Duality for Grassmann bundles and applications (1

(UJ, 17.10.2016) with L. Darondeau

$X$ alg. var over $k = \bar{k}$, $A_\circ X$ - Chow group

$A^\circ X$ - Chow ring $X$-smooth

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad \text{vector bundle of rk } n$$

flag : $E_\circ : E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$ $\quad$ $\mathrm{rk}\, E_i = i$

$G_d(E)$ Grassmann bundle of $d$-subspaces in the fibres of $E$ $\quad$ $P(E) = G_1 E$ proj. bundle of lines

$\downarrow \pi$

$X$ $\quad$ univ. seq. $0 \to \mathcal{U}^d \to E \to Q^{n-d} \to 0$ on $G_d E$

partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_d)$ $\quad$ $\lambda_1 \leqslant n-d$

$\square = (n-d)^d = (n-d, \ldots, n-d)$ $\quad$ $\lambda \subset \square$

technical numbers : $n_i = n-d+i-\lambda_i$, $1 \leqslant n_1 < n_2 < \cdots < n_d \leqslant n$

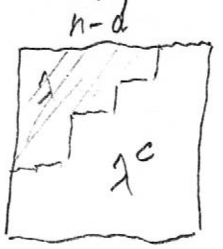
$\Omega_\lambda(E_\cdot) \subset G_d(E)$,

$\downarrow$ $\quad$ $\Omega_\lambda(E_\cdot)_x = \{ V \in G_d(E_x) : \dim(V \cap E_{n_i, x}) \geqslant i,\ i = 1, \ldots, d \}$

$X \ni x$ $\quad$ $\mathrm{codim}\, \Omega_\lambda(E_\cdot) = |\lambda|$

<u>Thm (Duality of Schubert calculus)</u> $\quad X = x$, $[\Omega_\lambda]$

$\lambda \subset \square$




$\lambda^c = (n-d-\lambda_d, \ldots, n-d-\lambda_1)$

$$[\Omega_\lambda] \cdot [\Omega_\mu] = \begin{cases} 1 & \mu = \lambda^c \\ 0 & \text{else} \end{cases}$$

$|\lambda| + |\mu| = d(n-d)$

<u>Basis thm</u> $A^\cdot G_d(E_x) = \bigoplus_{\lambda \subset \square} \mathbb{Z}\, \Omega_\lambda$

$W \subset G_d(E_x)$ subvariety, $[W] = \sum m_\lambda \Omega_\lambda$, duality $\Rightarrow$ $m_\lambda = [W] \cdot [\Omega_{\lambda^c}]$.

## Schur functions

$t_1, \dots, t_d$ variables of deg 1

$$s_\lambda(t_1, \dots, t_d) = \frac{\det\left(t_j^{\lambda_i + d - i}\right)}{\prod_{i<j}(t_i - t_j)} = \det\left(h_{\lambda_i - i + j}(t_1, \dots, t_d)\right)$$

$h$: complete homogeneous symm. fts.

Let $x_1, \dots, x_n$ be the Chern roots of $E$.

$s_\lambda(E) := s_\lambda(-x_1, \dots, -x_n)$ (Fulton's convention)

## Giambelli formula

$X = x$ $[\Omega_\lambda] = s_\lambda(\mathcal{U})$.

## Gysin homomorphisms

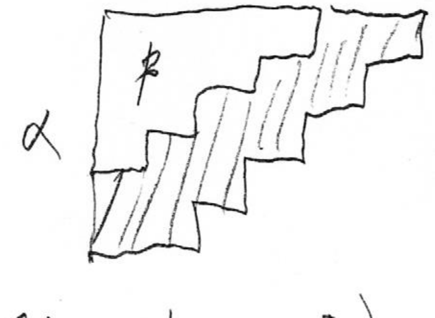
$f : X \to Y$ proper $\rightsquigarrow$

$f_* : A_\bullet X \to A_\bullet Y$ additive map induced by push forward of cycles: $V \subset X$ subvariety, $W = f(V)$ subvariety in $Y$. If $\dim W < \dim V$, then get $0$; if $\dim W = \dim V$, then $f_*[V] = [R(V) : R(W)]\,[W]$.

