Positivity of Legendrian Thom polynomials

Piotr Pragacz

pragacz@impan.pl

IM PAN Warszawa

Classical Thom polynomials for $f : M \to N$.

Theorem. *(PP+A.Weber, 2006) Let Σ be a singularity class. Then for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^{\Sigma} = \sum \alpha_I S_I(T^*M - f^*T^*N)\,,$$

is nonnegative.

Classical Thom polynomials for $f : M \to N$.

Theorem. *(PP+A.Weber, 2006) Let* Σ *be a singularity class. Then for any partition* I *the coefficient* α_I *in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^{\Sigma} = \sum \alpha_I S_I(T^*M - f^*T^*N)\,,$$

is nonnegative.

Theorem. *(M.Mikosz+PP+AW, 2007) For any Lagrange singularity class* Σ*, the Thom polynomial* $\mathcal{T}^{\Sigma}$ *is a nonnegative combination of* $\widetilde{Q}$*-functions.*

Classical Thom polynomials for $f : M \to N$.

Theorem. *(PP+A.Weber, 2006) Let* Σ *be a singularity class. Then for any partition* I *the coefficient* α_I *in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^\Sigma = \sum \alpha_I S_I(T^*M - f^*T^*N)\,,$$

is nonnegative.

Theorem. *(M.Mikosz+PP+AW, 2007) For any Lagrange singularity class* Σ*, the Thom polynomial* $\mathcal{T}^\Sigma$ *is a nonnegative combination of* $\widetilde{Q}$*-functions.*

Kazarian: What about Legendrian singularities?

Classical Thom polynomials for $f : M \to N$.

Theorem. *(PP+A.Weber, 2006) Let Σ be a singularity class. Then for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^{\Sigma} = \sum \alpha_I S_I(T^*M - f^*T^*N)\,,$$

is nonnegative.

Theorem. *(M.Mikosz+PP+AW, 2007) For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^{\Sigma}$ is a nonnegative combination of $\widetilde{Q}$-functions.*

Kazarian: What about Legendrian singularities?

Joint work with M. Mikosz and A. Weber

Legendrian geometry

Fix $n \in \mathbb{N}$.
W and ξ are vector spaces of dimension n and 1.

Legendrian geometry

Fix $n \in \mathbb{N}$.
W and ξ are vector spaces of dimension n and 1.

$$V := W \oplus (W^* \otimes \xi)\,.$$

Legendrian geometry

Fix $n \in \mathbb{N}$.
W and ξ are vector spaces of dimension n and 1.

$$V := W \oplus (W^* \otimes \xi)\,.$$

Standard *contact space*

$$V \oplus \xi = W \oplus (W^* \otimes \xi) \oplus \xi\,,$$

Legendrian geometry

Fix $n \in \mathbb{N}$.
W and ξ are vector spaces of dimension n and 1.

$$V := W \oplus (W^* \otimes \xi)\,.$$

Standard *contact space*

$$V \oplus \xi = W \oplus (W^* \otimes \xi) \oplus \xi\,,$$

equipped with the standard contact form

$$\alpha := dx - \sum_{i=1}^{n} p_i\, dq_i\,,$$

where x is a coordinate of ξ, q_i are the coordinates of W and p_i are dual coordinates of $W^* \otimes \xi$.

Legendrian submanifolds of $V \oplus \xi$ are maximal integral submanifolds of α, i.e. the manifolds of dimension n with tangent spaces contained in $\mathrm{Ker}(\alpha)$.

Legendrian submanifolds of $V \oplus \xi$ are maximal integral submanifolds of α, i.e. the manifolds of dimension n with tangent spaces contained in $\mathrm{Ker}(\alpha)$.

Example: The plane $x = const, z = const$ in the standard contact space with coordinates x, y, z and with form $\alpha = dz - ydz$ is Legendrian.

Legendrian submanifolds of $V \oplus \xi$ are maximal integral submanifolds of α, i.e. the manifolds of dimension n with tangent spaces contained in $\mathrm{Ker}(\alpha)$.

Example: The plane $x = const, z = const$ in the standard contact space with coordinates x, y, z and with form $\alpha = dz - ydz$ is Legendrian.

The space V is equipped with the symplectic form

$$\omega := \sum_{i=1}^{n} dp_i \wedge dq_i,$$

which again depends on the coordinate of ξ. It is well defined as an element of $\bigwedge^2 V^* \otimes \xi$. A submanifold of V is *Lagrangian* if ω restricted to its tangent spaces vanishes.

By Legendrian (resp. Lagrangian) submanifolds we shall mean the *germs* of such submanifolds through the origin in $V \oplus \xi$ and V, respectively.

By Legendrian (resp. Lagrangian) submanifolds we shall mean the *germs* of such submanifolds through the origin in $V \oplus \xi$ and V, respectively.

Lemma. *The projection of a Legendrian submanifold from* $V \oplus \xi$ *to* V *is a Lagrangian submanifold. All the Lagrangian submanifolds of* V *are obtainable in this way. Moreover, a Legendrian submanifold of* $V \oplus \xi$ *is uniquely determined by its Lagrangian image.*

By Legendrian (resp. Lagrangian) submanifolds we shall mean the *germs* of such submanifolds through the origin in $V \oplus \xi$ and V, respectively.

Lemma. *The projection of a Legendrian submanifold from $V \oplus \xi$ to V is a Lagrangian submanifold. All the Lagrangian submanifolds of V are obtainable in this way. Moreover, a Legendrian submanifold of $V \oplus \xi$ is uniquely determined by its Lagrangian image.*

We describe a space which parametrizes pairs of Legendrian submanifolds L_1, L_2. We say that two pairs of Legendrian submanifolds in $V \oplus \xi$ are *contact equivalent* if they differ by a holomorphic contactomorphism of $V \oplus \xi$.

