$\mathbb{A} = (a_1, \ldots, a_m)$, $\mathbb{B} = (b_1, \ldots, b_n)$ two sequences of elements in a commutative ring

$$R(\mathbb{A}, \mathbb{B}) := \prod_{i,j}(a_i - b_j)$$

E.g. $R(x, \mathbb{A})$ = polynomial in x with roots in $\mathbb{A}$

SYLVESTER'S DOUBLE SUMS

$\mathbb{A} = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_n)$ two sequences of variables

x an extra variable

Let $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$ be two integers.

$$\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; x) :=$$

$$\sum_{\mathbb{A}' \subset \mathbb{A}} \sum_{\mathbb{B}' \subset \mathbb{B}} R(x, \mathbb{A}') R(x, \mathbb{B}') \frac{R(\mathbb{A}', \mathbb{B}') R(\mathbb{A} \smallsetminus \mathbb{A}', \mathbb{B} \smallsetminus \mathbb{B}')}{R(\mathbb{A}', \mathbb{A} \smallsetminus \mathbb{A}') R(\mathbb{B}', \mathbb{B} \smallsetminus \mathbb{B}')},$$

the sum over all subsets $\mathbb{A}'$ of cardinality p

and $\mathbb{B}'$ of cardinality q.

Example. *For* $m = 3$, $n = 2$, $\quad \mathrm{Sylv}^{1,1}(\mathbb{A}, \mathbb{B}; x) =$

$$\frac{(a_1 - b_1)(a_2 - b_2)(a_3 - b_2)}{(a_1 - a_2)(a_1 - a_3)(b_1 - b_2)}(x - a_1)(x - b_1)$$

$$+\frac{(a_1 - b_2)(a_2 - b_1)(a_3 - b_1)}{(a_1 - a_2)(a_1 - a_3)(b_2 - b_1)}(x - a_1)(x - b_2)$$

$$+\frac{(a_2 - b_1)(a_1 - b_2)(a_3 - b_2)}{(a_2 - a_1)(a_2 - a_3)(b_1 - b_2)}(x - a_2)(x - b_1)$$

$$+\frac{(a_2 - b_2)(a_1 - b_1)(a_3 - b_1)}{(a_2 - a_1)(a_2 - a_3)(b_2 - b_1)}(x - a_2)(x - b_2)$$

$$+\frac{(a_3 - b_1)(a_1 - b_2)(a_2 - b_2)}{(a_3 - a_1)(a_3 - a_2)(b_1 - b_2)}(x - a_3)(x - b_1)$$

$$+\frac{(a_3 - b_2)(a_1 - b_1)(a_2 - b_1)}{(a_3 - a_1)(a_3 - a_2)(b_2 - b_1)}(x - a_3)(x - b_2)\,.$$

This is equal to $\quad -2(x - b_1)(x - b_2).$

Example. *For* $m = n = 3$, $\quad \mathrm{Sylv}^{2,1}(\mathbb{A}, \mathbb{B}; x) =$

$$\frac{(a_1 - b_1)(a_2 - b_1)(a_3 - b_2)(a_3 - b_3)}{(a_1 - a_3)(a_2 - a_3)(b_1 - b_2)(b_1 - b_3)}(x - a_1)(x - a_2)(x - b_1)$$
$$+\frac{(a_1 - b_1)(a_3 - b_1)(a_2 - b_2)(a_2 - b_3)}{(a_1 - a_2)(a_3 - a_2)(b_1 - b_2)(b_1 - b_3)}(x - a_1)(x - a_3)(x - b_1)$$
$$+\frac{(a_2 - b_1)(a_3 - b_1)(a_1 - b_2)(a_1 - b_3)}{(a_2 - a_1)(a_3 - a_1)(b_1 - b_2)(b_1 - b_3)}(x - a_2)(x - a_3)(x - b_1)$$
$$+\frac{(a_1 - b_2)(a_2 - b_2)(a_3 - b_1)(a_3 - b_3)}{(a_1 - a_3)(a_2 - a_3)(b_2 - b_1)(b_2 - b_3)}(x - a_1)(x - a_2)(x - b_2)$$
$$+\frac{(a_1 - b_2)(a_3 - b_2)(a_2 - b_1)(a_2 - b_3)}{(a_1 - a_2)(a_3 - a_2)(b_2 - b_1)(b_2 - b_3)}(x - a_1)(x - a_3)(x - b_2)$$
$$+\frac{(a_2 - b_2)(a_3 - b_2)(a_1 - b_1)(a_1 - b_3)}{(a_2 - a_1)(a_3 - a_1)(b_2 - b_1)(b_2 - b_3)}(x - a_2)(x - a_3)(x - b_2)$$
$$+\frac{(a_1 - b_3)(a_2 - b_3)(a_3 - b_1)(a_3 - b_2)}{(a_1 - a_3)(a_2 - a_3)(b_3 - b_1)(b_3 - b_2)}(x - a_1)(x - a_2)(x - b_3)$$
$$+\frac{(a_1 - b_3)(a_3 - b_3)(a_2 - b_1)(a_2 - b_2)}{(a_1 - a_2)(a_3 - a_2)(b_3 - b_1)(b_3 - b_2)}(x - a_1)(x - a_3)(x - b_3)$$
$$+\frac{(a_2 - b_3)(a_3 - b_3)(a_1 - b_1)(a_1 - b_2)}{(a_2 - a_1)(a_3 - a_1)(b_3 - b_1)(b_3 - b_2)}(x - a_2)(x - a_3)(x - b_3)$$