## Skew Schur functions

$\beta \subset \alpha$

$\alpha / \beta$




$$s_{\alpha/\beta} = \det\left(h_{\alpha_i - i + j - \beta_j}\right)$$

## Thm (Darondeau, P)

If $\beta \subset \square$, then

$$\pi_*\left(s_\alpha(\mathcal{U})\, s_\beta(\mathcal{U})\right) = s_{\alpha/\beta^c}(E).$$

<u>Thm</u> (Józefiak, Lascoux, P, 1978)

$$\pi_* s_\chi(\mathcal{U}) = s_{\chi - \square}(E).$$

We need <u>Littlewood - Richardson numbers</u> $c^{\gamma}_{\alpha\beta}$

$$s_\alpha s_\beta = \sum_\gamma c^{\gamma}_{\alpha\beta} s_\gamma \Rightarrow c^{\gamma}_{\alpha\beta} = c^{\gamma}_{\beta\alpha}.$$

described using tableaux

$$s_{\alpha/\beta} = \sum c^{\alpha}_{\beta\gamma} s_\gamma \Rightarrow c^{\alpha}_{\beta\gamma} \text{ depends on } \alpha/\beta \text{ and } \gamma$$

$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \dots)$

It follows that $c^{\alpha}_{\beta\gamma} = c^{\alpha}_{\gamma\beta}$ depends only on the partition $\beta$ and the skew partition $\alpha/\gamma$.

$$\Rightarrow c^{\alpha}_{\beta\gamma} = c^{\alpha+\square}_{\beta,\gamma+\square}. \quad \left(\alpha+\square/\gamma+\square = \alpha/\gamma\right)$$

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<u>Rk</u> Knutson-Tao $c^{\gamma}_{\alpha\beta} = \#$ puzzles ...

Buch, Fulton $c^{\gamma}_{\alpha\beta} = \#$ hives ...

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Have $\pi_*\left(s_\alpha(\mathcal{U}) s_\beta(\mathcal{U})\right) = \sum_\gamma c^{\gamma}_{\alpha\beta} s_{\gamma-\square}(E)$

$$= \sum c^{\gamma+\square}_{\alpha\beta} s_\gamma(E)$$

Prehistory: Naegelbask (1871) $s_\alpha = s_\alpha(t_1, \dots, t_d)$

$$s_\alpha \cdot s_\beta = \det\left(h_{\alpha_i + \beta_{d+1-j} - i + j}\right)$$

<u>Prop</u> $\beta \subset \square$, $\ell(\alpha) \le d$

$$s_{\alpha+\square/\beta^c} = s_\alpha \cdot s_\beta$$

Combinatorial proof using tableaux.

**Cor.** $c^{\gamma}_{\alpha\beta} = c^{\alpha+\square}_{\beta^c\gamma}$.

To prove thm, we need:

LHS $\to$ $c^{\gamma+\square}_{\alpha\beta} = c^{\alpha}_{\beta^c\gamma}$ $\leftarrow$ RHS

But $c^{\gamma+\square}_{\alpha\beta} \underset{\text{Cor.}}{=} c^{\alpha+\cancel{\square}}_{\beta^c(\gamma+\cancel{\square})} = c^{\alpha}_{\beta^c\gamma}$. $\square$

Thm (J, L, P, 1978)

$$\pi_*(s_\alpha(\mathcal{U})\cdot s_\beta(Q)) = s_{\alpha-\square,\beta}(E).$$

DP formula $\Rightarrow$ JLP formula

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$$\begin{tikzcd} Z \arrow[r, "\text{desing.}"'] \arrow[dr, "\theta"'] & W \arrow[r, phantom, "\subset_i"] \arrow[d] & G_d(E) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Relative basis thm.:

$A^\cdot G_d E$ has an $A^\cdot X$-basis $s_\beta \mathcal{U}$, where $\beta \subset \square$.

$[W] = \sum m_\beta s_\beta \mathcal{U}$.

$$\begin{tikzcd} A_\circ W \arrow[r, "i_*"] & A_\circ G_d(E) = A^\cdot G_d(E) \\ \arrow[u] & s_\alpha \mathcal{U} \arrow[u] \end{tikzcd}$$

$s_\alpha \mathcal{U} := s_\alpha(\mathcal{U}|_W) \cap [W]$

$$i_*(s_\alpha \mathcal{U}) = s_\alpha \mathcal{U}\cdot[W] = \sum m_\beta s_\alpha \mathcal{U}\cdot s_\beta \mathcal{U}$$

$(\alpha)$ $\theta_*(s_\alpha \mathcal{U}) = \pi_*\left(\sum m_\beta s_\alpha \mathcal{U} s_\beta \mathcal{U}\right) =$

$$= \sum m_\beta s_{\alpha/\beta^c} E.$$

<u>Pb</u> Generalize the Giambelli formula for vector bundles, i.e. for $[\Omega(E_\bullet)]$.