By Legendrian (resp. Lagrangian) submanifolds we shall mean the *germs* of such submanifolds through the origin in $V \oplus \xi$ and V, respectively.

Lemma. *The projection of a Legendrian submanifold from $V \oplus \xi$ to V is a Lagrangian submanifold. All the Lagrangian submanifolds of V are obtainable in this way. Moreover, a Legendrian submanifold of $V \oplus \xi$ is uniquely determined by its Lagrangian image.*

We describe a space which parametrizes pairs of Legendrian submanifolds L_1, L_2. We say that two pairs of Legendrian submanifolds in $V \oplus \xi$ are *contact equivalent* if they differ by a holomorphic contactomorphism of $V \oplus \xi$.

Lemma. *Any pair of Lagrangian submanifolds is symplectic equivalent to a pair (L_1, L_2) such that L_1 is a linear Lagrangian submanifold and the tangent space T_0L_2 is equal to W.*

A vector space ξ has no distinguished coordinate. We have to work with ξ – a line bundle over some base space X. The same applies to W – now possibly a nontrivial bundle.

A vector space ξ has no distinguished coordinate. We have to work with ξ – a line bundle over some base space X. The same applies to W – now possibly a nontrivial bundle.
Let

$$\tau : Leg(W, \xi) \to X$$

denote the Legendre Grassmann bundle parametrizing Legendrian submanifolds in $V_x \oplus \xi_x$, $x \in X$ whose projections to V_x are $linear$ spaces.

A vector space ξ has no distinguished coordinate. We have to work with ξ – a line bundle over some base space X. The same applies to W – now possibly a nontrivial bundle.
Let

$$\tau : Leg(W, \xi) \to X$$

denote the Legendre Grassmann bundle parametrizing Legendrian submanifolds in $V_x \oplus \xi_x$, $x \in X$ whose projections to V_x are $linear$ spaces.
We shall often identify $Leg(W, \xi)$ with the Lagrange Grassmann bundle

$$\tau : LG(V, \omega) \to X$$

since any Legendre submanifold in $V_x \oplus \xi_x$ is determined by its projection to V_x.

Tautological bundle over $Leg(W,\xi)$ is denoted by R.
We have the tautological sequence on $Leg(W,\xi)$:

$$0 \to R \to V \to R^* \otimes \xi \to 0\,.$$

Tautological bundle over $Leg(W,\xi)$ is denoted by R.
We have the tautological sequence on $Leg(W,\xi)$:

$$0 \to R \to V \to R^* \otimes \xi \to 0\,.$$

Fix $k \in \mathbb{N}$. We identify two Legendrian submanifolds if the degree of their tangency at 0 is greater than or equal to k. The equivalence class will be called "a k-jet of a submanifold".

Tautological bundle over $Leg(W,\xi)$ is denoted by R.
We have the tautological sequence on $Leg(W,\xi)$:

$$0 \to R \to V \to R^* \otimes \xi \to 0\,.$$

Fix $k \in \mathbb{N}$. We identify two Legendrian submanifolds if the degree of their tangency at 0 is greater than or equal to k. The equivalence class will be called “a k-jet of a submanifold”.

Let $\mathcal{C}^k(W,\xi)$ be the set of pairs of k-jets of Legendrian submanifolds $(L_1, L_2) \subset V_x \oplus \xi_x$ s.t. the projection of L_1 to V_x is a linear space and $T_0 L_2 = W_x$. Let

$$\pi : \mathcal{C}^k(W,\xi) \to Leg(W,\xi)$$

denote the projection such that $\pi(L_1, L_2) = L_1$.

Local study

Every k-jet of a Legendrian submanifold L in $V_x \oplus \xi_x$ is the graph of a 1-form

$$\alpha : W_x \to (W_x^* \otimes \xi_x) \oplus \xi_x \,.$$

The condition that L is Legendrian is equivalent to $d\alpha = 0$.

Local study

Every k-jet of a Legendrian submanifold L in $V_x \oplus \xi_x$ is the graph of a 1-form

$$\alpha : W_x \to (W_x^* \otimes \xi_x) \oplus \xi_x \, .$$

The condition that L is Legendrian is equivalent to $d\alpha = 0$. Therefore there exists a polynomial function

$$f : W_x \to \xi_x$$

of degree $k+1$ such that $\alpha = df$. The assumption $0 \in L$ is equivalent to $df(0) = 0$. In other words, L is the graph of the 1-jet of f.

Local study

Every k-jet of a Legendrian submanifold L in $V_x \oplus \xi_x$ is the graph of a 1-form

$$\alpha : W_x \to (W_x^* \otimes \xi_x) \oplus \xi_x \, .$$

The condition that L is Legendrian is equivalent to $d\alpha = 0$. Therefore there exists a polynomial function

$$f : W_x \to \xi_x$$

of degree $k+1$ such that $\alpha = df$. The assumption $0 \in L$ is equivalent to $df(0) = 0$. In other words, L is the graph of the 1-jet of f.
If, in addition, the tangency condition

$$T_0 L = W_x$$

holds, then the second jet of f vanish at 0.

