SYLVESTER'S DOUBLE SUMS

$\mathbb{A} = (a_1, \dots , a_m)$, $B = (b_1, \dots , b_n)$ two sequences of variables

x an extra variable

Let $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$ be two integers.

$$Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x) :=$$

$$\sum_{\mathbb{A}' \subset \mathbb{A}} \sum_{\mathbb{B}' \subset \mathbb{B}} R(x, \mathbb{A}')R(x, \mathbb{B}') \frac{R(\mathbb{A}', \mathbb{B}')R(\mathbb{A} \setminus \mathbb{A}', \mathbb{B} \setminus \mathbb{B}')}{R(\mathbb{A}', \mathbb{A} \setminus \mathbb{A}')R(\mathbb{B}', \mathbb{B} \setminus \mathbb{B}')} ,$$

the sum over all subsets $\mathbb{A}'$ of cardinality p

and $\mathbb{B}'$ of cardinality q.

Example. *For* $m=3$, $n=2$, $\quad Sylv^{1,1}(\mathbb{A},\mathbb{B};x) =$

$$\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_2)(a_3-b_2)}{(a_1-a_2)(a_1-a_3)(b_1-b_2)}(x-a_1)(x-b_1)\\
&+\frac{(a_1-b_2)(a_2-b_1)(a_3-b_1)}{(a_1-a_2)(a_1-a_3)(b_2-b_1)}(x-a_1)(x-b_2)\\
&+\frac{(a_2-b_1)(a_1-b_2)(a_3-b_2)}{(a_2-a_1)(a_2-a_3)(b_1-b_2)}(x-a_2)(x-b_1)\\
&+\frac{(a_2-b_2)(a_1-b_1)(a_3-b_1)}{(a_2-a_1)(a_2-a_3)(b_2-b_1)}(x-a_2)(x-b_2)\\
&+\frac{(a_3-b_1)(a_1-b_2)(a_2-b_2)}{(a_3-a_1)(a_3-a_2)(b_1-b_2)}(x-a_3)(x-b_1)\\
&+\frac{(a_3-b_2)(a_1-b_1)(a_2-b_1)}{(a_3-a_1)(a_3-a_2)(b_2-b_1)}(x-a_3)(x-b_2)\,.
\end{aligned}$$

This is equal to $\quad -2(x-b_1)(x-b_2)$.

Example. *For* $m = n = 3$, $\quad Sylv^{2,1}(\mathbb{A}, \mathbb{B}; x) =$

$$\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_2)(x-b_1)\\
+&\frac{(a_1-b_1)(a_3-b_1)(a_2-b_2)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_3)(x-b_1)\\
+&\frac{(a_2-b_1)(a_3-b_1)(a_1-b_2)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_1-b_2)(b_1-b_3)}(x-a_2)(x-a_3)(x-b_1)\\
+&\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_2)(x-b_2)\\
+&\frac{(a_1-b_2)(a_3-b_2)(a_2-b_1)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_3)(x-b_2)\\
+&\frac{(a_2-b_2)(a_3-b_2)(a_1-b_1)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_2-b_1)(b_2-b_3)}(x-a_2)(x-a_3)(x-b_2)\\
+&\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_3-b_2)}{(a_1-a_3)(a_2-a_3)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_2)(x-b_3)\\
+&\frac{(a_1-b_3)(a_3-b_3)(a_2-b_1)(a_2-b_2)}{(a_1-a_2)(a_3-a_2)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_3)(x-b_3)\\
+&\frac{(a_2-b_3)(a_3-b_3)(a_1-b_1)(a_1-b_2)}{(a_2-a_1)(a_3-a_1)(b_3-b_1)(b_3-b_2)}(x-a_2)(x-a_3)(x-b_3)
\end{aligned}$$

This is equal to $\quad 2(x-a_1)(x-a_2)(x-a_3) + (x-b_1)(x-b_2)(x-b_3).$

A formal series: $\sum_{j=0}^{\infty} z^j S_j$,

$S_j \in$ commutative ring, $S_0 = 1$.