This is equal to $\quad 2(x - a_1)(x - a_2)(x - a_3) + (x - b_1)(x - b_2)(x - b_3).$

MAIN THEOREM: *If $p+q<n$, then specializing $\mathbb{A}$ and $\mathbb{B}$ in the roots of the polynomials P and Q respectively, one has*

$$\mathrm{Sylv}^{p,q} = \pm \binom{p+q}{p} \mathcal{R}^{(p+q)}\,.$$

Example. *Let $m=5, n=4$. Then* $\mathrm{Sylv}^{2,1}(\mathbb{A},\mathbb{B};x) =$

$$\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_1-b_2)(b_1-b_3)(b_1-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_1)$$

$$+\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_2-b_1)(b_2-b_3)(b_2-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_2)$$

$$+\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_3-b_1)(b_3-b_2)(b_3-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_3)$$

$$+\frac{(a_1-b_4)(a_2-b_4)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_4-b_1)(b_4-b_2)(b_4-b_3)}$$
$$\times(x-a_1)(x-a_2)(x-b_4)$$

$+\ \ 9\times 4$ similar terms obtained by replacing (a_1,a_2) by (a_i,a_j) for $i<j$.

This is $3\mathcal{R}^{(3)}$ for the division of $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ by $(x-b_1)(x-b_2)(x-b_3)(x-b_4)$.

If $p = 0$ or $q = 0$, then double Sylvester sums

reduce to single ones:

$$\mathrm{Sylv}^{0,q}(\mathbb{A}, \mathbb{B}; x) = \sum_{\mathbb{B}' \subset \mathbb{B}} R(x, \mathbb{B}') \frac{R(\mathbb{A}, \mathbb{B} - \mathbb{B}')}{R(\mathbb{B}', \mathbb{B} - \mathbb{B}')} ,$$

the sum being over all subsets $\mathbb{B}'$ of cardinality q.

– the subresultant $\mathcal{R}^{(q)}$ associated with P and Q
(a result of Borchardt 1860).

Note that

$$R(\mathbb{A}, \mathbb{B} - \mathbb{B}') = \prod_{b \in \mathbb{B} \setminus \mathbb{B}'} P(b) .$$

For $q = n - 1$, one recovers Lagrange interpolation:

$$\pm \mathcal{R}_1 = \sum_{i=1}^{n} P(b_i) \prod_{j \neq i} \frac{x - b_j}{b_i - b_j} .$$

For example, for $n = 3$,

$$\mathcal{R}_1 = P(b_1) \frac{(x - b_2)(x - b_3)}{(b_1 - b_2)(b_1 - b_3)} + P(b_2) \frac{(x - b_1)(x - b_3)}{(b_2 - b_1)(b_2 - b_3)} + P(b_3) \frac{(x - b_1)(x - b_2)}{(b_3 - b_1)(b_3 - b_2)} .$$

A formal series: $\sum_{i=0}^{\infty} z^i S_i$,

$S_i \in$ a commutative ring, $S_0 = 1$.