Global picture

Thus we can identify $\pi^{-1}(W_x)$ with

$$\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W_x^*) \otimes \xi_x \,.$$

In fact, we obtain the following Cartesian square:

$$\begin{array}{ccc} \mathcal{C}^k(W,\xi) & \stackrel{\pi}{\rightarrow} & Leg(W,\xi) \\ \downarrow & & \downarrow \tau \\ \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi & \rightarrow & X \,. \end{array}$$

By a *Legendre singularity class* we mean a closed algebraic subset $\Sigma \subset \mathcal{C}^k(\mathbb{C}^n, \mathbb{C})$, invariant with respect to holomorphic contactomorphisms of $\mathbb{C}^{2n+1}$. (It is a union of contact equivalence classes.) Additionally, we assume that the singularity class Σ is *stable* with respect to enlarging the dimension of W. Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines a cycle

$$\Sigma(W, \xi) \subset \mathcal{C}^k(W, \xi).$$

By a *Legendre singularity class* we mean a closed algebraic subset $\Sigma \subset \mathcal{C}^k(\mathbb{C}^n, \mathbb{C})$, invariant with respect to holomorphic contactomorphisms of $\mathbb{C}^{2n+1}$. (It is a union of contact equivalence classes.) Additionally, we assume that the singularity class Σ is *stable* with respect to enlarging the dimension of W. Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines a cycle

$$\Sigma(W, \xi) \subset \mathcal{C}^k(W, \xi).$$

The element $PD[\Sigma(W, \xi)]$ of $H^*(\mathcal{C}^k(W, \xi), \mathbb{Z})$, which is the Poincaré dual of $[\Sigma(W, \xi)]$ is called the *Legendrian Thom polynomial* of Σ.

By a *Legendre singularity class* we mean a closed algebraic subset $\Sigma \subset \mathcal{C}^k(\mathbb{C}^n, \mathbb{C})$, invariant with respect to holomorphic contactomorphisms of $\mathbb{C}^{2n+1}$. (It is a union of contact equivalence classes.) Additionally, we assume that the singularity class Σ is *stable* with respect to enlarging the dimension of W. Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines a cycle

$$\Sigma(W, \xi) \subset \mathcal{C}^k(W, \xi).$$

The element $PD[\Sigma(W, \xi)]$ of $H^*(\mathcal{C}^k(W, \xi), \mathbb{Z})$, which is the Poincaré dual of $[\Sigma(W, \xi)]$ is called the *Legendrian Thom polynomial* of Σ.

To understand the structure of these polynomials, we need a bit of Schubert Calculus.

Legendrian Grassmann bundles

Let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimension one and let

$$W := \bigoplus_{i=1}^{n} \alpha_i \,, \qquad V := W \oplus (W^* \otimes \xi) \,.$$

Legendrian Grassmann bundles

Let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimension one and let

$$W := \bigoplus_{i=1}^{n} \alpha_i \,, \qquad V := W \oplus (W^* \otimes \xi) \,.$$

We have a symplectic form ω defined on V with values in ξ.
$LG(V, \omega)$ is a homogeneous space for the symplectic group $Sp(V, \omega) \subset \mathrm{End}(V)$.

Legendrian Grassmann bundles

Let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimension one and let

$$W := \bigoplus_{i=1}^{n} \alpha_i \,, \qquad V := W \oplus (W^* \otimes \xi)\,.$$

We have a symplectic form ω defined on V with values in ξ.
$LG(V,\omega)$ is a homogeneous space for the symplectic group $Sp(V,\omega) \subset \mathrm{End}(V)$.
Fix two "opposite" standard isotropic flags in V:

$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i \,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha_{n-i+1}^* \otimes \xi \,, \qquad (h = 1, 2, \ldots, n)$$

Legendrian Grassmann bundles

Let $\xi, \alpha_1, \alpha_2, \dots, \alpha_n$ be vector spaces of dimension one and let

$$W := \bigoplus_{i=1}^{n} \alpha_i \,, \qquad V := W \oplus (W^* \otimes \xi)\,.$$

We have a symplectic form ω defined on V with values in ξ.
$LG(V,\omega)$ is a homogeneous space for the symplectic group $Sp(V,\omega) \subset \mathrm{End}(V)$.
Fix two "opposite" standard isotropic flags in V:

$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i \,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha^*_{n-i+1} \otimes \xi \,, \qquad (h = 1, 2, \dots, n)$$

Consider two Borel groups $B^\pm \subset Sp(V,\omega)$, preserving the flags $F_\bullet^\pm$. The orbits of $B^\pm$ in $LG(V,\omega)$ form two "opposite" cell decompositions $\{\Omega_I(F_\bullet^\pm, \xi)\}$ of $LG(V,\omega)$.

The decompositions are indexed by strict partitions I contained in $\rho = (n, n-1, \ldots, 2, 1)$.

The "$+$" cells are transverse to the "$-$" cells.

The decompositions are indexed by strict partitions I contained in $\rho = (n, n-1, \ldots, 2, 1)$.

The "$+$" cells are transverse to the "$-$" cells.

All that is functorial w.r.t. the automorphisms of the lines ξ and α_i's, (they form a torus $(\mathbb{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles ξ and $\{\alpha_i\}_{i=1}^n$ over any base X. We get a Lagrange Grassmann bundle

$$\tau : LG(V, \omega) = Leg(W, \xi) \to X$$

together with two subgroup bundles $B^{\pm} \to X$.

The decompositions are indexed by strict partitions I contained in $\rho = (n, n-1, \dots, 2, 1)$.

The "$+$" cells are transverse to the "$-$" cells.

All that is functorial w.r.t. the automorphisms of the lines ξ and α_i's, (they form a torus $(\mathbb{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles ξ and $\{\alpha_i\}_{i=1}^n$ over any base X. We get a Lagrange Grassmann bundle

$$\tau : LG(V, \omega) = Leg(W, \xi) \to X$$

together with two subgroup bundles $B^{\pm} \to X$.