Desing. of $\Omega_\lambda(E_\bullet)$ (Kempf – Laksov 1974)

$x \in X$

$$F_x = \{ V_1 \subset \dots \subset V_d : \dim V_i = i,\ V_i \subset E_{n_i, x} \}$$

$$V \mapsto (V \cap E_{n_1,x}, \dots, V \cap E_{n_d,x})$$

$$V_d \in \Omega_\lambda(E_{\bullet,x}) \supset \overset{\circ}{\Omega}_\lambda = \{ V : \dim (V \cap E_{n_i,x}) = i \}$$

F as a chain of projective bundles:

$$F = P(E_{n_d} / \mathcal{U}_{d-1}) \xrightarrow{\xi_1} \cdots \xrightarrow{\xi_{d-1}} P(E_{n_2}/\mathcal{U}_1) \xrightarrow{\xi_d} P(E_{n_1}) \to X$$

Have $\mathcal{U}_1$ of rk 1 on $P(E_{n_1})$

$\mathcal{U}_2/\mathcal{U}_1$ of rk 1 on $P(E_{n_2}/\mathcal{U}_1)$

...

$\mathcal{U}_d/\mathcal{U}_{d-1}$ of rk 1 on $P(E_{n_d}/\mathcal{U}_{d-1})$

$\mathcal{U}_i$ is a universal rank $i$ bundle on the bundle of complete flags, restricted to F.

$\xi_i := -c_1(\mathcal{U}_{d+1-i}/\mathcal{U}_{d-i})$ – hyperplane class

$f$ – polynomial in $d$ variables with coeff. in $A^\bullet X$

Write $\nu = (n_d, n_{d-1}, \dots, n_1)$ $\nu_i = n_{d+1-i}$

Have a new type of a Gysin formula:

<u>Thm (D-P)</u> $\theta_* f(\xi_1, \dots, \xi_d) =$

$$\underbrace{[t_1^{\nu_1 - 1} \cdots t_d^{\nu_d - 1}]}_{\text{the coeff of}} \underbrace{\Big( f(t_1, \dots, t_d) \prod_{i<j} (t_i - t_j) \prod_i s_{1/t_i}(E_{\nu_i}) \Big)}_{\text{in}}$$

Explanations:

$\pi : P(E) \to X$ $\mathcal{O}_{P(E)}(-1)$ univ. line b.

$\xi = c_1 \mathcal{O}_{P(E)}(1)$

$(*)$ $\pi_* \xi^i = s_{i-n+1}(E)$ Segre class

$s_i(E) = (-1)^i h_i$ (Chern roots of $E$)

$s_1(E) = -c_1(E)$

Segre polynomial:

$s_x(E) = 1 + x s_1(E) + x^2 s_2(E) + \cdots$ $s_1(E) = s(E)$ – total Segre class

$s(E) = c(-E)$

$x = \frac{1}{t}$, rewrite $(*)$

$\pi_* \xi^{i} = [t^{n-1}]\big(t^{i} s_{1/t}(E)\big)$

$f(t) \in A^\bullet X[t]$, $\pi_* f(\xi) = [t^{n-1}]\big(f(t)\, s_{1/t}(E)\big)$

"fundamental formula" – good to iterate.

$d = 3$ (7)

$$P(E_{\nu_1}/U_2) \xrightarrow[p_1]{\xi_1} P(E_{\nu_2}/U_1) \xrightarrow[p_2]{\xi_2} P(E_{\nu_1}) \xrightarrow[p_3]{\xi_3} X$$

Chern polyn.

$$s_x(E+F) = s_x(E)\, s_x(F) \quad , \quad s_x(E/F) = s_x(E)\, c_x(F)$$

$$p_{1*} f(\xi_1, \xi_2, \xi_3) \overset{ff}{=} [t_1^{\nu_1 - 3}]\left( f(t_1, \xi_2, \xi_3)\, s_{1/t_1}(E_{\nu_1}/U_2)\right) =$$

$$s_{1/t_1}(E_{\nu_1}/U_2) = s_{1/t_1}(E_{\nu_1})\left(1 - \frac{\xi_2}{t_1}\right)\left(1 - \frac{\xi_3}{t_1}\right)$$

$$= s_{1/t_1}(E_{\nu_1}) \frac{(t_1 - \xi_2)(t_1 - \xi_3)}{t_1^2}$$

$$= [t_1^{\nu_1 - 1}]\left( f(t_1, \xi_2, \xi_3)\, s_{1/t_1}(E_{\nu_1})(t_1 - \xi_2)(t_1 - \xi_3)\right)$$

$$p_{2*} p_{1*} f(\xi_1, \xi_2, \xi_3) \overset{ff}{=}$$

$$[t_1^{\nu_1 - 1}][t_2^{\nu_2 - 2}]\left( f(t_1, t_2, \xi_3)\, s_{1/t_1}(E_{\nu_1})(t_1 - t_2)(t_1 - t_3)\, s_{1/t_2}(E_{\nu_2}/U_1)\right) =$$

$$s_{1/t_2}(E_{\nu_2})\left(1 - \frac{\xi_3}{t_2}\right)$$

$$= [t_1^{\nu_1 - 1} t_2^{\nu_2 - 1}]\left( f(t_1, t_2, \xi_3)\, s_{1/t_1}(E_{\nu_1})\, s_{1/t_2}(E_{\nu_2})(t_1 - t_2)(t_1 - \xi_3)(t_2 - \xi_3)\right)$$