For any $I = (i_1, \ldots, i_l) \in \mathbb{Z}^l$ one associates with the series
the *SCHUR FUNCTION* S_I:

$$S_I = |S_{i_k+k-h}|_{1 \leqslant h,k \leqslant l} .$$

For example,

$$S_{5,7,2,0,3} = \begin{vmatrix} S_5 & S_8 & S_4 & S_3 & S_7 \\ S_4 & S_7 & S_3 & S_2 & S_6 \\ S_3 & S_6 & S_2 & S_1 & S_5 \\ S_2 & S_5 & S_1 & 1 & S_4 \\ S_1 & S_4 & 1 & 0 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix} = S_{3,3,3,4,4}$$

We note that $S_{0,\ldots,0} = 1$

Given two alphabets $\mathbb{A}, \mathbb{B}$, we denote by $S_I(\mathbb{A} - \mathbb{B})$ the Schur functions associated with the rational series

$$\prod_{b \in \mathbb{B}} (1 - zb) / \prod_{a \in \mathbb{A}} (1 - za) .$$

One has

$$R(\mathbb{A}, \mathbb{B}) = S_{n^m}(\mathbb{A} - \mathbb{B}) .$$

E.g. $S_m(x - \mathbb{A}) =$ polynomial in x with roots in $\mathbb{A}$

$$= x^m + S_1(-\mathbb{A}) x^{m-1} + \cdots + S_m(-\mathbb{A})$$

We are given l virtual alphabets $\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}$, and a sequence $I \in \mathbb{Z}^l$.

MULTI-SCHUR FUNCTIONS:

$$S_I\left(\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}\right) = \left|S_{i_k+k-h}\left(\mathbb{A}^{(k)}\right)\right|_{1 \leqslant h,k \leqslant l}.$$

If

$$i_r = i_{r+1} = \ldots = i_t = i$$

are those parts of I that are equal to i, and that correspond to the argument $\mathbb{A}$, then we will write

$$S_{\ldots\,;i^{t-r};\,\ldots}(\ldots;\mathbb{A};\ldots).$$

For example, the multi-Schur function

$$S_{1,1,1,1,2,2,4,4,4,4,5}$$

associated with a sequence of virtual alphabets

$$\mathbb{A}^{(1)}, \mathbb{A}^{(2)}, \mathbb{A}^{(2)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(5)}, \mathbb{A}^{(5)}$$

is denoted by

$$S_{1;1^2;1;2^2;4^3;4;5}\left(\mathbb{A}^{(1)};\mathbb{A}^{(2)};\mathbb{A}^{(3)};\mathbb{A}^{(3)};\mathbb{A}^{(4)};\mathbb{A}^{(5)};\mathbb{A}^{(5)}\right).$$

Proposition. *For* $d = 0, 1, \ldots, n-1$,

$$\mathcal{R}^{(d)} = \pm S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$$

$$= \begin{vmatrix} S_1(\mathbb{B}-x) & \ldots & S_d(\mathbb{B}-x) & S_m(\mathbb{B}-\mathbb{A}) & \ldots & S_{m+n-d-1}(\mathbb{B}-\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{2-n}(\mathbb{B}-x) & \ldots & S_{d-n+1}(\mathbb{B}-x) & S_{m-n+1}(\mathbb{B}-\mathbb{A}) & \ldots & S_{m-d}(\mathbb{B}-\mathbb{A}) \end{vmatrix}.$$

(The determinant of order n.)

$$\mathcal{R}^{(d)} := \begin{vmatrix} S_0(-\mathbb{A}) & \ldots & S_{m+n-2d-2}(-\mathbb{A}) & S_{m+n-d-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{d-n+1}(-\mathbb{A}) & \ldots & S_{m-d-1}(-\mathbb{A}) & S_m(x-\mathbb{A}) \\ S_0(-\mathbb{B}) & \ldots & S_{m+n-2d-2}(-\mathbb{B}) & S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{d-m+1}(-\mathbb{B}) & \ldots & S_{n-d-1}(-\mathbb{B}) & S_n(x-\mathbb{B}) \end{vmatrix}$$

(The determinant of order $m+n-2d$.)

Both determinants as polynomials in x have degrees $< n$.

It suffices to show that the n specializations $x = b \in \mathbb{B}$

for both determinants, have the same values!

