, so the classes $[X^w]$ form an additive basis for the cohomology.

Our goal, in this section, is to give a closed formula for the constants c_{wv}^u, appearing in the decomposition of the product

$$[X^w] \cup [X^v] = \sum_u c_{wv}^u [X^u] \tag{2}$$

of Schubert classes.

We shall need a couple of tools that we describe now.

The characteristic map $c : S \to H^\bullet(G/B, \mathbf{Z})$ of the fibration $G \to G/B$ is usually called the *Borel characteristic map*. Its kernel is generated by positive degree W-invariants, and $c \otimes \mathbf{Q}$ is surjective (cf. [2]), so that the cohomology ring $H^\bullet(G/B, \mathbf{Q})$ is identified with the quotient of $S \otimes \mathbf{Q}$ modulo the ideal generated by positive degree W-invariants. By combining this last property with Lemma 2, we infer that the BGG-operators induce – via the characteristic map – operators A_w on $H^\bullet(G/B, \mathbf{Q})$ lowering the degree by $2l(w)$.

In particular, for $a, b \in H^\bullet(G/B, \mathbf{Q})$ and a simple root α, we have

$$A_\alpha(a \cup b) = A_\alpha(a) \cup b + s_\alpha(a) \cup A_\alpha(b)\,. \tag{3}$$

Iterations of this equation will play an important role in the present section and the next one.

Note also that the action of W on S induces – via the characteristic map – an action of W on $H^\bullet(G/B, \mathbf{Q})$. (This action will be described below in terms of Schubert classes – cf. Lemma 4.)

We record the following equation relating three "heroes" of the present note: the characteristic map, BGG-operators, and Schubert classes (cf. [7], Section 4 and [1], Section 4): for $f \in S^k$, in $H^\bullet(G/B, \mathbf{Z})$ we have

$$c(f) = \sum_{l(w)=k} A_w(f)[X^w]\,. \tag{4}$$

This equation is closely related to the question of finding polynomial representatives of Schubert classes – a problem that we do not address in the present note (cf. [11] for a discussion of this issue).

The next result says how the operators A_w act on Schubert classes (cf. [1], Theorem 3.14 (i)):

Lemma 3 *For $l(vw^{-1}) = l(v) - l(w)$, we have*

$$A_w([X^v]) = [X^{vw^{-1}}]\,, \tag{5}$$

and in the opposite case, $A_w([X^v]) = 0$.

We have also the following formula for the action of a simple reflection on a Schubert class (cf. [1], Theorem 3.12 (iv) and [7], Proposition 3):

Lemma 4 *For a simple root α and $w \in W$,*

$$s_\alpha([X^w]) = [X^w] \qquad \text{if } \; l(ws_\alpha) = l(w) + 1\,; \tag{6}$$

$$s_\alpha([X^w]) = -[X^w] - \sum (\beta^\vee, \alpha)[X^{ws_\alpha s_\beta}] \qquad \text{if } \; l(ws_\alpha) = l(w) - 1\,, \tag{7}$$

where the sum is over all positive roots $\beta \neq \alpha$ such that $l(ws_\alpha s_\beta) = l(w)$.

We now proceed towards computing the structure constants c^u_{wv}. By combining Equations (2) and (5), we can express the coefficient c^v_{wv} as follows:

$$c^u_{wv} = A_u([X^w] \cup [X^v])\,. \tag{8}$$

Suppose that $l(w) = k$ and $l(v) = l$. Take a reduced decomposition of u:

$$u = s_{\alpha_1} \cdots s_{\alpha_{k+l}}\,.$$

Iterating (3) we obtain

$$c^u_{wv} = A_{\alpha_1} \cdots A_{\alpha_{k+l}}([X^w] \cup [X^v]) = \sum A_I([X^w]) \cup A^I_\alpha([X^v])\,,$$

where the sum is over all subsequences $I = (i_1 < \cdots < i_k) \subset \{1, 2, \ldots, k+l\}$, $A_I := A_{\alpha_{i_1}} \cdots A_{\alpha_{i_k}}$, and A^I_α is obtained by replacing in $A_{\alpha_1} \cdots A_{\alpha_{k+l}}$ each A_{α_i} by s_{α_i} for $i \in I$. By Lemma 3 we infer the following result.