For any $J = (j_1, \dots, j_l) \in \mathbb{Z}^l$ one associates with the series the *SCHUR FUNCTION* S_J:

$$S_J = \left| S_{j_k + k - h} \right|_{1 \leqslant h,k \leqslant l} .$$

For example,

$$S_{5,7,2,0,3} = \begin{vmatrix} S_5 & S_8 & S_4 & S_3 & S_7 \\ S_4 & S_7 & S_3 & S_2 & S_6 \\ S_3 & S_6 & S_2 & S_1 & S_5 \\ S_2 & S_5 & S_1 & 1 & S_4 \\ S_1 & S_4 & 1 & 0 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix} = S_{3,3,3,4,4}$$

We note that $S_{0,\dots,0} = 1$

Given two alphabets $\mathbb{A}, \mathbb{B}$, we denote by $S_J(\mathbb{A} - \mathbb{B})$ the Schur functions associated with the rational series

$$\prod_{b \in \mathbb{B}} (1 - zb) / \prod_{a \in \mathbb{A}} (1 - za) .$$

One has

$$R(\mathbb{A}, \mathbb{B}) = S_{n^m}(\mathbb{A} - \mathbb{B}) .$$

Use MAPLE (or better: ACE) to divide x^4 by $S_3(x - \mathbb{B})$ (here $\mathbb{A} = 0$):

$$\mathcal{R}_3 = S_{4,4,4} \cdot \frac{S_2^2}{S_{3,3}^2}$$

Sylvester: $\mathcal{R}_3 = R(\mathbb{B}, \mathbb{A}) = S_{4,4,4}$

i.e. he uses normalized remainders to get polynomials in $\mathbb{A}$, $\mathbb{B}$,

and not rational functions.

So do we.

We normalize (after Sylvester):

$$\mathcal{R}_n = R(\mathbb{B}, \mathbb{A}) = S_{m^n}(\mathbb{B} - \mathbb{A})$$

and

$$\mathcal{R}_i = x^{n-i} S_{(m-n+i)^i}(\mathbb{B} - \mathbb{A}) + \text{lower degree terms}$$

Example. *Let* $m=5, n=4$. *Then* $Sylv^{2,1}(\mathbb{A},\mathbb{B};x) =$

$$\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_1-b_2)(b_1-b_3)(b_1-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_1)$$

$$+\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_2-b_1)(b_2-b_3)(b_2-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_2)$$

$$+\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_3-b_1)(b_3-b_2)(b_3-b_4)}$$
$$\times(x-a_1)(x-a_2)(x-b_3)$$

$$+\frac{(a_1-b_4)(a_2-b_4)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_4-b_1)(b_4-b_2)(b_4-b_3)}$$
$$\times(x-a_1)(x-a_2)(x-b_4)$$

$+\;\; 9\times 4$ similar terms obtained by replacing (a_1,a_2) by (a_i,a_j) for $i<j$.

This is $3\mathcal{R}^{(3)}$ for the division of

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$$

by

$$(x-b_1)(x-b_2)(x-b_3)(x-b_4)\,.$$

Euler: $\mathcal{R}_1$ is the unique polynomial (up to normalization) of degree $\leqslant n-1$ in the space $\operatorname{Lin}(P, Q, xQ, \dots, x^{m-n}Q)$.

$\mathcal{R}^{(d)}$ is the unique polynomial (up to normalization) of degree $\leqslant d$ in the space $\operatorname{Lin}(P, xP, \dots, x^{n-d-1}P, Q, xQ, \dots, x^{m-d-1}Q)$.

$$\mathcal{D}^{(d)} := \begin{vmatrix} S_0(-\mathbb{A}) & \dots & S_{m+n-2d-2}(-\mathbb{A}) & \S_{m+n-d-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{d-n+1}(-\mathbb{A}) & \dots & S_{m-d-1}(-\mathbb{A}) & S_m(x-\mathbb{A}) \\ S_0(-\mathbb{B}) & \dots & S_{m+n-2d-2}(-\mathbb{B}) & S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{d-m+1}(-\mathbb{B}) & \dots & S_{n-d-1}(-\mathbb{B}) & S_n(x-\mathbb{B}) \end{vmatrix}$$

$$= \pm \begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{m+n-2d} & x^{n-d-1}P \\ 0 & 1 & \alpha_1 & \alpha_2 & \dots & \alpha_{m+n-2d-1} & x^{n-d-2}P \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & & \alpha_{m-d-1} & P \\ 1 & \beta_1 & \beta_2 & \beta_3 & \dots & \beta_{m+n-2d} & x^{m-d-1}Q \\ 0 & 1 & \beta_1 & \beta_2 & \dots & \beta_{m+n-2d-1} & x^{m-d-2}Q \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & & \beta_{n-d-1} & Q \end{vmatrix}.$$

(The determinant of order $m+n-2d$.)

