On diagonals of flag bundles   IMPA Rio 22.05.

with Shizuo Kaji   1

$\pi : F \to X$ map of varieties, diagonal $\Delta \subset F \times_X F$

fibre bundles in topology

smooth proper morphisms in alg. geom.

Graham, Fulton, P

Knowing $[\Delta]$, can compute the class of a subscheme of F

Today $\Delta \subset F \times F$, study their global equations

With Pati and Srinivas: which varieties X have the "diagonal property" (D):

$\exists$ E vector bundle, $\mathrm{rk}\, E = \dim X$ and section s of E s.t. $Z(s) = \Delta$.

$$\begin{array}{c} E \\ \downarrow \uparrow s \\ X \times X \supset \Delta \text{ diagonal} \end{array}$$

X has (D) $\Rightarrow$ X is nonsingular

Any nonsingular curve has (D)

If X_1, X_2 have (D), then $X_1 \times X_2$ has (D)

Starting point of this theory:

<u>Thm</u> (Fulton, P) The flag varieties SL_n/P have (D).

- proof later
- related to the theory of Schubert polynomials of Lascoux - Schützenberger

Schubert polys are polynomial lifts of the classes of Schubert varieties in cohomology of SL_n/B.

There is a scalar product on the pol. ring for which the Schubert polys and their duals form adjoint bases.

The reprod. kernel of this scalar product is the top Schubert polynomial = top Chern class of the bundle realizing (D) for SL_n/B.

"Weak point property" (P) : for some $x \in X$ $\exists$ $\begin{matrix} E \\ \downarrow \uparrow s \\ X \end{matrix}$ $\mathrm{rk}\, E = \dim X$ ②

and a section s of E s.t. $Z(s) = x$

$(D) \Rightarrow (P)$ but in general $(P) \not\Rightarrow (D)$ e.g. $Q_3 \subset \mathbb{P}^4$

Similar properties in topology: X sm. cpt. conn mfd

(D_c) : $\dim X = 2m$, E sm. complex v.b. of complex rank m on $X \times X$, s smooth section of E transverse to the zero section s_E of E s.t. $Z(s) = \Delta$

(P_c) : $\exists x \in X$ $\exists$ $\begin{matrix} E \\ \downarrow \uparrow s \pitchfork s_E \\ X \end{matrix}$ $Z(s) = x$

Lemma If a complex bundle E of complex rank m has $c_m E = \pm 1$ then one can use it to realize (P_c).

$(D) \Rightarrow (D_c)$, $(P) \Rightarrow (P_c)$ — useful for counterexample

G/B for other groups

G simple, simply connected alg. gp

$B \subset G$ Borel, $T \subset B$ max. torus, G/B flag mfd

$\dim G/B = m$; when there exists a complex vector bundle E of complex rank m on G/B s.t. $c_m(E)$ is the class of a point in $H^{2m}(G/B; \mathbb{Z})$?

<u>Thm</u> (Kaji, P) For G of type B_i $(i \geq 3)$, D_i $(i \geq 4)$, G_2, F_4 and E_i $(i = 6, 7, 8)$, the flag manifold G/B has not (P_c), and hence it has not (D_c).

<u>Pf</u> $X(T)$ group of characters of T

$K(G/B)$ Grothendieck group

Atiyah – Hirzebruch homomorphism:

$$\beta_1 : S^{\cdot}(X(T)) \longrightarrow K(G/B)$$

$\lambda \in X(T) \quad e^\lambda \mapsto L_\lambda = G \times_B \mathbb{C}_\lambda$, line bundle on G/B

<u>Thm</u> (A-H, ..., Kostant-Kumar) β_1 is surjective.