Theorem 1 *With the above notation,*

$$c^u_{wv} = \sum A^I_\alpha([X^v])\,, \tag{9}$$

where the sum runs over all I such that $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$ is a reduced decomposition of w.

Applying successively to the summands in (9) the formulas (5), (6), and (7), we get an expression for the constants c^u_{wv}.

Recall the following formula for multiplication by the classes of Schubert divisors in $H^\bullet(G/B, \mathbf{Z})$:

Theorem 2 *(Chevalley, [5]) For $w \in W$, and a simple root α,*

$$[X^w] \cup [X^{s_\alpha}] = \sum (\beta^\vee, \omega_\alpha)[X^{ws_\beta}]\,, \tag{10}$$

where β runs over positive roots such that $l(ws_\beta) = l(w) + 1$ and ω_α denotes the fundamental weight associated with α.

Proof. We prove Equation (10) using Theorem 1. By the definition of a fundamental weight, we have for $\gamma \in \Delta$, $(\omega_\alpha, \gamma^\vee) = \delta_{\alpha\gamma}$, the Kronecker delta. This implies that $A_\gamma(\omega_\alpha) = \delta_{\alpha\gamma}$, and using Equation (4) we get $c(\omega_\alpha) = [X^{s_\alpha}]$. Fix $w \in W$ and pick $f \in S \otimes \mathbf{Q}$ such that $(c \otimes \mathbf{Q})(f) = [X^w]$. Then in $H^\bullet(G/B, \mathbf{Q})$,

$$[X^{s_\alpha}] \cup [X^w] = (c \otimes \mathbf{Q})(\omega_\alpha \cdot f)\,, \tag{11}$$

and by Theorem 1 we obtain that the coefficient of the Schubert class $[X^u]$ in the expansion of (11) can be evaluated as the sum (9) with $[X^v]$ replaced by ω_α.

Take a reduced decomposition $u = s_{\alpha_1} \cdots s_{\alpha_h}$. By the "Exchange Condition" (cf. [18], pp.14–15), a reduced decomposition for w can be gotten from the one for u by omitting one simple reflection if $u = ws_\beta$ for some (positive) root β. Conversely, if $w = s_{\alpha_1} \cdots s_{\alpha_{p-1}} s_{\alpha_{p+1}} \cdots s_{\alpha_h}$, then

$$w^{-1}u = s_{\alpha_h} \cdots s_{\alpha_p} \cdots s_{\alpha_h} = s_\beta$$

for $\beta = s_{\alpha_h} \cdots s_{\alpha_{p+1}}(\alpha_p)$. The root β is positive by, e.g., [15], Proposition 3.6 because $s_{\alpha_h} \cdots s_{\alpha_1}$ is reduced.

Since the omitted simple reflection is unique, the looked at sum (9) has exactly one summand

$$s_{\alpha_1} \cdots s_{\alpha_{p-1}} A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha) = A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha)\,.$$

The latter expression equals $(\beta^\vee, \omega_\alpha)$ because $A_{\alpha_p}(g) = (\alpha_p^\vee, g)$ for $g \in S^1$, the inner product (,) is W-invariant, and $s_{\alpha_h} \cdots s_{\alpha_{p+1}}(\alpha_p^\vee) = \beta^\vee$. This proves the theorem.

For an algebraic proof in the SL_n-case[3] along these lines, see [20]. A geometric proof in the SL_n-case is given in the notes by Brion [3].