Proposition. *For $d = 0, 1, \dots, n-1$, $\mathcal{R}^{(d)} = \pm\mathcal{D}^{(d)}$.*

We are given l virtual alphabets $\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}$, and a sequence $J \in \mathbb{Z}^l$.

SCHUR MULTIFUNCTIONS:

$$S_J\big(\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}\big) = |S_{j_k+k-h}(\mathbb{A}^{(k)})|_{1 \leqslant h,k \leqslant l} \,.$$

If

$$i_r = i_{r+1} = \ldots = i_t = i$$

are those parts of I that are equal to i, and that correspond to the argument $\mathbb{A}$, then we will write

$$S_{\ldots\,;i^{t-r};\,\ldots}(\ldots;\mathbb{A};\ldots)\,.$$

For example, the Schur multifunction

$$S_{1,1,1,1,2,2,4,4,4,4,5}$$

associated with a sequence of virtual alphabets

$$\mathbb{A}^{(1)}, \mathbb{A}^{(2)}, \mathbb{A}^{(2)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(5)}, \mathbb{A}^{(5)}$$

is denoted by

$$S_{1;1^2;1;2^2;4^3;4;5}\big(\mathbb{A}^{(1)};\mathbb{A}^{(2)};\mathbb{A}^{(3)};\mathbb{A}^{(3)};\mathbb{A}^{(4)};\mathbb{A}^{(5)};\mathbb{A}^{(5)}\big)\,.$$

Proposition. *(Lascoux, 1987) For* $d = 0, 1, \ldots, n-1$,

$$\mathcal{R}^{(d)} = S_{1^d;(m-d)^{n-d}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})$$

$$= \begin{vmatrix} S_1(\mathbb{B}-x) & \ldots & S_d(\mathbb{B}-x) & S_m(\mathbb{B}-\mathbb{A}) & \ldots & S_{m+n-d-1}(\mathbb{B}-\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{2-n}(\mathbb{B}-x) & \ldots & S_{d-n+1}(\mathbb{B}-x) & S_{m-n+1}(\mathbb{B}-\mathbb{A}) & \ldots & S_{m-d}(\mathbb{B}-\mathbb{A}) \end{vmatrix} .$$

(The determinant of order n.)

Using the relation

$$S_i(\mathbb{A} - x) = S_i(\mathbb{A}) - xS_{i-1}(\mathbb{A}) \, ,$$

where $\mathbb{A}$ is a virtual alphabet and x is a single variable, and performing elementary operations on the rows (or columns) of a determinant defining a Schur multifunction, we arrive at the following:

Transformation Lemma. *Let* $q \leqslant l$ *be a positive integer and* $\mathbb{X}$ *an alphabet of cardinality* $r \leqslant q$.

(i) The replacement of $S_i(\mathbb{A}^{(k)})$ *by* $S_i(\mathbb{A}^{(k)} - \mathbb{X})$ *in the first* $q-r$ *rows of a sequence of* q *consecutive rows of the determinant, does not change the value of the determinant.*

(ii) Suppose that there exists k *such that* $k + q \leqslant l$, $j_{k+1} = j_{k+2} = \ldots = j_{k+q}$, *and* $\mathbb{A}^{(k+1)} = \ldots \mathbb{A}^{(k+q)} = \mathbb{A}$. *Then the replacement* $S_i(\mathbb{A})$ *by* $S_i(\mathbb{A} - \mathbb{X})$ *in the* $(k+q)$*-th,* $(k+q-1)$*-th ,...,*$(k+1+r+1)$*-th column of the determinant, does not change the value of the determinant.*