In $S^\cdot(X(T))$, every element is a $\mathbb{Z}$-comb. of monomials $e^{\lambda_1} \cdot \ldots \cdot e^{\lambda_k}$, $\lambda_i \in X(T)$; $\beta_1(e^{\lambda_1} \cdots e^{\lambda_k}) = L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_k}$

<u>Cor.</u> In $K(G/B)$, the class of any bundle is a $\mathbb{Z}$-comb. of classes of line bundles L_μ for some $\mu \in X(T)$

Borel characteristic homomorphism:

$$c: S(X(T)) \to H^*(G/B; \mathbb{Z})$$

$$\lambda \in X(T), \quad e^\lambda \mapsto c_1(L_\lambda)$$

<u>Cor</u> The Chern classes of any vector bundle on G/B are in the image of c.

<u>Def</u>. The smallest positive integer t_G s.t. $t_G \cdot$ (class of a point) $\in \mathrm{Im}(c)$ is called the torsion index of G.

<u>Thm</u> (Borel, ...) $t_G = 1 \iff G$ is of type A_i or C_i

This implies the theorem.

Type C_i ?

<u>Prop</u> For G of type C_n, $Sp(2n, \mathbb{C})/B$ has (P_c).

(probably not (D_c)).

Surfaces with (D) / $k=\bar{k}$ include <u>ruled surfaces</u>:

$P(E)$ $\operatorname{rk} E = 2$
$\downarrow$
C curve

Try to generalize this result

Flag bundles in type A

E vector bundle of rank n on X over a field

$d_\bullet : 0 < d_1 < d_2 < \dots < d_{k-1} < d_k = n$

$d_\bullet$-flag : $E_1 \subset E_2 \subset \dots \subset E_{k-1} \subset E_k = E$ $\operatorname{rk} E_i = i$

$\pi : Fl_{d_\bullet}(E) \to X$ flag bundle of $d_\bullet$-flags

E.g. $d_1 = d < d_2 = n$ $\rightsquigarrow$ $G_d(E)$ Grassmann bundle of d-subbundles

$d=1$ $P(E)$ projectivization of line subbundles

$$S_1 \subset S_2 \subset \dots \subset S_{k-1} \subset S_k = \pi^* E \xrightarrow{q_1} Q_1 \xrightarrow{q_2} Q_2 \to \dots \xrightarrow{q_k} Q_k = 0$$

$\operatorname{rk} S_i = d_i$, $Q_i = E/S_i$

Composition of Grassmann bundles:

$$Fl_{d_\bullet}(E) \to \dots \to G_{d_3-d_2}(Q_2) \to G_{d_2-d_1}(Q_1) \to G_{d_1}(E) \to X$$

$$\dim Fl_{d_\bullet}(E) = \dim X + \sum_{i=1}^{k-1} (d_i - d_{i-1})(n-d_i)$$

$F = Fl_{d_\bullet}(E)$, $F_1 = F_2 = F$, $p_i : F_1 \times F_2 \to F_i$

Construct a vector bundle H on $F_1 \times F_2$.

$k=2$ $\quad H = \operatorname{Hom}(p_1^* S_1, p_2^* Q_1)$

$k \geq 3$ $\quad \varphi : \bigoplus_{i=1}^{k-1} \operatorname{Hom}(p_1^* S_i, p_2^* Q_i) \to \bigoplus_{i=1}^{k-2} \operatorname{Hom}(p_1^* S_i, p_2^* Q_{i+1})$

$$\varphi\left(\sum_{i=1}^{k-1} h_i\right) = \sum_{i=1}^{k-2} \left(h_{i+1}|_{S_i} - q_{i+1} \circ h_i\right)$$

Lemma φ is surjective

$$H = \mathrm{Ker}(\varphi)\,,\quad \mathrm{rk}(H) = \sum_{i=1}^{k-1} (d_i - d_{i-1})(n - d_i).$$

X has (D') if $\exists$ v.b. $A, B \to X$, $\mathrm{rk}\,A + \mathrm{rk}\,B = \dim X$, a section s of X and a section t of $B_{Z(s)}$ s.t. $Z(t) = \Delta$. Analogous (P') $\exists\, x \in X \ldots$ $Z(t) = x$