$Leg(W, \xi)$ admits two (relative) stratifications

$$\{\Omega_I(F_\bullet^{\pm}, \xi) \to X\}_I$$

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W, \xi) \to X$.

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W, \xi) \to X$. The classes of their closures give a basis of homology.

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W, \xi) \to X$. The classes of their closures give a basis of homology. Each $Z^-_{I\lambda}$ is transverse to each stratum $\Omega_J(F^+_\bullet, \xi)$.

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W,\xi) \to X$. The classes of their closures give a basis of homology. Each $Z^-_{I\lambda}$ is transverse to each stratum $\Omega_J(F^+_\bullet, \xi)$.

Consider $F := \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \to X$.

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W, \xi) \to X$. The classes of their closures give a basis of homology. Each $Z^-_{I\lambda}$ is transverse to each stratum $\Omega_J(F^+_\bullet, \xi)$.

Consider $F := \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \to X$.
Pulling back F to $Leg(W, \xi) \to X$, we get the following jet bundle:

$$\pi : E := \tau^* F \to Leg(W, \xi).$$

Assume that X is a compact homogeneous space under G; $X = G/P$. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-_\bullet, \xi)$$

form an algebraic cell decomposition of the space of $Leg(W, \xi) \to X$. The classes of their closures give a basis of homology. Each $Z^-_{I\lambda}$ is transverse to each stratum $\Omega_J(F^+_\bullet, \xi)$.

Consider $F := \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \to X$.
Pulling back F to $Leg(W, \xi) \to X$, we get the following jet bundle:

$$\pi : E := \tau^* F \to Leg(W, \xi).$$

The space of the bundle E is equal to $\mathcal{C}^k(W, \xi)$.

A singularity class Σ defines the subset $\Sigma(W, \xi) \subset E$, which is Zariski fibered over X.

A singularity class Σ defines the subset $\Sigma(W, \xi) \subset E$, which is Zariski fibered over X.

Theorem. *Fix $I \subset \rho$ and λ. Suppose that the vector bundle E is generated by its global sections. Then, in E, the intersection of $\Sigma(W, \xi)$ with the closure of any $\pi^{-1}(Z^{-}_{I\lambda})$ is represented by a nonnegative cycle.*

A singularity class Σ defines the subset $\Sigma(W,\xi) \subset E$, which is Zariski fibered over X.

Theorem. *Fix* $I \subset \rho$ *and* λ*. Suppose that the vector bundle* E *is generated by its global sections. Then, in* E*, the intersection of* $\Sigma(W,\xi)$ *with the closure of any* $\pi^{-1}(Z^{-}_{I\lambda})$ *is represented by a nonnegative cycle.*

Let $\iota : Leg(W,\xi) \to \mathcal{C}^k(W,\xi)$ be the zero section.

A singularity class Σ defines the subset $\Sigma(W,\xi) \subset E$, which is Zariski fibered over X.

Theorem. *Fix $I \subset \rho$ and λ. Suppose that the vector bundle E is generated by its global sections. Then, in E, the intersection of $\Sigma(W,\xi)$ with the closure of any $\pi^{-1}(Z^{-}_{I\lambda})$ is represented by a nonnegative cycle.*

Let $\iota : Leg(W,\xi) \to \mathcal{C}^k(W,\xi)$ be the zero section.
For the induced map ι^* on integral cohomology rings,

$$\iota^* PD[\Sigma(W,\xi)] =: \sum_{I,\lambda} \gamma_{I\lambda} [\overline{Z^{-}_{I\lambda}}]^*$$

A singularity class Σ defines the subset $\Sigma(W,\xi) \subset E$, which is Zariski fibered over X.

Theorem. *Fix $I \subset \rho$ and λ. Suppose that the vector bundle E is generated by its global sections. Then, in E, the intersection of $\Sigma(W,\xi)$ with the closure of any $\pi^{-1}(Z^-_{I\lambda})$ is represented by a nonnegative cycle.*

Let $\iota : Leg(W,\xi) \to \mathcal{C}^k(W,\xi)$ be the zero section.
For the induced map ι^* on integral cohomology rings,

$$\iota^* PD[\Sigma(W,\xi)] =: \sum_{I,\lambda} \gamma_{I\lambda} [\overline{Z^-_{I\lambda}}]^*$$

If F is globally generated, then $\gamma_{I\lambda} \geq 0$:

A singularity class Σ defines the subset $\Sigma(W,\xi) \subset E$, which is Zariski fibered over X.

Theorem. *Fix $I \subset \rho$ and λ. Suppose that the vector bundle E is generated by its global sections. Then, in E, the intersection of $\Sigma(W,\xi)$ with the closure of any $\pi^{-1}(Z^-_{I\lambda})$ is represented by a nonnegative cycle.*

Let $\iota : Leg(W,\xi) \to \mathcal{C}^k(W,\xi)$ be the zero section.
For the induced map ι^* on integral cohomology rings,

$$\iota^* PD[\Sigma(W,\xi)] =: \sum_{I,\lambda} \gamma_{I\lambda} [\overline{Z^-_{I\lambda}}]^*$$

If F is globally generated, then $\gamma_{I\lambda} \geq 0$:

$$\gamma_{I\lambda} = \langle \iota^* PD[\Sigma(W,\xi)], [\overline{Z^-_{I\lambda}}] \rangle = [\Sigma(W,\xi)] \cdot [\overline{Z^-_{I\lambda}}] \geq 0\,.$$

Suppose that $W = \alpha^{\oplus n}$ and $\alpha^{-m} \otimes \xi$ is g.g. for $m \geq 3$.