Example. *For virtual alphabets $\mathbb{A}$, $\mathbb{B}$, ... , and an alphabet $\mathbb{X}$ of cardinality 2, the following equalities hold:*

$$\begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}) & S_8(\mathbb{B}) & S_4(\mathbb{C}) & S_3(\mathbb{D}) & S_7(\mathbb{E}) \\ S_4(\mathbb{A}) & S_7(\mathbb{B}) & S_3(\mathbb{C}) & S_2(\mathbb{D}) & S_6(\mathbb{E}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix} = \begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{B}-\mathbb{X}) & S_4(\mathbb{C}-\mathbb{X}) & S_3(\mathbb{D}-\mathbb{X}) & S_7(\mathbb{E}-\mathbb{X}) \\ S_4(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{B}-\mathbb{X}) & S_3(\mathbb{C}-\mathbb{X}) & S_2(\mathbb{D}-\mathbb{X}) & S_6(\mathbb{E}-\mathbb{X}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix}$$

and

$$\begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_9(\mathbb{A}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_2(\mathbb{C}) \end{vmatrix}$$

$$= \begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}-\mathbb{X}) & S_9(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{A}-\mathbb{X}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{A}-\mathbb{X}) & S_2(\mathbb{C}) \end{vmatrix} .$$

We specialize the determinant $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$:
We will perform operations on alphabets only, and therefore will write only them.
We subtract $\mathbb{B}-b$ in the top row.
So, starting from

$$\begin{vmatrix} \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix},$$

we get

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & \dots & b-\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix}.$$

Now, we subtract b in the last indicated columns; we get

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & -\mathbb{A} & \dots & -\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \end{vmatrix}.$$

This determinant clearly factorizes, because in the first row, the elements:

$$S_{m+1}(-\mathbb{A}), \dots, S_{m+n-d-1}(-\mathbb{A})$$

are null. It is equal to

$$(-1)^d R(b,\mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

We specialize the determinant $\mathcal{D}^{(d)}$:

$$\mathcal{D}^{(d)} = \begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \end{vmatrix},$$

in $x = b \in \mathbb{B}$. We get

$$\begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

Now, we subtract b in the first indicated rows; we get

$$\begin{vmatrix} -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

By Laplace expansion along the last column, we get

$$(-1)^{(m-d)(n-d)} R(b, \mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B} - b - \mathbb{A}).$$

I. "LIMIT CASE" : $Sylv^{p,q}$ for $p+q=n$.

Proposition. *(i) Suppose that $p+q=n<m$. Then*

$$Sylv^{p,q}(\mathbb{A},\mathbb{B};x) = \pm\binom{p+q}{p} R(x,\mathbb{B})\,.$$

(ii) Suppose that $p+q=n=m$. Then

$$Sylv^{p,q}(\mathbb{A},\mathbb{B};x) = \binom{p+q-1}{q} R(x,\mathbb{A}) + \binom{p+q-1}{p} R(x,\mathbb{B})\,.$$

II. DIVIDED DIFFERENCE FORMLA FOR $\mathcal{R}_1$.

Let $\mathbb{B} = \{b_1, \ldots, b_n\}$ be a set of indeterminates.

Divided differences, due to Newton, are linear operators

$$\partial_i : \mathbb{Z}[\mathbb{B}] \to \mathbb{Z}[\mathbb{B}] \qquad \text{(of degree } -1)$$

defined by

$$\partial_i(f) = (f - \tau_i f)/(b_i - b_{i+1}) \qquad i = 1, \ldots, n-1,$$

where τ_i denotes the i-th simple transposition exchanging i with $i+1$.

More generally, to every reduced decomposition

$$\sigma = \tau_{i_1} \cdots \tau_{i_k}$$

one associates the composition

$$\partial_\sigma = \partial_{i_1} \cdots \partial_{i_k}$$

which is independent of the choice of the reduced decomposition of σ.

Lemma. *For every $k \in \mathbb{Z}$, and a virtual alphabet $\mathbb{A}$ which is independent,of $\mathbb{B}$, one has*

$$\partial_i S_k(\mathbb{B}_i - \mathbb{A}) = S_{k-1}(\mathbb{B}_{i+1} - \mathbb{A}) \,.$$

Proposition. *The first remainder $\mathcal{R}_1$ of the division of $P(x) = S_m(x - \mathbb{A})$ by $Q(x) = S_n(x - \mathbb{B})$ equals*

$$\mathcal{R}_1 = \partial_{n-1} \cdots \partial_1 \big(P(b_1) R(x, \mathbb{B} - b_1)\big) \,.$$