Thm (K, P) If X has (D), then $Fl_{d_\bullet}(E)$ has (D') for any $d_\bullet$

$$\pi : Fl_{d_\bullet}(E) \to X$$

$$\begin{array}{c} G^{\dim X} \\ \downarrow \uparrow s \\ X \times X \end{array} \qquad Z(s) = \Delta_X \qquad G' = (\pi_1 \times \pi_2)^* G\,,\ s'$$

$$Z = Z(s') = (\pi_1 \times \pi_2)^{-1}(\Delta_X) \subset F_1 \times F_2$$

$q_1, q_2 : X \times X \to X$ projections

$$(q_1^* E)_{\Delta_X} = (q_2^* E)_{\Delta_X} \implies$$

$$h_i : (p_1^* S_i)_Z \to (p_1^* E_{F_1})_Z = (p_2^* E_{F_2})_Z \to (p_2^* Q_i)_Z$$

$i = 1, \ldots, k-1$ These homomorphisms give rise to a section $\sum h_i$ of $\bigoplus_{i=1}^{k-1} \mathrm{Hom}(p_1^* S_i, p_2^* Q_i)_Z \to Z$

Have on Z: $h_{i+1}|S_i = q_{i+1} \circ h_i$ (indeed since h_i and h_{i+1} factorize through E, the two displayed homomorphisms from $(p_1^* S_i)_Z$ to $(p_2^* Q_{i+1})_Z$ are equal)

We get a section t of $H_Z \to Z$.

$\mathrm{rk}(G') + \mathrm{rk}(H) = \dim(F)$.

$Z(t) = \Delta$:

$\Delta \subset Z(t)$ taut. seq. on Grassannians are complexes

$f \in Z$; if $t(f) = 0$ then $f \in \Delta_F$.

Having defined s', t globally it suffices the assertion

$$Fl_{d.}(E) = \text{base} \times \underset{\substack{\parallel \\ F_x}}{Fl_{d.}(E_x)}$$

For $X = pt$ this is a proof that SL_m/P has (D).

$\pi_1(f) = \pi_2(f) = x$

$f = (L_1 \subset \cdots \subset L_{k-1} \subset L_k = E_x,\ M_1 \subset \cdots \subset M_{k-1} \subset M_k = E_x) \in F_x \times F_x$

restriction h_i to $F_x \times F_x$:

$$p_1^* S_i \to p_1^* V_{F_x} = V_{F_x \times F_x} = p_2^* V_{F_x} \to p_2^* Q_i'$$

At $f = ((L_i), (M_i))$ becomes

$$L_i \hookrightarrow V \twoheadrightarrow V/M_i$$

$$t(f) = 0 \Rightarrow L_i = M_i \Rightarrow Z(t) = \Delta_F . \ \square$$

Thm (K-P) If X has (P), then $Fl_{d.}(E)$ has (P').

$LG(\mathbb{C}^4) = Q_3$

X - variety, L line bundle on X

L-point property : $\forall_{x \in X}$ $\exists$ v.b. F on X and $s \in \Gamma(X, F)$ such that $d = \operatorname{rank} F = \dim X$, $\det F = L$, $Z(s) = x$.

Fact: X nonsing. proper $\operatorname{Pic} X$ - fin. gen. If for any coh. triv. line bundle L on X, either L^{-1}-point property fails or $L \otimes \omega_X^{-1}$ point prop. fails, then X has not (D).

$Q_3 \subset \mathbb{P}^4$ $L_1 = \mathcal{O}(-1)$, $L_2 = \mathcal{O}(-2)$ $\omega = \mathcal{O}(-3)$, $L_1^{-1} = L_2 \otimes \omega^{-1}$

Suff.: L_1^{-1}-point property fails : $c(E) = 1 + d_1[Q_2] + d_2[L] + d_3[P]$

There is no rank 3 v.b. E on Q_3 with $d_3 = 1$ and $d_1 = 1$

HRR: $\chi(Q_3, E) = \frac{15}{2} - 2d_2 \notin \mathbb{Z}$.