Thom polynomials and Schur functions

Piotr Pragacz

pragacz@impan.pl

IM PAN Warszawa

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If a *singularity class* Σ is "stable" (e.g. closed under the contact equivalence), then $\mathcal{T}^\Sigma$ depends on $c_i(TM - f^*TN)$.

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(R_m "parametrizes" TM for $\dim M = m$, similarly for R_n.)

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$A_i, k = 0$:

$(x, u_1, \ldots, u_{i-1}) \to (x^{i+1} + \sum_{j=1}^{i-1} u_j x^j, u_1, \ldots, u_{i-1})$

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$$\mathcal{T}^{A_1}(c_1(M), c_1(N)) = f^* c_1(N) - c_1(M).$$

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Well defined up to conjugacy; it can be chosen so that the images of its projections to the factors are *linear*. Its representations on the source and target will be denoted by

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$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)} \quad \text{and} \quad e(\eta) := e(E'_\eta).$$

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(i) if $\xi \neq \eta$ and $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$, then $\mathcal{T}^{\eta}(c(\xi)) = 0$;

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(i) if $\xi \neq \eta$ and $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$, then $\mathcal{T}^{\eta}(c(\xi)) = 0$;

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This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

For $k = 0$:

$A_1, \ldots, A_8, I_{2,2}, I_{2,3}, I_{2,4}, I_{3,3}, I_{2,5,}, I_{3,4}, I_{2,6}, I_{3,5}, I_{4,4}$.

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$\mathcal{T}_r^\eta$ = Thom polynomial of $\eta(r)$.

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$$\sum S_i(\mathbb{A} - \mathbb{B}) z^i = \prod_{b \in \mathbb{B}} (1 - bz) / \prod_{a \in \mathbb{A}} (1 - az) \, .$$

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Giambelli's formula: The dual of the class of a *Schubert variety* in a Grassmannian is given by a Schur polynomial of the tautological bundle on it.

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- proved by Berele-Regev in their study of polynomial characters of Lie superalgebras; particular cases known to 19th century algebraists: Pomey etc.

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Theorem. *(PP+AW, 2006) Let Σ be a singularity class. Then for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial*

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- interpreting $\mathcal{P}^i$ as a "generalized resultant" and using some specialization trick.

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Equations characterizing the Thom polynomial: A_0, A_1, A_2:

$$\mathcal{T}_r(-\mathbb{B}_{r-1}) = \mathcal{T}_r(x - \boxed{2x} - \mathbb{B}_{r-1}) = \mathcal{T}_r(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0\,,$$

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$I_{2,2}$:

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(The variables here correspond now to the Chern roots of the *cotangent* bundles).

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Lemma. *A partition appearing in the Schur function expansion of* $\mathcal{T}_r$ *contains* $(r+1, r+1)$ *and has at most three parts.*

Linear endomorphism Φ: $S_{i_1,i_2,i_3} \mapsto S_{i_1+1,i_2+1,i_3+1}$.

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Lemma. $\mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\mathcal{T}_{r-1})$.

Proposition. $\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1 x_2)^{r+1}\ S_{r-1}(\mathbb{D})$.

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The Segre class $s_{r-1}(\mathrm{Sym}^2(E))$ is:

$$\sum_{p\le q,\ p+q=r-1} \left[\binom{r}{p+1}+\binom{r}{p+2}+\cdots+\binom{r}{q+1}\right]\ S_{p,q}(E)\,.$$

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One gets a parametric (in "r") expression: $\mathcal{T}_r^{I_{2,2}} = \sum \alpha_I S_I$

Morin singularities $A_i(r)$. We define:

$$F_r^{(i)}(-) := \sum_J S_J(\boxed{2} + \boxed{3} + \cdots + \boxed{i}) S_{r-j_{i-1},\dots,r-j_1,r+|J|}(-)\,,$$

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where the sum is over partitions $J \subset (r^{i-1})$, and $F_r^{(1)}(-) = S_r(-)$.

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Theorem. $F_r^{(1)} = S_r$ *and* $F_r^{(2)} = \sum_{j \leq r} 2^j S_{r-j,r+j}$ *are the Thom polynomials of* $A_1(r)$ *and* $A_2(r)$.

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– results of Thom and Ronga.

Theorem. *(PP) Suppose that* $\Sigma^j(f) = \emptyset$ *for* $j \geq 2$. *(This says that on* $\Sigma^1(f)$*, the kernel of* $df : TM \to f^*TN$ *is a line bundle.) Then, for any* $r \geq 1$,

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The Schur expansions of the Thom polynomial $\mathcal{T}_r^{A_4}$ are not known (apart from $r = 1, 2, \mathbf{3}, \mathbf{4}$ – Ozer Ozturk).

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Complex case: Kazarian.

These authors used *monomials* in the Chern classes.

Every germ of a Lagrangian submanifold of V

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One has also the "Gauss fibration" $\mathcal{L}(V) \to LG(V)$ (which is not a vector bundle for $k \geq 3$).

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Geometric insight: The fundamental classes of the Schubert varieties in the *Lagrangian Grassmannian* $LG(V)$ are given by the appropriate $\widetilde{Q}$*-functions* of the tautological bundle on that Grassmannian (PP, 1986).