The same method works for all spaces G/P, where P is a parabolic subgroup of G. Let θ be a subset of Δ and let W_θ be the subgroup of W generated by $\{s_\alpha\}_{\alpha\in\theta}$. We set $P_\theta := BW_\theta B$. Denote by W^θ the set

$$W^\theta := \{w \in W : \ l(ws_\alpha) = l(w) + 1 \quad \forall \alpha \in \theta\}.$$

[3]This case was obtained by Monk [21] using different methods.

This last set is the set of *minimal length left coset representatives* of W_θ in W.

The projection $G/B \to G/P_\theta$ induces an injection

$$H^\bullet(G/P_\theta, \mathbf{Z}) \hookrightarrow H^\bullet(G/B, \mathbf{Z})$$

which additively identifies $H^\bullet(G/P_\theta, \mathbf{Z})$ with $\bigoplus_{w \in W^\theta} \mathbf{Z}[X^w]$. Multiplicatively, $H^\bullet(G/P_\theta, \mathbf{Q})$ is identified with the ring of invariants $H^\bullet(G/B, \mathbf{Q})^{W_\theta}$. We refer for details to [1], Sect. 5.

The restriction $c : S^{W_\theta} \to H^\bullet(G/P_\theta, \mathbf{Z})$ of the Borel characteristic map satisfies, for any W_θ-invariant f from S^k, the following equation in $H^\bullet(G/P_\theta, \mathbf{Z})$:

$$c(f) = \sum_{\substack{w \in W^\theta \\ l(w) = k}} A_w(f)[X^w]\,. \tag{12}$$

For maximal parabolic subgroups P of the symplectic and orthogonal groups, this method led to combinatorial expressions for the structure constants in the products of arbitrary Schubert classes by some "special Schubert classes" in $H^\bullet(G/P, \mathbf{Z})$ (cf. [24], [25], [26], [9], and [23]).

Remark 1 *Equation (3) is often called the "Leibniz-type formula". Kostant and Kumar [19] discovered independently, in the context of the "nil Hecke ring", that the structure constants can be computed via the iteration of the Leibniz-type formula.*

4 A combinatorial proof of the Pieri formula

In this section, we give a proof of the classical Pieri formula for the Grassmannian $\mathrm{Gr}(n, m)$ of n-dimensional subspaces in $\mathbf{C}^m$ via the above method. In fact, there are two Pieri formulas: for multiplication by the Chern classes[14] of the tautological subbundle on $\mathrm{Gr}(n, m)$, and for multiplication by the Chern classes of the tautological quotient bundle on $\mathrm{Gr}(n, m)$. The latter version appears more often mainly because the Chern classes of the tautological quotient bundle enjoy a simple interpretation in terms of the classical "Schubert conditions": the kth Chern class is represented by the locus of all n-planes in $\mathbf{C}^m$ which have positive dimensional intersection with a fixed $(m - n - k + 1)$-plane in $\mathbf{C}^m$. By passing to the dual Grassmannian, we see that both formulas are, in fact, equivalent. We shall treat in detail the latter case. We also make a link with the ring of symmetric functions, known since Giambelli (cf. [12] and [13]).

For the remainder of this note, we set $q := m - n$.

In the following, I, J will denote *strict* partitions contained in the partition $(m, m-1, \dots, q+1)$ with exactly n parts [4]. (We identify partitions with their Young diagrams, as is customary.) Note that such partitions contain the "upper-left triangle"

$$\delta = (n, n-1, \dots, 1)\,.$$

[4]In other words, $I = (i_1, \dots, i_n)$ where $m \geq i_1 > \dots > i_n \geq 1$.

On the other hand, λ, μ will denote "ordinary" partitions contained in (q^n). In fact, there is a bijection between these two sets: with I, we associate λ defined by $\lambda_p = i_p - n + p - 1$ for $p = 1, \dots, n$.

