# Thom polynomials and Schur functions

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(Often $\mathcal{T}^\Sigma$ depends on $c_i(TM - f^*TN)$, $i = 1, 2, \ldots$)

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($R_m$ "parametrizes" $TM$ for $\dim M = m$, similarly for $R_n$.)

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$A_i, p = 0$:

$$(x, u_1, \ldots, u_{i-1}) \to (x^{i+1} + \sum_{j=1}^{i-1} u_j x^j, u_1, \ldots, u_{i-1})$$

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$\operatorname{codim}(\eta) = \operatorname{codim} \eta(f)$.

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Well defined up to conjugacy; it can be chosen so that the images of its projections to the factors are *linear*. Its representations on the source and target will be denoted by

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The *Chern class* and *Euler classs* of $\eta$ are defined by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)} \quad \text{and} \quad e(\eta) := e(E'_\eta).$$

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This system of equations (taken for all such $\xi$'s) determines the Thom polynomial $\mathcal{T}^{\eta}$ in a unique way.

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$\mathcal{T}_r^\eta =$ Thom polynomial of $\eta(r)$.

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*Alphabet* $\mathbb{A}$: a finite set of indeterminates.

We identify an alphabet $\mathbb{A} = \{a_1, \ldots, a_m\}$ with the sum $a_1 + \cdots + a_m$.

Take another alphabet $\mathbb{B}$.

$$\sum S_i(\mathbb{A}-\mathbb{B}) z^i = \prod_{b \in \mathbb{B}} (1-bz) / \prod_{a \in \mathbb{A}} (1-az) \, .$$

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$$S_I(\mathbb{A}-\mathbb{B}) := \Big| S_{i_q+q-p}(\mathbb{A}-\mathbb{B}) \Big|_{1 \le p,q \le h} \, .$$

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We have

$$S_{(j_1,\ldots,j_k,i_1+n,\ldots,i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A})\; R(\mathbb{A},\mathbb{B})\; S_J(-\mathbb{B})\,.$$

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- important in study of polynomial characters of Lie superalgebras; particular cases known to 19th century algebraists: Pomey etc.

$I_{2,2}$: $c_2^2 - c_1c_3$

$I_{2,3}$: $2c_1c_2^2 - c_1^2c_3 + 2c_2c_3 - 2c_1c_4$

$I_{2,4}$: $2c_1^2c_2^2 + c_2^3 - 2c_1^3c_3 + 2c_1c_2c_3 - 3c_3^3 - 5c_1^2c_4 + 9c_2c_4 - 6c_1c_5$

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$I_{2,2}$: $S_{22}$
$I_{2,3}$: $4S_{23} + 2S_{122}$
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**Theorem.** *(PP+AW, 2006) Let $\Sigma$ be a singularity class. Then for any partition $I$ the coefficient $\alpha_I$ in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^{\Sigma} = \sum \alpha_I S_I(T^*M - f^*T^*N)\,,$$

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(..., Usui-Tango, Fulton-Lazarsfeld)

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Equations characterizing the Thom polynomial: $A_0$, $A_1$, $A_2$:

$$\mathcal{T}_r(-\mathbb{B}_{r-1}) = \mathcal{T}_r(x - \boxed{2x} - \mathbb{B}_{r-1}) = \mathcal{T}_r(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0\,,$$

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$$= x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1)\ R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1})$$

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(The variables here correspond now to the Chern roots of the *cotangent* bundles).

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**Lemma.** *A partition appearing in the Schur function expansion of* $\mathcal{T}_r$ *contains* $(r+1, r+1)$ *and has at most three parts.*

Linear endomorphism $\Phi$: $S_{i_1,i_2,i_3} \mapsto S_{i_1+1,i_2+1,i_3+1}$.

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$\overline{\mathcal{T}_r}$ = sum of terms " $\alpha_{ij}S_{ij}$ " in $\mathcal{T}_r$.

**Lemma.** $\mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\mathcal{T}_{r-1})$.

**Proposition.** $\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1 x_2)^{r+1}\ S_{r-1}(\mathbb{D})$.

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The Segre class $s_{r-1}(\mathrm{Sym}^2(E))$ is:

$$\sum_{p\le q,\ p+q=r-1} \left[\binom{r}{p+1} + \binom{r}{p+2} + \cdots + \binom{r}{q+1}\right] S_{p,q}(E)\,.$$

**Morin singularities** $A_i(r)$. We define:

$$F_r^{(i)}(-) := \sum_{J \subset (r^{i-1})} S_J(\boxed{2} + \boxed{3} + \cdots + \boxed{i})\, S_{r-j_{i-1},\ldots,r-j_1,r+|J|}(-)\,,$$

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The Schur expansions of the Thom polynomial $\mathcal{T}_r^{A_4}$ are not known (apart from $r = 1, 2, \mathbf{3}, \mathbf{4}$ – Ozer Ozturk).

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These types are defined by imposing conditions on the higher order jets of $L$.

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# Lagrangian Thom polynomials

Let $L$ be a Lagrangian submanifold in the linear symplectic space $V = W \oplus W^*$ equipped with the standard symplectic form.
Classically, in real symplectic geometry, the *Maslov class* is represented by the cycle

$$\Sigma = \{x \in L \ : \ \dim(T_x L \cap W^*) > 0\}.$$

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Consider the subgroup of $\mathrm{Aut}(V)$ consisting of holomorphic symplectomorphisms preserving the fibration $V \to W$. This group defines the *Lagrangian equivalence* of jets of Lagrangian submanifolds.

*A Lagrange singularity class* is a closed algebraic set which is a sum of Lagrangian equivalence classes.

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**Theorem.** *(MM+PP+AW, 2007) For any Lagrange singularity class* $\Sigma$*, the Thom polynomial* $\mathcal{T}^{\Sigma}$ *is a nonnegative combination of* $\widetilde{Q}$*-functions.*

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Geometric insight: The fundamental classes of the Schubert varieties in the *Lagrangian Grassmannian* $LG(V)$ are given by the appropriate $\widetilde{Q}$*-functions* of the tautological bundle on that Grassmannian (PP, 1986).

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For the Legendre singularity classes, MK+MM+PP+AW generalized the last result to a one-parameter basis with positivity property.

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A localization formula was used earlier for Morin singularities by Berczi-Szenes. Their formulas involve residues; we do not see how to get Schur expansions from them.

THE END