.

Suppose that $W = \alpha^{\oplus n}$ and $\alpha^{-m} \otimes \xi$ is g.g. for $m \geq 3$.

Let $X = \mathbf{P}^n$ and $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$

$$\xi_2 = \mathcal{O}(1), \alpha_2 = \mathbf{1} \quad \xi_3 = \mathcal{O}(-3), \alpha_3 = \mathcal{O}(-1)\,,$$

.

Suppose that $W = \alpha^{\oplus n}$ and $\alpha^{-m} \otimes \xi$ is g.g. for $m \geq 3$.

Let $X = \mathbf{P}^n$ and $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$

$$\xi_2 = \mathcal{O}(1), \alpha_2 = \mathbf{1} \quad \xi_3 = \mathcal{O}(-3), \alpha_3 = \mathcal{O}(-1)\,,$$

Our naive proof of the positivity property in the first case was the starting point of this project. The positivity property of the last two cases was suggested to us by Kazarian, and checked by him on computer up to degree 7.

Suppose that $W = \alpha^{\oplus n}$ and $\alpha^{-m} \otimes \xi$ is g.g. for $m \geq 3$.

Let $X = \mathbf{P}^n$ and $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$

$$\xi_2 = \mathcal{O}(1), \alpha_2 = \mathbf{1} \quad \xi_3 = \mathcal{O}(-3), \alpha_3 = \mathcal{O}(-1)\,,$$

Our naive proof of the positivity property in the first case was the starting point of this project. The positivity property of the last two cases was suggested to us by Kazarian, and checked by him on computer up to degree 7.

To overlap all these three cases we consider the product

$$X := \mathbf{P}^n \times \mathbf{P}^n$$

$$W := p_1^*\mathcal{O}(-1)^{\oplus n}\,, \qquad \xi := p_1^*\mathcal{O}(-3) \otimes p_2^*\mathcal{O}(1)\,,$$

where $p_i : X \to \mathbf{P}^n$, $i = 1, 2$, are the projections.

Suppose that $W = \alpha^{\oplus n}$ and $\alpha^{-m} \otimes \xi$ is g.g. for $m \geq 3$.

Let $X = \mathbf{P}^n$ and $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$

$$\xi_2 = \mathcal{O}(1), \alpha_2 = \mathbf{1} \quad \xi_3 = \mathcal{O}(-3), \alpha_3 = \mathcal{O}(-1)\,,$$

Our naive proof of the positivity property in the first case was the starting point of this project. The positivity property of the last two cases was suggested to us by Kazarian, and checked by him on computer up to degree 7.

To overlap all these three cases we consider the product

$$X := \mathbf{P}^n \times \mathbf{P}^n$$

$$W := p_1^*\mathcal{O}(-1)^{\oplus n}\,, \qquad \xi := p_1^*\mathcal{O}(-3) \otimes p_2^*\mathcal{O}(1)\,,$$

where $p_i : X \to \mathbf{P}^n$, $i = 1, 2$, are the projections. Restricting the bundles W and ξ to the diagonal, or to the factors we obtain our three cases.

The space $Leg(W, \xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, The space $Leg(W, \xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, where strict $I \subset \rho$, $a \leq n$, $b \leq n$.

The space $Leg(W,\xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, where strict $I \subset \rho$, $a \leq n$, $b \leq n$.
The classes of closures of the cells of this decomposition give a basis of the homology of $Leg(W,\xi)$.

The space $Leg(W,\xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, where strict $I \subset \rho$, $a \leq n$, $b \leq n$.
The classes of closures of the cells of this decomposition give a basis of the homology of $Leg(W,\xi)$.
The dual basis of cohomology is denoted by

$$e_{I,a,b} := [\overline{Z^-_{I(a,b)}}]^* \,.$$

The space $Leg(W,\xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, where strict $I \subset \rho$, $a \leq n$, $b \leq n$.
The classes of closures of the cells of this decomposition give a basis of the homology of $Leg(W,\xi)$.
The dual basis of cohomology is denoted by

$$e_{I,a,b} := [\overline{Z^-_{I(a,b)}}]^* \,.$$

Theorem. *Let* Σ *be a Legendre singularity class. Then* $PD[\Sigma(W,\xi)]$ *has nonnegative coefficients in the basis* $\{e_{I,a,b}\}$.

The space $Leg(W,\xi)$ has a cell decomposition $Z^-_{I\lambda} = Z^-_{I(a,b)}$, where strict $I \subset \rho$, $a \leq n$, $b \leq n$.
The classes of closures of the cells of this decomposition give a basis of the homology of $Leg(W,\xi)$.
The dual basis of cohomology is denoted by

$$e_{I,a,b} := [\overline{Z^-_{I(a,b)}}]^* \,.$$

Theorem. *Let Σ be a Legendre singularity class. Then $PD[\Sigma(W,\xi)]$ has nonnegative coefficients in the basis $\{e_{I,a,b}\}$.*
The vector bundle F on $X = \mathbf{P}^n \times \mathbf{P}^n$ is globally generated:

$$F = \bigoplus_{j=3}^{k+1} \mathrm{Sym}^j(W^*) \otimes \xi = \bigoplus_{j=3}^{k+1} \mathrm{Sym}^j(\mathbf{1}^n) \otimes p_1^*\mathcal{O}(j-3) \otimes p_2^*\mathcal{O}(1).$$

Let for $i = 1, 2$, $v_i = c_1(p_i^*(\mathcal{O}(1))$.