Also, we associate with I the following permutation w_I in the symmetric group S_m:

$$w_I = \cdots (s_{q-\lambda_3+3} \cdots s_{q+1} s_{q+2})(s_{q-\lambda_2+2} \cdots s_q s_{q+1})(s_{q-\lambda_1+1} \cdots s_{q-1} s_q) . \quad (13)$$

It is easy to see, that the right-hand side of (13) gives a reduced decomposition of w_I.

Take for example $m = 7$, $n = 3$, and $I = (6,4,3)$. Then $\lambda = (3,2,2)$ and $w_I = s_5 s_6 s_4 s_5 s_2 s_3 s_4$ which is the permutation $[1,3,6,7,2,4,5]$ (we display a permutation as the sequence of its consecutive values).

In general, for $I = (m \geq i_1 > \cdots > i_n \geq 1)$, we have in S_m,

$$w_I = [j_1 < \cdots < j_q \,, m+1-i_n < \cdots < m+1-i_1] ,$$

where $j_1, \dots, j_q$ are uniquely determined by I.

Let $B \subset SL_m(\mathbf{C})$ be the Borel group of lower triangular matrices. Using the notation of the previous section, we set $P = P_\theta$, where θ is obtained by omitting the simple root $\varepsilon_n - \varepsilon_{n+1}$ in the basis $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{m-1} - \varepsilon_m$ of the root system of type (A_{m-1}):

$$\{\varepsilon_i - \varepsilon_j \mid i \neq j\} \subset \oplus_{i=1}^m \mathbf{R}\varepsilon_i .$$

We have an identification $SL_m(\mathbf{C})/P = \mathrm{Gr}(n,m)$. We set $X^I := \overline{Bw_0 w_I P/P}$, where $w_0 = [m, m-1, \dots, 1]$, and $X^\lambda := X^I$ for λ associated with I as above. Note that $[X^\lambda] \in H^{2|\lambda|}(\mathrm{Gr}(n,m), \mathbf{Z})$, where $|\lambda|$ denotes the sum of the parts of λ.

Denote by $(k+)$ the strict partition $(k+n, n-1, \dots, 1)$, so that its associated λ is a one-row partition (k).

We want to compute the coefficients c_J in the expansion:

$$[X^I] \cup [X^{(k+)}] = \sum_J c_J [X^J] .$$

Set $x_i := -\varepsilon_{m+1-i}$ for $i = 1, \dots, m$, so that $c(x_1), \dots, c(x_q)$ are the Chern roots of the tautological quotient bundle on on $\mathrm{Gr}(n,m)$. The Borel characteristic map allows us to treat $H^\bullet(\mathrm{Gr}(n,m), \mathbf{Z})$ as a quotient of the ring S' of polynomials symmetric in $x_1, \dots, x_q$ and in $x_{q+1}, \dots, x_m$. (Recall that for type (A_{m-1}), the characteristic map is surjective without tensoring by $\mathbf{Q}$.) The operators s_α and A_α indexed by the simple roots corresponding to P are induced by the following operators s_i and A_i, $i = 1, \dots, q-1, q+1, \dots, m-1$, on S'. The operator s_i interchanges x_i with x_{i+1}, leaving other variables invariant, and A_i is the ith simple (Newton's) divided difference ∂_i: for $f \in S'$,

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} .$$

The operator A_w on S', in this case ($w \in S_m$), will be denoted by ∂_w, as is customary.

Let $e_k = e_k(x_1, \ldots, x_q)$ be the kth elementary symmetric polynomial in $x_1, \ldots, x_q$. We now record:

Lemma 5 *For any $k = 1, \ldots, q$, the following equation holds in $H^\bullet(\mathrm{Gr}(n,m), \mathbf{Z})$:*

$$c(e_k) = [X^{(k)}]\,.$$

Proof. By virtue of Equation (12), it suffices to show that

$$\partial_w(e_k) = 0 \ \text{ unless } \ w = w_{(k+)}\,, \quad \text{and} \ \ \partial_{w_{(k+)}}(e_k) = 1\,.$$

Note that $w_{(k+)} = s_{q-k+1} \cdots s_{q-1} s_q$. The displayed assertion follows by induction on the number of variables, by invoking the following properties of ∂_i:

$$\partial_i(e_r(x_1, \ldots, x_j)) \neq 0 \quad \text{only if} \ \ j = i\,,$$

$$\partial_i(e_r(x_1, \ldots, x_i)) = e_{r-1}(x_1, \ldots, x_{i-1})\,.$$

The lemma is proved.