Thom polynomials of Lagrange and Legendre singularities up to codim 6.

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A_5: $\mathbf{27}\,\widetilde{\mathbf{Q}}_{\mathbf{31}} + \mathbf{60}\,\widetilde{\mathbf{Q}}_{\mathbf{4}} + t(22\,\widetilde{Q}_{21} + 86\,\widetilde{Q}_3) + t^2 40\,\widetilde{Q}_2 + t^3 6\,\widetilde{Q}_1$

D_5: $\mathbf{6}\,\widetilde{\mathbf{Q}}_{\mathbf{31}} + t\,4\,\widetilde{Q}_{21}$

A_6: $\mathbf{87\,\widetilde{Q}_{32}} + \mathbf{228\,\widetilde{Q}_{41}} + \mathbf{360\,\widetilde{Q}_{5}} + t(343\,\widetilde{Q}_{31} + 756\,\widetilde{Q}_{4}) + t^2(151\,\widetilde{Q}_{21} + 584\,\widetilde{Q}_{3}) + t^3 196\,\widetilde{Q}_{2} + t^4 24\,\widetilde{Q}_{1}$

A_6: $\mathbf{87}\,\widetilde{\mathbf{Q}}_{\mathbf{32}} + \mathbf{228}\,\widetilde{\mathbf{Q}}_{\mathbf{41}} + \mathbf{360}\,\widetilde{\mathbf{Q}}_{\mathbf{5}} + t(343\,\widetilde{Q}_{31} + 756\,\widetilde{Q}_4) +$
$t^2(151\,\widetilde{Q}_{21} + 584\,\widetilde{Q}_3) + t^3 196\,\widetilde{Q}_2 + t^4 24\,\widetilde{Q}_1$

D_6: $\mathbf{12}\,\widetilde{\mathbf{Q}}_{\mathbf{32}} + \mathbf{24}\,\widetilde{\mathbf{Q}}_{\mathbf{41}} + t32\,\widetilde{Q}_{31} + t^2 12\,\widetilde{Q}_{21}$

A_6: $\mathbf{87}\,\widetilde{\mathbf{Q}}_{\mathbf{32}} + \mathbf{228}\,\widetilde{\mathbf{Q}}_{\mathbf{41}} + \mathbf{360}\,\widetilde{\mathbf{Q}}_{\mathbf{5}} + t(343\,\widetilde{Q}_{31} + 756\,\widetilde{Q}_4) +$
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A_7: $\mathbf{135\,\widetilde{Q}_{321}} + \mathbf{1275\,\widetilde{Q}_{42}} + \mathbf{2004\,\widetilde{Q}_{51}} + \mathbf{2520\,\widetilde{Q}_{6}} + t(7092\,\widetilde{Q}_{5} + 4439\,\widetilde{Q}_{41} + 1713\,\widetilde{Q}_{32}) + t^2(3545\,\widetilde{Q}_{31} + 7868\,\widetilde{Q}_{4}) + t^3(1106\,\widetilde{Q}_{21} + 4292\,\widetilde{Q}_{3}) + t^4 1148\,\widetilde{Q}_{2} + t^5 120\,\widetilde{Q}_{1}$

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A_6: $\mathbf{87\,\widetilde{Q}_{32}} + \mathbf{228\,\widetilde{Q}_{41}} + \mathbf{360\,\widetilde{Q}_{5}} + t(343\,\widetilde{Q}_{31} + 756\,\widetilde{Q}_{4}) +$
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E_7: $\mathbf{9\,\widetilde{Q}_{321}} + \mathbf{60\,\widetilde{Q}_{42}} + \mathbf{24\,\widetilde{Q}_{51}} + t(56\,\widetilde{Q}_{41} + 66\,\widetilde{Q}_{32}) +$
$t^2 42\,\widetilde{Q}_{31} + t^3 10\,\widetilde{Q}_{21}$

A_6: $\mathbf{87}\,\widetilde{\mathbf{Q}}_{\mathbf{32}} + \mathbf{228}\,\widetilde{\mathbf{Q}}_{\mathbf{41}} + \mathbf{360}\,\widetilde{\mathbf{Q}}_{\mathbf{5}} + t(343\,\widetilde{Q}_{31} + 756\,\widetilde{Q}_4) +$
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P_8: $\widetilde{\mathbf{Q}}_{\mathbf{321}}$.

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Then MK+MM+PP+AW generalized that to a 1-parameter basis with nonnegativity property. By specializing the parameters, we recover the previous bases.

Our methods are based on nonnegativity of cone classes in gg vector bundles and on the Bertini-Kleiman “general translate theorem”.

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We also prove positivity; this ameliorates our former result for the Lagrange singularities.

Final comments

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A localization formula was used earlier for Morin singularities by Berczi-Szenes.

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A good sign that localization formulas can be also used to find S.e. of Thom polynomials, is the following translation of a recent result of Feher and Rimanyi proved using l.f. (they state the result using monomials in Chern classes) :

Theorem. *Let η be a stable singularity.*

1. By erasing the maximal columns from the S.e. of $\mathcal{T}_r^\eta$ we get $\mathcal{T}_{r-1}^\eta$.

2. The length of any partition in S.e. of $\mathcal{T}_r^\eta$ is $\leq \dim(Q_\eta) - 1$.

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THE END