Let for $i = 1, 2$, $v_i = c_1(p_i^*(\mathcal{O}(1))$.

$$\mathcal{T}^{\Sigma} = \sum_{I,a,b} \gamma_{I,a,b}\; e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} PD[\Omega_I(F_\bullet^+, \xi)] v_1^a v_2^b\,.$$

Let for $i = 1, 2$, $v_i = c_1(p_i^*(\mathcal{O}(1))$.

$$\mathcal{T}^{\Sigma} = \sum_{I,a,b} \gamma_{I,a,b}\, e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} PD[\Omega_I(F_{\bullet}^{+}, \xi)] v_1^a v_2^b \,.$$

Want: an additive basis of the ring of Legendrian characteristic classes with the property that any Legendrian Thom polynomial is a nonnegative combination of basis elements.

Let for $i = 1, 2$, $v_i = c_1(p_i^*(\mathcal{O}(1))$.

$$\mathcal{T}^{\Sigma} = \sum_{I,a,b} \gamma_{I,a,b}\, e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} PD[\Omega_I(F_\bullet^+, \xi)] v_1^a v_2^b\,.$$

Want: an additive basis of the ring of Legendrian characteristic classes with the property that any Legendrian Thom polynomial is a nonnegative combination of basis elements.

Take a pair of integers p, q.

Let for $i = 1, 2$, $v_i = c_1(p_i^*(\mathcal{O}(1))$.

$$\mathcal{T}^{\Sigma} = \sum_{I,a,b} \gamma_{I,a,b}\, e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} PD[\Omega_I(F_\bullet^+, \xi)] v_1^a v_2^b \,.$$

Want: an additive basis of the ring of Legendrian characteristic classes with the property that any Legendrian Thom polynomial is a nonnegative combination of basis elements.

Take a pair of integers p, q.

$$\xi^{(p,q)} = \xi_2^{\otimes p} \otimes \xi_3^{\otimes q}$$

$$\alpha = \alpha^{(p,q)} = \alpha_2^{\otimes p} \otimes \alpha_3^{\otimes q} = \alpha_3^{\otimes q}$$

Divide $H^*(Leg(W,\xi),\mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2$$

Divide $H^*(Leg(W,\xi),\mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2$$

that is specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(Leg(W^{(p,q)},\xi^{(p,q)}),\mathbb{Q})$ isomorphic to the ring of Legendrian characteristic classes in degrees up to n (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0 and $(p,q) \neq (0,0)$.)

Divide $H^*(Leg(W,\xi),\mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2$$

that is specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(Leg(W^{(p,q)},\xi^{(p,q)}),\mathbb{Q})$ isomorphic to the ring of Legendrian characteristic classes in degrees up to n (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0 and $(p,q) \neq (0,0)$.)

Theorem. *If p and q are nonnegative, $q - 3p \neq 0$ and $(p,q) \neq (0,0)$ then the Thom polynomial is a nonnegative combination of the $PD[\Omega_I(F_\bullet^+,\xi)]\, t^i$'s.*

Divide $H^*(Leg(W,\xi),\mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2$$

that is specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(Leg(W^{(p,q)},\xi^{(p,q)}),\mathbb{Q})$ isomorphic to the ring of Legendrian characteristic classes in degrees up to n (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0 and $(p,q) \neq (0,0)$.)

Theorem. *If p and q are nonnegative, $q - 3p \neq 0$ and $(p,q) \neq (0,0)$ then the Thom polynomial is a nonnegative combination of the $PD[\Omega_I(F_\bullet^+,\xi)]\, t^i$'s.*

The family $PD[\Omega_I(F_\bullet^+,\xi)]\, t^i$ is a one-parameter family of bases depending on the parameter p/q.

Case 1. $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$. This corresponds to fixing the parameter to be 1; $p = 1$ and $q = 1$; $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles W and ξ to the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$.

Case 1. $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$. This corresponds to fixing the parameter to be 1; $p = 1$ and $q = 1$; $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles W and ξ to the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$.

In the next theorem A is a virtual bundle $W^* \otimes \xi - R$, and t is half the first Chern class of ξ^*.

Case 1. $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$. This corresponds to fixing the parameter to be 1; $p = 1$ and $q = 1$; $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles W and ξ to the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$.

In the next theorem A is a virtual bundle $W^* \otimes \xi - R$, and t is half the first Chern class of ξ^*.

Theorem. *The Thom polynomial of a Legendre singularity class Σ is a combination:*

$$\mathcal{T}^{\Sigma} = \sum_{j \geq 0} \sum_{I} \alpha_{I,j} \; \widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \cdot t^j \, .$$

Here I runs over strict partitions in ρ, and $\alpha_{I,j}$ are nonnegative integers.

Legendrian vs. classical

$t = v_1 = v_2$

Legendrian vs. classical

$t = v_1 = v_2$

Proposition. *For a nonempty stable Legendre singularity class Σ, the Lagrangian Thom polynomial (i.e. $\mathcal{T}^\Sigma$ evaluated at $t = 0$) is nonzero. (So, also $\mathcal{T}^\Sigma$ is nonzero.)*

Legendrian vs. classical

$t = v_1 = v_2$

Proposition. *For a nonempty stable Legendre singularity class Σ, the Lagrangian Thom polynomial (i.e. $\mathcal{T}^\Sigma$ evaluated at $t = 0$) is nonzero. (So, also $\mathcal{T}^\Sigma$ is nonzero.)*

Kazarian: The classification of Legendre singularities is parallel to the classification of critical point singularities w.r.t. stable right equivalence. For a Legendre singularity class Σ consider the associated singularity class of maps $f : M \to C$ from n-dimensional manifolds to curves. We denote the related Thom polynomial by Tp^Σ.