This lemma says that $X^{(k)}$ represents the kth Chern class of the tautological rank q quotient bundle on $\mathrm{Gr}(n,m)$.

Number the successive columns of J from left to right with $m, m-1, \ldots, 1$, the successive rows from top to bottom with $1, \ldots, n$, and use the matrix coordinates for boxes in J.

Let J^* be the effect of subtracting the triangle δ from J. In the following, D will denote a subset of J^*.

Definition 2 *Read J^* row by row from left to right and from top to bottom. Every box from D (resp. from $J^* \setminus D$) in column i gives us s_i (resp. ∂_i). Then ∂_J^D is the composition of the resulting s_i's and ∂_i's (the composition written from right to left), and r_D is the word obtained by erasing all the ∂_i's from ∂_J^D.*

In particular, r_{J^*} is the reduced decomposition (13) of w_J, and $\partial_J^\emptyset = \partial_{w_J}$.

Take for example $m = 8$, $n = 3$, and $J = (8,6,5)$. In the following picture, "$\bullet$" depicts a box in D and "$\circ$" stands for a box in $J^* \setminus D$. Moreover, the row-numbers and column-numbers are displayed.

$$\begin{array}{ccccccccc}
 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
1 & \times & \times & \times & \bullet & \bullet & \bullet & \bullet & \bullet \\
2 & \times & \times & \bullet & \circ & \circ & \circ & & \\
3 & \times & \bullet & \circ & \bullet & \bullet & & &
\end{array}$$

Then we have

$$\partial_J^D = s_4 s_5 \partial_6 s_7 \partial_3 \partial_4 \partial_5 s_6 s_1 s_2 s_3 s_4 s_5 \qquad \text{and} \qquad r_D = s_4 s_5 s_7 s_6 s_1 s_2 s_3 s_4 s_5\,.$$

If r_D is a reduced decomposition of w_I, then D is a disjoint union of the following "p-ribbons". For fixed $p = 1, \ldots, n$, the *p-ribbon* consists of all boxes of D giving rise to those s_i (in r_D) which "transport" the item "$m+1-i_p$" from its position in $[1, 2, \ldots, m]$ to its position in the sequence w_I.

In the above example, for $I = (7, 5, 2)$, the 1-ribbon consists of the dots in the first row, the 2-ribbon is $\{(3,4), (3,5), (2,6)\}$, and the 3-ribbon is $\{(3,7)\}$.

It can happen that some p-ribbon is empty. Suppose that p is such that the p-ribbon is not empty (this is equivalent to the fact that the box $(p, n+p-1)$ belongs to the p-ribbon). Then the column-numbers of boxes in the p-ribbon are $m+1-i_p, \ldots, n+p-2, n+p-1$, and their row-numbers weakly increase while reading D from left to right and from top to bottom.

By Theorem 1 and Lemma 5, we have

$$c_J = \sum \partial_J^D(e_k)\,, \tag{14}$$

where the sum is over all subsets $D \subset J^*$ such that r_D is a reduced decomposition of w_I.

We need the following lemma.