For example, for $n = 3$,

$$\mathcal{R}_1 = P(b_1) \frac{(x-b_2)(x-b_3)}{(b_1-b_2)(b_1-b_3)} + P(b_2) \frac{(x-b_1)(x-b_3)}{(b_2-b_1)(b_2-b_3)} + P(b_3) \frac{(x-b_1)(x-b_2)}{(b_3-b_1)(b_3-b_2)} \,.$$

III. A SPECIALIZATION LEMMA.

Lemma. *(i) The specialization* $Sylv^{p,q}(\mathbb{A},\mathbb{B};a)$*,* $a\in\mathbb{A}$*, is equal to*

$$(-1)^p R(a,\mathbb{B})\ c\,,$$

where c *is the top coefficient of* $Sylv^{p,q}(\mathbb{A}-a,\mathbb{B};x)$*.*

(ii) The specialization $Sylv^{p,q}(\mathbb{A},\mathbb{B};b)$*,* $b\in\mathbb{B}$*, is equal to*

$$(-1)^{m-p+q} R(b,\mathbb{A})\ c\,,$$

where c *is the top coefficient of* $Sylv^{p,q}(\mathbb{A},\mathbb{B}-b;x)$*.*

PROOF OF THE THEOREM Write $d := p+q$.

To show:

$$Sylv^{p,q}(\mathbb{A},\mathbb{B};x) = \pm\binom{d}{p} S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$$

if $d<n$.

Since $\deg_x Sylv^{p,q} < n$, it coincides with its first remainder modulo $R(x,\mathbb{B})$.

By II:

$$Sylv^{p,q}(x) = \partial_{n-1}\cdots\partial_1\big(Sylv^{p,q}(b_1)R(x,\mathbb{B}-b_1)\big)\,.$$

By III,

$$Sylv^{p,q}(b_1) = \pm R(b_1,\mathbb{A}) \times \text{the top coefficient } \mathbf{c} \text{ in } Sylv^{p,q}(\mathbb{A},\mathbb{B}-b_1;x)$$

If $d < n-1$,then , by the induction assumption,

$$\mathbf{c} = \pm \binom{d}{p} S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) \,,$$

with $\square = (m-d)^{n-d-1}$.

If $d = n-1$ then $\mathbf{c}$ is given by the same formula by the limit case I.

We are left with computation

$$\partial_{n-1} \cdots \partial_1 \big(R(x, \mathbb{B} - b_1) R(b_1, \mathbb{A}) S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) \big) \,.$$

Use three simple identities:

$$R(x, \mathbb{B} - b_1) = \pm \sum_{i=0}^{n-1} (-1)^i S_{1^{n-i-1}}(\mathbb{B} - x) b_1^i \,;$$

$$b_1^i R(b_1, \mathbb{A}) = S_{m+i}(b_1 - \mathbb{A}) \,;$$

$$S_{m+i}(b_1 - \mathbb{A}) S_{\square}(\mathbb{B} - b_1 - \mathbb{A}) = S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) \,.$$

Finally

$$Sylv^{p,q}(b_1) R(x, \mathbb{B} - b_1) = \pm \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) \,.$$

is sent via $\partial_{n-1} \ldots \partial_1$ to

$$\pm \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square; m+i-(n-1)}(\mathbb{B} - \mathbb{A}) \,.$$

In this last sum, the terms for $i = 0, \ldots, n-d-2$ disapear.

In the remaining sum for $i = n-d-1, \ldots, n-1$:

$$\sum_{j=0}^{d} (-1)^j S_{1^{d-j}}(\mathbb{B} - x) S_{\square; m-d+j}(\mathbb{B} - \mathbb{A}) \,,$$

one recognizes a Laplace expansion of $S_{1^d; (m-d)^{n-d}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})$.

Another possible method for getting Sylvester's formulas.

Lemma. *Let $i \leqslant n-1$ be a nonnegative integer and let $\mathbb{C}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}) \leqslant i$. *Then for a nonnegative integer j and a virtual alphabet $\mathbb{D}$,*

$$\partial_{n-1}\cdots\partial_1\big(S_i(b_1-\mathbb{C})S_j(b_1+\mathbb{D})\big) = S_{i+j-n+1}(\mathbb{B}-\mathbb{C}+\mathbb{D})\,,$$

provided $\mathbb{C}$ and $\mathbb{D}$ are independent of $\mathbb{B}$.