Legendrian vs. classical

$t = v_1 = v_2$

Proposition. *For a nonempty stable Legendre singularity class Σ, the Lagrangian Thom polynomial (i.e. $\mathcal{T}^\Sigma$ evaluated at $t = 0$) is nonzero. (So, also $\mathcal{T}^\Sigma$ is nonzero.)*

Kazarian: The classification of Legendre singularities is parallel to the classification of critical point singularities w.r.t. stable right equivalence. For a Legendre singularity class Σ consider the associated singularity class of maps $f : M \to C$ from n-dimensional manifolds to curves. We denote the related Thom polynomial by Tp^Σ.

We have

$$Tp^\Sigma = \mathcal{T}^\Sigma \cdot c_n(T^*M \otimes f^*TC)\,.$$

Legendrian vs. classical

$t = v_1 = v_2$

Proposition. *For a nonempty stable Legendre singularity class* Σ*, the Lagrangian Thom polynomial (i.e.* $\mathcal{T}^\Sigma$ *evaluated at* $t=0$*) is nonzero. (So, also* $\mathcal{T}^\Sigma$ *is nonzero.)*

Kazarian: The classification of Legendre singularities is parallel to the classification of critical point singularities w.r.t. stable right equivalence. For a Legendre singularity class Σ consider the associated singularity class of maps $f : M \to C$ from n-dimensional manifolds to curves. We denote the related Thom polynomial by Tp^Σ.

We have

$$Tp^\Sigma = \mathcal{T}^\Sigma \cdot c_n(T^*M \otimes f^*TC)\,.$$

We know that Tp^Σ is nonzero. One shows that Tp^Σ, specialized with $f^*TC = \mathbf{1}$ i.e. $t = 0$, is also nonzero. The assertion follows from the equation.

Final remarks

Chern class formula for

$$PD[\Omega_I(F_\bullet^+, \xi)]$$

depending on the Chern classes of ξ, R, W, F_i^+ $i = 1, \ldots, n$
– still to be found: PP(1986), PP-Ratajski, Lascoux-PP, Buch-Kresch-Tamvakis, Kazarian (2009).

Final remarks

Chern class formula for

$$PD[\Omega_I(F^+_\bullet, \xi)]$$

depending on the Chern classes of ξ, R, W, F^+_i $i = 1, \ldots, n$
– still to be found: PP(1986), PP-Ratajski, Lascoux-PP, Buch-Kresch-Tamvakis, Kazarian (2009).

The Legendrian coefficients are difficult to compute. Knowing that they are nonnegative, find upper bounds for them!

Final remarks

Chern class formula for

$$PD[\Omega_I(F_\bullet^+, \xi)]$$

depending on the Chern classes of ξ, R, W, F_i^+ $i = 1, \ldots, n$
– still to be found: PP(1986), PP-Ratajski, Lascoux-PP, Buch-Kresch-Tamvakis, Kazarian (2009).

The Legendrian coefficients are difficult to compute. Knowing that they are nonnegative, find upper bounds for them!

First method uses the equation relating Legendrian and classical Thom polynomials. Algebraically it is some instance of the “factorization formula” for super Schur functions.

Final remarks

Chern class formula for

$$PD[\Omega_I(F^+_\bullet, \xi)]$$

depending on the Chern classes of ξ, R, W, F^+_i $i = 1, \ldots, n$
– still to be found: PP(1986), PP-Ratajski, Lascoux-PP, Buch-Kresch-Tamvakis, Kazarian (2009).

The Legendrian coefficients are difficult to compute. Knowing that they are nonnegative, find upper bounds for them!

First method uses the equation relating Legendrian and classical Thom polynomials. Algebraically it is some instance of the "factorization formula" for super Schur functions.

Second method combines different specializations in the one parameter family of positive bases.

Examples

$\mathbf{A_2}$: $\mathbf{\widetilde{Q}_1}$ $\mathbf{A_3}$: $\mathbf{3\widetilde{Q}_2} + v_2\widetilde{Q}_1$

$\mathbf{A_4}$: $\mathbf{12\widetilde{Q}_3} + \mathbf{3\widetilde{Q}_{21}} + (3v_1 + 7v_2)\widetilde{Q}_2 + (v_1v_2 + v_2^2)\widetilde{Q}_1$

$\mathbf{D_4}$: $\mathbf{\widetilde{Q}_{21}}$.

Examples

$\mathbf{A_2}$: $\widetilde{\mathbf{Q}}_{\mathbf{1}}$ $\mathbf{A_3}$: $\mathbf{3\widetilde{Q}_2} + v_2\widetilde{Q}_1$

$\mathbf{A_4}$: $\mathbf{12\widetilde{Q}_3} + \mathbf{3\widetilde{Q}_{21}} + (3v_1 + 7v_2)\widetilde{Q}_2 + (v_1v_2 + v_2^2)\widetilde{Q}_1$

$\mathbf{D_4}$: $\widetilde{\mathbf{Q}}_{\mathbf{21}}$.

$\mathbf{P_8} = \widetilde{\mathbf{Q}}_{\mathbf{321}}$.