Lemma 6 *If there are boxes (i,j) and $(i-1, j-1)$ in $J^* \setminus D$, then $\partial_J^D(e_k) = 0$.*

Proof. We set $E := \prod_{i=1}^n (1 + x_i)$ and we shall prove that $\partial_J^D(E) = 0$. To compute with compositions of the s_i's and ∂_i's in ∂_J^D, it is handy to introduce the following more general functions. For $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \{0,1\}^m$, we set $E_{\mathbf{a}} := \prod_{i=1}^m (1 + a_i x_i)$, so that $E = E_{(1,\ldots,1,0,\ldots,0)}$ with q 1's. We have:

$$s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'} \quad \text{where } \mathbf{a}' = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_m)\,; \tag{15}$$

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \quad \text{if } a_{i+1} = a_i + d\,, \tag{16}$$

where $\mathbf{a}' = (a_1, \ldots, 0, 0, \ldots, a_n)$ is $\mathbf{a}$ with a_i, a_{i+1} replaced by zeros. Using (15) and (16), we see that the operator ∂_j in ∂_J^D, corresponding to the box (i,j) "kills" the function $E_{\mathbf{a}}$ that has been obtained by applying the previous operators s_r and ∂_r (in ∂_J^D) to E. This proves the lemma.

It follows from this lemma that there is *at most one* $D \subset J^*$ such that r_D is a reduced decomposition of w_I and $\partial_I^D(e_k) \neq 0$, namely $D = I^*$. (Indeed, the p-ribbon must exactly coincide with the pth row of I^*.) In other words, the sum in (14) has at most one summand.

Second, applying Lemma 6 again, we see that $D = I^*$ gives a non-zero contribution to the sum in (14) iff $J \setminus I$ is a *horizontal strip with pairwise separated rows*[5]. In this case, using (15) and (16), we obtain $\partial_J^{I^*}(e_k) = 1$.

We rewrite the outcome of the above considerations in terms of Schubert classes $[X^\lambda] \in H^\bullet(\mathrm{Gr}(n,m), \mathbf{Z})$ in part (i) of the following theorem. Part (ii) follows from part (i) by passing to the dual Grassmannian.

[5] Recall that a *horizontal strip* is a skew diagram with at most one box in each column, and a *vertical strip* is a skew diagram with at most one box in each row.

Theorem 3 *(Pieri, [22]) (i) For any partition* $\lambda \subset (q^n)$ *and* $k = 1, \ldots, q$,

$$[X^\lambda] \cup [X^{(k)}] = \sum_\mu [X^\mu] \,, \tag{17}$$

where $|\mu| = |\lambda| + k$ *and* $\mu \setminus \lambda$ *is a horizontal strip.*
(ii) For any partition $\lambda \subset (q^n)$ *and* $p = 1, \ldots, n$,

$$[X^\lambda] \cup [X^{(1,\ldots,1)}] = \sum_\mu [X^\mu] \,, \tag{18}$$

where 1 *appears* p *times,* $|\mu| = |\lambda| + p$ *and* $\mu \setminus \lambda$ *is a vertical strip.*

For example, we have in $H^\bullet(\mathrm{Gr}(3,8), \mathbf{Z})$:

$$[X^{(4,2)}] \cup [X^{(3)}] = [X^{(5,4)}] + [X^{(5,3,1)}] + [X^{(5,2,2)}] + [X^{(4,4,1)}] + [X^{(4,3,2)}] \,.$$

Remark 2 *There are several (really) different proofs of the Pieri formula. We do not attempt to make a survey here. The proof that appears most often in monographs is based on studying the triple intersection of general translates of Schubert varieties. This proof appeared originally in Hodge's paper [16]. Cf. also [10], §9.4 , and the notes of Brion [3], where this proof is discussed in the context of Richardson varieties.*

Remark 3 *The Schubert classes* $[X^{(k)}]$ *and* $[X^{(1,\ldots,1)}]$ *are often called "special". These classes and the "special Schubert classes" in [24], [25], [26], and [9] have the following property: the corresponding* $w \in W$ *has a unique reduced decomposition. This seems to be a proper group-theoretic characterization of a "special Schubert class", and was also remarked by Kirillov and Maeno.*

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