JOINT WORK WITH V. SRINIVAS AND V. PATI.

Suppose that s is a section of a vector bundle $\mathcal{E} \to X$. Consider the *zero scheme* of s

$$Z(s) = \{x \in X : s(x) = 0\} .$$

We will say that a variety X has the

Weak point property (P) – if for a point $x \in X$ there exist a vector bundle $\mathcal{E}$ on X of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\{x\} = Z(s)$.

Diagonal property (D) – if there exist a vector bundle $\mathcal{E}$ on $X \times X$ of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\Delta = Z(s)$, where Δ denotes the diagonal in $X \times X$.

Notice that (D) implies (P) – in fact, for any point $x \in X$ – via restriction from $X \times X$ to $X \times \{x\}$.

Consider the ideal sheaf $\mathcal{J}_\Delta \subset \mathcal{O}_{X \times X}$. The diagonal property (D) implies that

$$\mathcal{E}^*_{|\Delta} = \mathcal{J}_\Delta / \mathcal{J}^2_\Delta \cong \Omega^1_X$$

(the last isomorphism uses the isomorphism $\Delta \cong X$). Therefore, Ω^1_X is locally free of rank equal to dimension of X. Hence X is smooth.

(D) holds for Grassmannians and, more generally, for flag varieties SL_n/P.

Cohomologically trivial line bundles

A line bundle $\mathcal{L}$ on a scheme X is called *cohomologically trivial* (we shall write "c.t.") if $H^i(X, \mathcal{L}) = 0$ for all $i \geq 0$.

Example 1 Any smooth projective curve supports a c.t. line bundle.

Any abelian variety supports a c.t. line bundle.

Theorem 2 *(i) Let $p_1, p_2 : X \times X \to X$ be the two projections. If X has (D), and moreover, the following isomorphism $(*)$ holds:*

$$\operatorname{Pic}(X \times X) \cong p_1^* \operatorname{Pic}(X) \oplus p_2^* \operatorname{Pic}(X),$$

then there exists a c.t. line bundle $\mathcal{L}$ over X such that

$$\det(\mathcal{E}) = p_1^* \mathcal{L}^{-1} \otimes p_2^* (\mathcal{L} \otimes \omega_X^{-1}) \,.$$

(ii) If $\dim X = 2$ and there exists a c.t. line bundle on X, then X has (D).

Corollary 3 *Suppose that the isomorphism $(*)$ holds for X, and X supports no c.t. bundle. Then X has not (D).*

Let X be a smooth proper variety over an algebraic closed field. The isomorphism $(*)$ holds for X if and only if $\operatorname{Pic} X$ is a finitely generated abelian group. Also, if $H^1(X, \mathcal{O}_X) = 0$ then $(*)$ holds for X.

When a smooth projective surface has (D)?

We remind that for a surface X to have (D) is almost equivalent to the existence of a c.t. line bundle on X.

1. There exists a surface Y having (D), and a birational proper map $f : Y \to X$.
2. If $f : Y \to X$ is a birational map, X has (D) and $\operatorname{Pic} X$ is finitely generated, then Y has (D).
3. If X is birational to one of the following: a ruled or an abelian surface or a K3 surface with 2 disjoint smooth rational curves or an elliptic surface with a section or a complex Enriques or hyperelliptic surface, then X has (D).
4. If $\operatorname{Pic} X = \mathbb{Z}$, $\Gamma(X, \mathcal{O}_X(1)) \neq 0$ and X has (D), then $X = \mathbb{P}^2$.

More precisely, the first item says that any surface – after blowing up sufficiently many points – becomes a surface having (D).

The last item implies that (D) fails for general K3 surfaces or general hypersurfaces in $\mathbb{P}^3$ of degree greater than or equal to 4.

When a higher dimensional variety has (D)?

We now look at varieties of dimension ≥ 3.

We first consider the varieties with Picard group $\mathbb{Z}$.

Proposition 4 *Let X be a smooth projective variety of dimension $d \geq 3$ over a field with* $\operatorname{Pic} X = \mathbb{Z}$. *If X has (D) and $H^0(X, \mathcal{O}_X(1)) \neq 0$ then X is a Fano variety and $\omega_X \cong \mathcal{O}_X(-n)$ for some $n \geq 2$.*

This result has two useful consequences.

Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of multidegree $(d_1, \ldots, d_r)$ such that $r < n - 3$ and $\sum d_i \geq n$. Then X has not (D).

In particular, the non-Fano hypersurfaces in the projective spaces have not (D).

Also, let X be a smooth projective Fano variety such that $b_2(X) = 1$ and $\omega_X = \mathcal{O}_X(-1)$ (i.e. X is of index 1). Then X has not (D).

Let X be a scheme and $\mathcal{L}$ be a line bundle over X. We say that X has the

$\mathcal{L}$-point property if for every (closed) point $x \in X$ there exists a vector bundle $\mathcal{F}$ over X such that $d = \operatorname{rank} \mathcal{F} = \dim X$, $\det(\mathcal{F}) = \mathcal{L}$, and there exists a section s of $\mathcal{F}$ such that $\{x\} = Z(s)$.

Note that in this case $c_1(\mathcal{F}) = c_1(\mathcal{L})$ and $c_d(\mathcal{F}) = [x]$.

