Singularities and positivity

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red herring: it was thought that $c_1^2 - 2c_2$ is positive but is not.

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Whenever we speak about the classes of algebraic cycles, we always mean their *Poincaré dual classes* in cohomology.

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If a *singularity class* Σ is "stable" (e.g. closed under the contact equivalence), then $\mathcal{T}^\Sigma$ depends on $c_i(TM - f^*TN)$.

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(R_m "parametrizes" TM for $\dim M = m$, similarly for R_n.)

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$A_i, k = 0$:

$(x, u_1, \ldots, u_{i-1}) \to (x^{i+1} + \sum_{j=1}^{i-1} u_j x^j, u_1, \ldots, u_{i-1})$

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Well defined up to conjugacy; it can be chosen so that the images of its projections to the factors are *linear*. Its representations on the source and target will be denoted by

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Since $\mathcal{J}(E,F) = F^N$ is ample, the latter polynomial is positive for ample v.b., so is a positive combination of Schur polynomials.

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- discovered by Berele-Regev in their study of polynomial characters of Lie superalgebras; particular cases known to 19th century algebraists: Pomey etc.

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(The variables here correspond now to the Chern roots of the *cotangent* bundles).

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The Segre class $s_{r-1}(\mathrm{Sym}^2(E))$ is:

$$\sum_{p \le q,\ p+q=r-1} \left[\binom{r}{p+1} + \binom{r}{p+2} + \cdots + \binom{r}{q+1} \right] S_{p,q}(E)\,.$$

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Theorem. *(PP, 1988) Let η be of Thom-Boardman type $\Sigma^{i,\cdots}$. Then all summands in the Schur function expansion of $\mathcal{T}_r^\eta$ are indexed by partitions containing the rectangle partition $(r+i-1, \ldots, r+i-1)$ (i times).*

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Every germ of a Lagrangian submanifold of V is the image of W via a certain germ symplectomorphism.

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A Lagrange singularity class is any closed pure dimensional algebraic subset of $\mathcal{J}^k(V)$ which is invariant w.r.t. the action of H.

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Theorem. *(MM+PP+AW, 2007) For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^\Sigma$ is a nonnegative combination of $\widetilde{Q}$-functions.*

Proposition. *For a strict partition* $I \subset \rho$*, there exists only one strict partition* $I' \subset \rho$ *and* $|I'| = \dim LG(V) - |I|$ *, for which* $\widetilde{Q}_I(R^*) \cdot \Omega_{I'} \neq 0$*. (*$I'$ *complements* I *in* ρ*).*

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Lemma. *Let* $\pi : E \to X$ *be a globally generated bundle on a proper homogeneous variety* X*. Let* C *be a cone in* E*, and let* Z *be any algebraic cycle in* X *of the complementary dimension. Then the intersection* $[C] \cdot [Z]$ *is nonnegative.*

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Lemma. *We have a natural isomorphism*

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Let

$$C = C_{G \cap \Sigma} \Sigma \subset N_G \mathcal{J}$$

be the $normal\ cone$ of $G \cap \Sigma$ in Σ. Denote by $j : G \hookrightarrow N_G \mathcal{J}$ the zero-section inclusion.

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The Lagrangian Grassmannian $G = LG(V)$ is a homogeneous space with respect to the action of the symplectic group $Sp(V)$. The lemma applied to the bundle $N_G\mathcal{J} \to G$, entails $[C] \cdot \Omega_{I'}$ nonnegative.

Some Legendrian geometry

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Get 2 types of submanifolds: *linear subspaces*, *the submanifolds which have the tangent space at the origin equal to W*; they are the graphs of the differentials of the functions $f : W \to \xi$ satisfying $df(0) = 0$ and $d^2 f(0) = 0$

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Additionally, we assume that Σ is stable with respect to enlarging the dimension of W.

Jet bundle $\mathcal{J}^k(W, \xi)$

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The space $\mathcal{J}^k(W,\xi)$ fibers over X. It is equal to the pull-back:

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Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines $\Sigma(W,\xi) \subset \mathcal{J}^k(W,\xi)$.

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The Chern classes $a_i = c_i(A)$ generate the cohomology $H^*(LG(V,\omega),\mathbf{Z}) \cong H^*(\mathcal{J}^k(W,\xi),\mathbf{Z})$ as an algebra over $H^*(X,\mathbf{Z})$.

Let us fix an approximation of $BU(1) = \bigcup_{n\in\mathbf{N}} \mathbf{P}^n$, that is we set $X = \mathbf{P}^n$, $\xi = \mathcal{O}(1)$. Let $W = \mathbf{1}^n$ be the trivial bundle of rank n.

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and is often denoted by $\mathcal{T}^{\Sigma}$. It is written in terms of the generators a_i and $s = c_1(\xi)$.

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Details to appear in Journal of Differential Geometry (accepted yesterday).

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Also, the corresponding Lagrangian Thom polynomial is nonzero.

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Diverio, Merker, Rousseau: for a general hypersurface $X \subset \mathbf{P}^{n+1}$, the Green-Griffiths conjecture is true if $\deg(X) > 2^{n^5}$.

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Localization techniques, iterated residues

THE END