Lemma. *Let i_0 be a nonnegative integer and let $\mathbb{C}^{(0)}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}^{(0)}) \leqslant i_0$. *Then for integers $i_1,\ldots,i_{n-1}$, and virtual alphabets $\mathbb{C}^{(1)},\ldots,\mathbb{C}^{(n-1)}$,*

$$\begin{aligned}&\partial_{n-1}\cdots\partial_1\big(S_{i_0}(b_1-\mathbb{C}^{(0)})S_{i_1,\ldots,i_{n-1}}(\mathbb{B}^2-\mathbb{C}^{(1)},\ldots,\mathbb{B}^2-\mathbb{C}^{(n-1)})\big)\\ &\qquad = (-1)^{n-1}S_{i_0,i_1-1,\ldots,i_{n-1}-1}(\mathbb{B}-\mathbb{C}^{(0)},\mathbb{B}-\mathbb{C}^{(1)},\ldots,\mathbb{B}-\mathbb{C}^{(n-1)})\,.\end{aligned}$$

We fix $\mathbb{A}' := \mathbb{A}_p$, $\mathbb{A}" := \mathbb{A}^{p+1}$, $\mathbb{B}' := \mathbb{B}_q$, and $\mathbb{B}" := \mathbb{B}^{q+1}$.

To show:

$$\begin{aligned}&\partial_{\mathbb{A}"\mathbb{A}'}\partial_{\mathbb{B}"\mathbb{B}'}\left[R(x,\mathbb{A}'+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\right]\\ &= \pm\binom{p+q}{p}S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,.\end{aligned}$$

Since

$$\begin{aligned}&\partial_{\mathbb{A}'}\partial_{\mathbb{B}'}\big((1/p!)(1/q!)\Delta(x+\mathbb{A}')(x+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\big)\\ &= R(x,\mathbb{A}'+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\,,\end{aligned}$$

to show is:

$$\nabla\big(\Delta(x+\mathbb{A}'+\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\big) = \pm(p+q)!S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$$

where

$$\nabla := \partial_{\mathbb{B}^2 b_1}\cdots\partial_{\mathbb{B}^{q-p+1} b_{q-p}}\partial_{\mathbb{A}^2 a_1}\partial_{\mathbb{B}^{q-p+2} b_{q-p+1}}\cdots\partial_{\mathbb{A}^p a_{p-1}}\partial_{\mathbb{B}^q b_{q-1}}\partial_{A" a_p}\partial_{B" b_q}$$

Gysin maps for Grassmann bundles (or "integration over a fibre" in Grassmann fibrations).

Let E and F be vector bundles on a variety X of ranks m and n. Let

$$\pi_E : Gr^p(E) \to X \qquad \text{and} \qquad \pi_F : Gr^q(F) \to X$$

be two Grassmann bundles on X parametrizing rank p quotients of E and rank q quotients of F, endowed with:

$$0 \to R_E \to E \to Q_E \to 0 \qquad \text{and} \qquad 0 \to R_F \to F \to Q_F \to 0\,.$$

Form a product over X:

$$\pi := \pi_E \times_X \pi_F : Gr^p(E) \times_X Gr^q(F) \to X\,.$$

We have a Gysin map

$$\pi_* : H(Gr^p(E) \times_X Gr^q(F)) \to H(X)\,.$$

Proposition. *Let L be a line bundle on X. Then,*

$$\pi_*\Big(c_{\text{top}}\,\mathrm{Hom}(Q_E, L)\ c_{\text{top}}\,\mathrm{Hom}(Q_F, L)\ c_{\text{top}}\,\mathrm{Hom}(Q_F, Q_E)\ c_{\text{top}}\,\mathrm{Hom}(R_F, R_E)\Big)$$
$$= \pm\binom{p+q}{p} S_{1^{p+q};(m-p-q)^{n-p-q}}(F-L; F-E)\,.$$

Corollary.

$$\pi_*\Big(c_{\text{top}}\,\mathrm{Hom}(Q_F, Q_E)\ c_{\text{top}}\,\mathrm{Hom}(R_F, R_E)\Big) = \pm\binom{p+q}{p} S_{(m-p-q)^{n-p-q}}(F-E)\,.$$