Theorem 5 *Let X be a smooth proper variety over an algebraically closed field. If* $\operatorname{Pic} X$ *is finitely generated and X has (D), then there exists a c.t. line bundle $\mathcal{L}$ on X such that*

(i) X has the $\mathcal{L}^{-1}$-point property, and

(ii) X has the $\mathcal{L} \otimes \omega_X^{-1}$-point property.

Corollary 6 *Let X be a scheme with finitely generated* $\operatorname{Pic} X$. *If for any c.t. line bundle $\mathcal{L}$, either $\mathcal{L}^{-1}$-point property fails or $\mathcal{L} \otimes \omega_X^{-1}$-point property fails then X has not (D).*

For example, if X is a smooth complex projective quadric of dimension 3, then $\mathcal{O}_X(-1)$ and $\mathcal{O}_X(-2)$ are the unique c.t. line bundles on X. One checks – with the help of the corollary – that X has not (D).

Proposition 7 *Let X be a group variety over an algebraically closed field. Then X has (D) iff it has (P). In particular, an abelian variety has (D) iff it has (P).*

Oliver Debarre (letter of 12.12.2006) :

1. The Jacobian of a smooth projective curve has (P).

2. There exist non-principally polarized abelian varieties in dimension greater than 2, which fail to have (P).

3. The point property (P) **may characterize** *Jacobians among all principally polarized abelian varieties!*

Affine case

Let k be an alg. closed field and A be a finitely generated k-algebra. Let $X = \operatorname{Spec} A$. If $\dim X = 2$, then X has (D) by Serre.

Theorem 8 *An affine algebraic group over an alg. closed field has (D).*

Indeed, M. Kumar and P. Murthy: an affine algebraic group over an alg. closed field has (P).

Conjecture 9 *There exists smooth affine complex varieties of any dimension ≥ 3, for which (D) fails.*

Diagonal property in topology

Let M be a compact connected oriented smooth manifold of real dimension n, and Δ be the diagonal submanifold of $M \times M$.

We say that M has property (D_r) if there exist a smooth *real* vector bundle $\mathcal{E}$ over $M \times M$ with $\mathrm{rank}(\mathcal{E}) = n$ and a smooth section s of $\mathcal{E}$ s.t. s is transverse to the zero section $0_{\mathcal{E}}$ of $\mathcal{E}$ and $s^{-1}(0_{\mathcal{E}}) = \Delta$.

If the vector bundle $\mathcal{E}$ is *orientable*, then we say that M has the property (D_o).

If $\dim_{\mathbb{R}} M = 2m$ and the vector bundle $\mathcal{E}$ is a smooth *complex* vector bundle of $rank_{\mathbb{C}}(\mathcal{E}) = m$ then we say that M has the property (D_c).

We have the following relations between these properties:

$$(D) \Rightarrow (D_c) \Rightarrow (D_o) \Rightarrow (D_r).$$

Our results:

1. S^n has (D_r) if and only if $n = 1, 2, 4$ or 8 (all except the first have (D_o)).

2. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 2$. Then M has (D_c). (This is in contrast with the algebraic situation.)

3. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 3$. Assume that $H^1(M, \mathbb{Z}) = 0$ and $H^2(M, \mathbb{Z}) = \mathbb{Z}$. Then if M satisfies (D_c), the second Stiefel-Whitney class $w_2(M)$ vanishes (i.e. M is spin).

4. Let $M \subset \mathbb{CP}^N$ be a smooth projective variety of $\dim_{\mathbb{C}} M = 3$. Assume that M is a strict complete intersection or a set-theoretically complete intersection with $H_1(M, \mathbb{Z}) = 0$. Then M has (D_c) if and only if M is spin.

5. Let M be a smooth strict complete intersection of $\dim_{\mathbb{C}} M = 3$ in $\mathbb{CP}^n$ with $M = X_1 \cap \cdots \cap X_{n-3}$ where X_i is a smooth hypersurfaces of degree d_i. Then M has (D_c) only if $(n + 1 - \sum d_i)$ is even. In particular, a smooth hypersurface M in $\mathbb{CP}^4$ has (D_c) only if it is of odd degree.

6. Let $m \geq 2$. Then a smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}$ has not (D_c).

In fact, $Q_{2m-1} \subset \mathbb{P}^{2m}$ over *any alg. closed field*, has not (D).

Conjecture 10 *Let Q_n be a smooth quadric hypersurface in $\mathbb{P}^{n+1}$ (over an alg. closed field). Then Q_n has (D) if and only if $n = 1, 2$ or 4.*

Three other conjectures:

1. From the results on surfaces, it follows that any smooth projective *toric* surface has (D). We conjecture that any smooth toric variety has (D).

2. The Grassmannian of Lagrangian 2-planes in $\mathbb{C}^4$ is identified with the quadric Q_3, which has not (D). Also 3-dimensional orthogonal Grassmannian fails to have (D).

We conjecture that, in general, the homogeneous spaces

$$Sp_n/P \quad \text{and} \quad SO(n)/P$$

(P being any parabolic) have not (D).

The question: "Do the flag varieties for the symplectic and orthogonal groups have (D)?" arose in discussions of the speaker with William Fulton, at the University of Chicago in 1996, in connection with *Schubert polynomials.*

3. We conjecture that (D) fails for a general cubic threefold X, though X has (P) and even the $\mathcal{O}_X(1)$-point property ($\mathcal{O}_X(-1)$ is the unique c.t. line bundle on X).