$\mathbf{A_5}$: $\mathbf{60\widetilde{Q}_4} + \mathbf{27\widetilde{Q}_{31}} + (6v_1 + 16v_2)\widetilde{Q}_{21} + (39v_1 + 47v_2)\widetilde{Q}_3 +$
$(6v_1^2 + 22v_1v_2 + 12v_2^2)\widetilde{Q}_2 + (2v_1^2v_2 + 3v_1v_2^2 + v_2^3)\widetilde{Q}_1$

$\mathbf{D_5}$: $\mathbf{6\widetilde{Q}_{31}} + 4v_2\widetilde{Q}_{21}$,

$\mathbf{P_9}$: $\mathbf{12\widetilde{Q}_{421}} + 12v_2\widetilde{Q}_{321}$.

Examples

$\mathbf{A_2}$: $\widetilde{\mathbf{Q}}_{\mathbf{1}}$ $\mathbf{A_3}$: $\mathbf{3\widetilde{Q}_2} + v_2\widetilde{Q}_1$

$\mathbf{A_4}$: $\mathbf{12\widetilde{Q}_3} + \mathbf{3\widetilde{Q}_{21}} + (3v_1 + 7v_2)\widetilde{Q}_2 + (v_1v_2 + v_2^2)\widetilde{Q}_1$

$\mathbf{D_4}$: $\widetilde{\mathbf{Q}}_{\mathbf{21}}$.

$\mathbf{P_8} = \widetilde{\mathbf{Q}}_{\mathbf{321}}$.

$\mathbf{A_5}$: $\mathbf{60\widetilde{Q}_4} + \mathbf{27\widetilde{Q}_{31}} + (6v_1 + 16v_2)\widetilde{Q}_{21} + (39v_1 + 47v_2)\widetilde{Q}_3 +$
$(6v_1^2 + 22v_1v_2 + 12v_2^2)\widetilde{Q}_2 + (2v_1^2v_2 + 3v_1v_2^2 + v_2^3)\widetilde{Q}_1$

$\mathbf{D_5}$: $\mathbf{6\widetilde{Q}_{31}} + 4v_2\widetilde{Q}_{21}$,

$\mathbf{P_9}$: $\mathbf{12\widetilde{Q}_{421}} + 12v_2\widetilde{Q}_{321}$.

$$\mathbf{A_8} : \mathbf{18840\widetilde{Q}_{61}} + \mathbf{20160\widetilde{Q}_7} + \mathbf{3123\widetilde{Q}_{421}} + \mathbf{5556\widetilde{Q}_{43}} + \mathbf{15564\widetilde{Q}_{52}} +$$
$$t(71856\widetilde{Q}_6 + 3999\widetilde{Q}_{321} + 55672\widetilde{Q}_{51} + 34780\widetilde{Q}_{42}) +$$
$$t^2(64524\widetilde{Q}_{41} + 24616\widetilde{Q}_{32} + 105496\widetilde{Q}_5) + t^3(36048\widetilde{Q}_{31} + 81544\widetilde{Q}_4) +$$
$$t^4(8876\widetilde{Q}_{21} + 34936\widetilde{Q}_3) + t^5 7848\widetilde{Q}_2 + t^6 720\widetilde{Q}_1 ;$$

$\mathbf{E_8}$:

$$\mathbf{93\widetilde{Q}_{421} + 108\widetilde{Q}_{43} + 204\widetilde{Q}_{52} + 72\widetilde{Q}_{61}} +$$

$$t(99\widetilde{Q}_{321} + 216\widetilde{Q}_{51} + 414\widetilde{Q}_{42}) +$$

$$t^2(246\widetilde{Q}_{41} + 246\widetilde{Q}_{32}) + t^3 126\widetilde{Q}_{31} + t^4 24\widetilde{Q}_{21} \,;$$

$\mathbf{E_8}$:

$$\mathbf{93\widetilde{Q}_{421} + 108\widetilde{Q}_{43} + 204\widetilde{Q}_{52} + 72\widetilde{Q}_{61}} +$$

$$t(99\widetilde{Q}_{321} + 216\widetilde{Q}_{51} + 414\widetilde{Q}_{42}) +$$

$$t^2(246\widetilde{Q}_{41} + 246\widetilde{Q}_{32}) + t^3 126\widetilde{Q}_{31} + t^4 24\widetilde{Q}_{21} \,;$$

$\mathbf{X_9}$:

$$\mathbf{18\widetilde{Q}_{52} + 27\widetilde{Q}_{43}} + t(42\widetilde{Q}_{42} + 6\widetilde{Q}_{51}) + t^2(21\widetilde{Q}_{32} + 11\widetilde{Q}_{41}) +$$

$$t^3 6\widetilde{Q}_{31} + t^4\widetilde{Q}_{21} \,;$$

$\mathbf{E_8}$:

$$\mathbf{93\widetilde{Q}_{421} + 108\widetilde{Q}_{43} + 204\widetilde{Q}_{52} + 72\widetilde{Q}_{61}} +$$

$$t(99\widetilde{Q}_{321} + 216\widetilde{Q}_{51} + 414\widetilde{Q}_{42}) +$$

$$t^2(246\widetilde{Q}_{41} + 246\widetilde{Q}_{32}) + t^3 126\widetilde{Q}_{31} + t^4 24\widetilde{Q}_{21} \,;$$

$\mathbf{X_9}$:

$$\mathbf{18\widetilde{Q}_{52} + 27\widetilde{Q}_{43}} + t(42\widetilde{Q}_{42} + 6\widetilde{Q}_{51}) + t^2(21\widetilde{Q}_{32} + 11\widetilde{Q}_{41}) +$$

$$t^3 6\widetilde{Q}_{31} + t^4\widetilde{Q}_{21} \,;$$

$\mathbf{P_9}$:

$$\mathbf{12\widetilde{Q}_{421}} + t12\widetilde{Q}_{321} \,.$$