Using

$$S_i(\mathbb{A}-x) = S_i(\mathbb{A}) - xS_{i-1}(\mathbb{A})\,,$$

and performing elementary operations on the rows (or columns)

of the determinant defining a multi-Schur function, we arrive at:

TRANSFORMATION LEMMA: *For virtual alphabets* $\mathbb{A}$, $\mathbb{B}$, *... , and an alphabet* $\mathbb{X}$ *of cardinality 2,*

$$\begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}) & S_8(\mathbb{B}) & S_4(\mathbb{C}) & S_3(\mathbb{D}) & S_7(\mathbb{E}) \\ S_4(\mathbb{A}) & S_7(\mathbb{B}) & S_3(\mathbb{C}) & S_2(\mathbb{D}) & S_6(\mathbb{E}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix} = \begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{B}-\mathbb{X}) & S_4(\mathbb{C}-\mathbb{X}) & S_3(\mathbb{D}-\mathbb{X}) & S_7(\mathbb{E}-\mathbb{X}) \\ S_4(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{B}-\mathbb{X}) & S_3(\mathbb{C}-\mathbb{X}) & S_2(\mathbb{D}-\mathbb{X}) & S_6(\mathbb{E}-\mathbb{X}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix}$$

and

$$\begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_9(\mathbb{A}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_2(\mathbb{C}) \end{vmatrix}$$

$$= \begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}-\mathbb{X}) & S_9(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{A}-\mathbb{X}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{A}-\mathbb{X}) & S_2(\mathbb{C}) \end{vmatrix}\,.$$

We specialize the determinant $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$:

We will perform operations on alphabets only, and therefore will write only them.

We subtract $\mathbb{B}-b$ in the top row.

So, starting from

$$\begin{vmatrix} \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix},$$

we get

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & \dots & b-\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix}.$$

Now, we subtract b in the last indicated columns; we get

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & -\mathbb{A} & \dots & -\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \end{vmatrix}.$$

This determinant clearly factorizes, because in the first row, the elements:

$$S_{m+1}(-\mathbb{A}),\dots,S_{m+n-d-1}(-\mathbb{A})$$

are null. It is equal to

$$\pm R(b,\mathbb{A})S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

We specialize the determinant $\mathcal{R}^{(d)}$:

$$\mathcal{R}^{(d)} = \begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \end{vmatrix},$$

in $x = b \in \mathbb{B}$. We get

$$\begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

Now, we subtract b in the first indicated rows; we get

$$\begin{vmatrix} -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

By Laplace expansion along the last column, we get

$$\pm R(b,\mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

I. "LIMIT CASE" : $\mathrm{Sylv}^{p,q}$ for $p+q=n$.

Proposition. *(i) Suppose that $p+q=n<m$. Then*

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)=\pm\binom{p+q}{p}R(x,\mathbb{B})\,.$$

(ii) Suppose that $p+q=n=m$. Then

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)=\binom{p+q-1}{q}R(x,\mathbb{A})+\binom{p+q-1}{p}R(x,\mathbb{B})\,.$$

II. DIVIDED DIFFERENCE FORMULA FOR $\mathcal{R}_1$.

Let $\mathbb{B} = \{b_1, \ldots, b_n\}$ be a set of indeterminates.

Divided differences, invented by Newton, are linear operators

$$\partial_i : \mathbb{Z}[\mathbb{B}] \to \mathbb{Z}[\mathbb{B}] \qquad \text{(of degree } -1)$$

defined by

$$\partial_i(f) = (f - \tau_i f)/(b_i - b_{i+1}) \qquad i = 1, \ldots, n-1,$$

where τ_i denotes the i-th simple transposition exchanging i with $i+1$.

More generally, to every reduced decomposition

$$\sigma = \tau_{i_1} \cdots \tau_{i_k}$$

one associates the composition

$$\partial_\sigma = \partial_{i_1} \cdots \partial_{i_k}$$

which is independent of the choice of the reduced decomposition of σ .

Lemma. *For every $k \in \mathbb{Z}$, and a virtual alphabet $\mathbb{A}$ which is independent,of $\mathbb{B}$, one has*

$$\partial_i S_k(\mathbb{B}_i - \mathbb{A}) = S_{k-1}(\mathbb{B}_{i+1} - \mathbb{A}) .$$

Proposition. *The first remainder $\mathcal{R}_1$ of the division of $P(x) = S_m(x - \mathbb{A})$ by $Q(x) = S_n(x - \mathbb{B})$ equals*

$$\mathcal{R}_1 = \partial_{n-1} \cdots \partial_1 \big(P(b_1) R(x, \mathbb{B} - b_1)\big) .$$