...

$$\xi_3 \to t_3$$

Apply the formula to $\quad f = s_\alpha = \dfrac{\det\left(t_j^{\alpha_i + d - i}\right)}{\prod_{i<j}(t_i - t_j)}$

$$\theta_* s_\alpha(\mathcal{U}) = [\prod t_j^{\nu_j - 1}]\left(\det\left(t_j^{\alpha_i + d - i} s_{1/t_j}(E_{\nu_j})\right)\right) =$$

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<u>Lemma</u> If $f_{ij} \in A^\circ X[t_j]$ for any $i$, then

$$[\prod_j t_j^{e_j}] \det(f_{ij}) = \det([t_j^{e_j}] f_{ij}).$$

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$$= \det\left([t_j^{\nu_j - 1}]\left(t_j^{\alpha_i + d - i} s_{1/t_j}(E_{\nu_j})\right)\right)$$

$$= \det\left(s_{\alpha_i - i + d + 1 - \nu_j}(E_{\nu_j})\right)_{1 \le i,j \le d}.$$

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Come back to $[\Omega_\lambda(E_\bullet)]$

Hence, $[\Omega_\lambda(E_\bullet)] = \sum_{\beta \subset \square} m_\beta s_\beta(\mathcal{U})$, $\underset{\uparrow \text{ wanted}}{m_\beta} \in A^\circ X$.

We do:

$$\begin{tikzcd}
F \arrow[r, "\text{desing.}"] \arrow[dr, "\theta"'] & \Omega_\lambda(E_\bullet) \arrow[r, hook] \arrow[d] & G_d(E) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

push generators of $A_\circ \Omega_\lambda(E_\bullet)$ in two ways and get a system of equations.

K-L used $F$ in a different way:

$$\begin{tikzcd} F \arrow[r, hook, shift left, "\psi"] \arrow[d, "\text{desing.}"'] & G_d(E) \times F \arrow[l, shift left, "p_2"] \arrow[d, "p_1"] \\ \Omega_\lambda(E_\bullet) \arrow[r, hook] & G_d(E) \end{tikzcd}$$

express $\psi(F)$ as the zero scheme of a vector bundle on $G \times F$ of proper rank, and push-forward its $c_{top}$ along $p_1$.

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We do:

$$(\alpha) \quad \det\left(s_{\alpha_i - i + d - 1 - \nu_j}(E_{\nu_j})\right) = \underbrace{\sum m_\beta \, s_{\alpha/\beta^c}(E)}_{\text{duality}}$$

as long as $\alpha \leq \beta^c$ (lex. order) $\quad s_{\alpha/\beta^c}(E) = 0$

if $\alpha = \beta^c \quad s_{\alpha/\beta^c}(E) = 1$

– get invertible triangular system in $m_\beta$.

Want to solve it in $m_\beta$ and substitute to $\sum m_\beta s_\beta \mathcal{U}$ to get $[\Omega_\lambda(E_\bullet)]$

The most popular formula for $[\Omega_\lambda(E_\bullet)]$ is

Thm (K-L, Lascoux, 1974)

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$$[\Omega_\lambda(E_\bullet)] = \det\left(c_{\lambda_i - i + j}(Q - E_{n_i})\right) \quad 1 \leq i,j \leq d.$$

v.b. $E_{n_1} \subset \ldots \subset E_{n_d}$ are present

This formula can be obtained from $\{(\alpha)\}_{\alpha < \square}$.

Very rough sketch:

To LHS and RHS of $(\alpha)$ look alike expand

$$s(E_{\nu_j}) = s(E_{\nu_j} - E + E) = s(E_{\nu_j} - E)\, s(E) =$$
$$= c(E/E_{\nu_j})\, s(E)$$

so presumably $m_\beta$ depend on $c_i(E/E_{\nu_j})$.

Now $\sum_\beta m_\beta s_\beta(\mathcal{U}) = \sum_\beta P_\beta(c_i(E/E_{\nu_j}))\, s_\beta(\mathcal{U})$

$$\underset{\text{very likely}}{=} P(c_i(E - E_{\nu_j} - \mathcal{U}))$$

$$\underset{\text{after comput.}}{=} \det\Big(c_{\lambda_i - i + j}(\underbrace{E - \mathcal{U}}_{Q} - E_{\nu_i})\Big)$$

last 2 pages of arXiv:1602.01983