For example, for $n = 3$,

$$\mathcal{R}_1 = P(b_1)\frac{(x-b_2)(x-b_3)}{(b_1-b_2)(b_1-b_3)} + P(b_2)\frac{(x-b_1)(x-b_3)}{(b_2-b_1)(b_2-b_3)} + P(b_3)\frac{(x-b_1)(x-b_2)}{(b_3-b_1)(b_3-b_2)} .$$

III. A SPECIALIZATION LEMMA.

Lemma. *(i) The specialization* $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};a)$, $a\in\mathbb{A}$, *is equal to*

$$(-1)^p R(a,\mathbb{B})\ c\,,$$

where c *is the top coefficient of* $\mathrm{Sylv}^{p,q}(\mathbb{A}-a,\mathbb{B};x)$.

(ii) The specialization $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};b)$, $b\in\mathbb{B}$, *is equal to*

$$(-1)^{m-p+q} R(b,\mathbb{A})\ c\,,$$

where c *is the top coefficient of* $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B}-b;x)$.

PROOF OF THE THEOREM Write $d:=p+q$.

To show:

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)=\pm\binom{d}{p}S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$$

if $d<n$.

Since $\deg_x \mathrm{Sylv}^{p,q}<n$, it coincides with its first remainder modulo $R(x,\mathbb{B})$.

By II:

$$\mathrm{Sylv}^{p,q}(x)=\partial_{n-1}\cdots\partial_1\big(\mathrm{Sylv}^{p,q}(b_1)R(x,\mathbb{B}-b_1)\big)\,.$$

By III,

$$\mathrm{Sylv}^{p,q}(b_1)=\pm R(b_1,\mathbb{A})\times\text{the top coefficient }\mathbf{c}\text{ in }\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B}-b_1;x)$$

If $d < n-1$,then , by the induction assumption,

$$\mathbf{c} = \pm \binom{d}{p} S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) \,,$$

with $\square = (m-d)^{n-d-1}$.

If $d = n-1$ then $\mathbf{c}$ is given by the same formula by the limit case I.

We are left with computation

$$\partial_{n-1} \cdots \partial_1 \big(R(x, \mathbb{B} - b_1) R(b_1, \mathbb{A}) S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) \big) \,.$$

Use three simple identities:

$$R(x, \mathbb{B} - b_1) = \pm \sum_{i=0}^{n-1} (-1)^i S_{1^{n-i-1}}(\mathbb{B} - x) b_1^i \,;$$

$$b_1^i R(b_1, \mathbb{A}) = S_{m+i}(b_1 - \mathbb{A}) \,;$$

$$S_{m+i}(b_1 - \mathbb{A}) S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) = S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) \,.$$

Finally

$$\mathrm{Sylv}^{p,q}(b_1) R(x, \mathbb{B} - b_1) = \pm \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) \,.$$

is sent via $\partial_{n-1} \ldots \partial_1$ to

$$\pm \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square; m+i-(n-1)}(\mathbb{B} - \mathbb{A}) \,.$$

In this last sum, the terms for $i = 0, \ldots, n-d-2$ disapear.

In the remaining sum for $i = n-d-1, \ldots, n-1$:

$$\sum_{j=0}^{d} (-1)^j S_{1^{d-j}}(\mathbb{B} - x) S_{\square; m-d+j}(\mathbb{B} - \mathbb{A}) \,,$$

one recognizes a Laplace expansion of $S_{1^d; (m-d)^{n-d}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})$.

Another possible method for getting Sylvester's formulas.

Lemma. *Let $i \leqslant n-1$ be a nonnegative integer and let $\mathbb{C}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}) \leqslant i$. *Then for a nonnegative integer j and a virtual alphabet $\mathbb{D}$,*

$$\partial_{n-1} \cdots \partial_1 \big(S_i(b_1 - \mathbb{C}) S_j(b_1 + \mathbb{D}) \big) = S_{i+j-n+1}(\mathbb{B} - \mathbb{C} + \mathbb{D}) ,$$

provided $\mathbb{C}$ and $\mathbb{D}$ are independent of $\mathbb{B}$.

Lemma. *Let i_0 be a nonnegative integer and let $\mathbb{C}^{(0)}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}^{(0)}) \leqslant i_0$. *Then for integers $i_1, \ldots, i_{n-1}$, and virtual alphabets $\mathbb{C}^{(1)}, \ldots, \mathbb{C}^{(n-1)}$,*

$$\begin{aligned} &\partial_{n-1} \cdots \partial_1 \big(S_{i_0}(b_1 - \mathbb{C}^{(0)}) S_{i_1,\ldots,i_{n-1}}(\mathbb{B}^2 - \mathbb{C}^{(1)}, \ldots, \mathbb{B}^2 - \mathbb{C}^{(n-1)}) \big) \\ &\qquad = (-1)^{n-1} S_{i_0, i_1 - 1, \ldots, i_{n-1}-1}(\mathbb{B} - \mathbb{C}^{(0)}, \mathbb{B} - \mathbb{C}^{(1)}, \ldots, \mathbb{B} - \mathbb{C}^{(n-1)}) . \end{aligned}$$

We fix $\mathbb{A}' := \mathbb{A}_p$, $\mathbb{A}\text{"} := \mathbb{A}^{p+1}$, $\mathbb{B}' := \mathbb{B}_q$, and $\mathbb{B}\text{"} := \mathbb{B}^{q+1}$.

To show:

$$\begin{aligned} &\partial_{\mathbb{A}\text{"}\,\mathbb{A}'} \partial_{\mathbb{B}\text{"}\,\mathbb{B}'} [R(x, \mathbb{A}' + \mathbb{B}') R(\mathbb{A}', \mathbb{B}') R(\mathbb{A}\text{"}, \mathbb{B}\text{"})] \\ &= \pm \binom{p+q}{p} S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}) . \end{aligned}$$

Since

$$\begin{aligned} &\partial_{\mathbb{A}'} \partial_{\mathbb{B}'} \big((1/p!)(1/q!) \Delta(x + \mathbb{A}')(x + \mathbb{B}') R(\mathbb{A}', \mathbb{B}') R(\mathbb{A}\text{"}, \mathbb{B}\text{"}) \big) \\ &= R(x, \mathbb{A}' + \mathbb{B}') R(\mathbb{A}', \mathbb{B}') R(\mathbb{A}\text{"}, \mathbb{B}\text{"}) , \end{aligned}$$

to show is:

$$\nabla \big(\Delta(x + \mathbb{A}' + \mathbb{B}') R(\mathbb{A}\text{"}, \mathbb{B}\text{"}) \big) = \pm (p+q)! S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})$$

where

$$\nabla := \partial_{\mathbb{B}^2 b_1} \cdots \partial_{\mathbb{B}^{q-p+1} b_{q-p}} \partial_{\mathbb{A}^2 a_1} \partial_{\mathbb{B}^{q-p+2} b_{q-p+1}} \cdots \partial_{\mathbb{A}^p a_{p-1}} \partial_{\mathbb{B}^q b_{q-1}} \partial_{A\text{"} a_p} \partial_{B\text{"} b_q}$$

Gysin maps for Grassmann bundles (or "integration over a fibre" in Grassmann fibrations).

Let E and F be vector bundles on a variety X of ranks m and n. Let

$$\pi_E : Gr^p(E) \to X \qquad \text{and} \qquad \pi_F : Gr^q(F) \to X$$

be two Grassmann bundles on X parametrizing rank p quotients of E and rank q quotients of F, endowed with:

$$0 \to R_E \to E \to Q_E \to 0 \qquad \text{and} \qquad 0 \to R_F \to F \to Q_F \to 0\,.$$

Form a product over X:

$$\pi := \pi_E \times_X \pi_F : Gr^p(E) \times_X Gr^q(F) \to X\,.$$

We have a Gysin map

$$\pi_* : H(Gr^p(E) \times_X Gr^q(F)) \to H(X)\,.$$

Proposition. *Let L be a line bundle on X. Then,*

$$\pi_*\Big(c_{\rm top}\,{\rm Hom}(Q_E, L)\ c_{\rm top}\,{\rm Hom}(Q_F, L)\ c_{\rm top}\,{\rm Hom}(Q_F, Q_E)\ c_{\rm top}\,{\rm Hom}(R_F, R_E)\Big)$$
$$= \pm\binom{p+q}{p} S_{1^{p+q};(m-p-q)^{n-p-q}}(F-L; F-E)\,.$$

Corollary.

$$\pi_*\Big(c_{\rm top}\,{\rm Hom}(Q_F, Q_E)\ c_{\rm top}\,{\rm Hom}(R_F, R_E)\Big) = \pm\binom{p+q}{p} S_{(m-p-q)^{n-p-q}}(F-E)\,.$$