Multiplying Schubert classes

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1 Characteristic map and BGG-operators

G a semisimple algebraic group, $B \subset G$ - Borel.

X a variety on which B acts freely (from the right).

Suppose that X/B exists so that $p : X \to X/B$ is a principal B-bundle.

Let $\mu : B \to GL(V)$ be a linear representation.

$\mathcal{L}_\mu$:= the vector bundle $X \times^B V$ that is the quotient of $X \times V$ by the relation

$$(x, v) \sim \big(xb, \mu(b)^{-1}v\big).$$

Equivalently, if U is an open subset of X/B, then $\Gamma(U, \mathcal{L}_\mu)$ is the set of morphisms $\varphi : p^{-1}(U) \to V$ such that

$$\varphi(xb) = \mu(b)^{-1}\varphi(x)\,.$$

In particular, with any character χ of B, there is associated a line bundle $\mathcal{L}_\chi$ (with transition functions induced by χ).

This induces a homo of groups

$$X^*(B) \to \mathrm{Pic}(X/B)\,,$$

where $X^*(B) =$ group of characters of B.

Composing this with the homo of the first Chern class

$$\mathrm{Pic}(X/B) \to H^2(X/B, \mathbf{Z})\,,$$

one gets a homo

$$X^*(B) \to H^2(X/B, \mathbf{Z})\,,$$

which extends to a homo of graded rings

$$c : S^\bullet\big(X^*(B)\big) \to H^\bullet(X/B, \mathbf{Z})$$

This homo is called the *characteristic map* of the fiber bundle $p : X \to X/B$.

$S = \oplus S^k$ will abbreviate $S^\bullet\big(X^*(B)\big) = \oplus S^k\big(X^*(B)\big)$.

Choose a maximal torus $T \subset B$ with the Weyl group $W = N_G(T)/T$ of (G, T).

W acts on the group of characters $X^*(T)$ of T, and since $X^*(B) = X^*(T)$, this induces an action of W on S.

The root system of (G, T) is denoted by R.

The set R^+ of positive roots consists of the opposities of roots of (B, T).

Let $\Delta \subset R^+$ be the associated basis of R.

W is generated by simple reflections, i.e. by the reflections associated with the elements of Δ.

$s_\alpha =$ the reflection associated with a root α.

s_α can be realized as a linear endo of the Euclidean space $X^*(T) \otimes \mathbf{R}$, equipped with a W-invariant inner product $(\ ,\ )$:

$s_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

reduced decomposition of $w \in W$: a presentation $w = s_{\alpha_1} \cdots s_{\alpha_l}$ where all $\alpha_p \in \Delta$, and l is the smallest number occuring in such a presentation, called the *length* of w and denoted $l(w)$.

$w_0 =$ the longest element of W, the unique element of W with length equal to $\mathrm{card}(R^+)$.

Need: "BGG-operators" A_w, $w \in W$, acting on the ring S.

Definition 1 *Given a root α and $f \in S$,*

$$A_\alpha(f) := \frac{f - s_\alpha(f)}{\alpha}\,.$$

A_α is a well defined (group) endo on S lowering the degree by 1. Note that $A_\alpha(f) = (\alpha^\vee, f)$ for $f \in S^1$.

Lemma 1 *If $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ are simple roots such that*

$$s_{\alpha_1} \cdots s_{\alpha_k} = s_{\beta_1} \cdots s_{\beta_k}$$

are two reduced decompositions, then

$$A_{\alpha_1} \cdots A_{\alpha_k} = A_{\beta_1} \cdots A_{\beta_k}\,.$$

For $w = s_{\alpha_1} \cdots s_{\alpha_k}$ (reduced decomposition), the operator

$$A_w := A_{\alpha_1} \cdots A_{\alpha_k}$$

is *well-defined* (i.e. doesn't depend on a reduced decomposition of w).

Lemma 2 *We have for $f, g \in S$ and a simple root α,*

$$A_\alpha(fg) = A_\alpha(f)g + s_\alpha(f)A_\alpha(g)\,. \tag{1}$$

Geometric interpretations of BGG-operators are related to correspondences in flag bundles, and Gysin maps for *Bott-Samelson schemes.*

2 Structure constants for Schubert classes

In the geometry of flag manifolds G/B a large role is played by the *Schubert cells* BwB/B and their closures called *Schubert varieties.*

$X^w := \overline{Bw_0wB/B}$. The cohomology class $[X^w]$ of X^w lies in $H^{2l(w)}(G/B, \mathbf{Z})$.

The Schubert cells form a cellular decomposition of G/B, so the classes $[X^w]$ form an additive basis for the cohomology.

Goal: give a closed formula for the constants c_{wv}^u, appearing in

$$[X^w] \cup [X^v] = \sum_u c_{wv}^u [X^u]\,. \tag{2}$$

Bertini-Kleiman: c_{wv}^u are nonnegative.

The characteristic map $c : S \to H^\bullet(G/B, \mathbf{Z})$ of $G \to G/B$ is called the *Borel characteristic map.*

Its kernel is generated by positive degree W-invariants, and $c \otimes \mathbf{Q}$ is surjective, so $H^\bullet(G/B, \mathbf{Q})$ = quotient of $S \otimes \mathbf{Q}$ modulo the ideal generated by positive degree W-invariants.

Get: an action of W on $H^\bullet(G/B, \mathbf{Q})$ and induced BGG-operators A_w on $H^\bullet(G/B, \mathbf{Q})$ (W-invariants are scalars for BGG-operators on S).

Leibniz-type formula: for $a, b \in H^\bullet(G/B, \mathbf{Q})$ and a simple root α,

$$A_\alpha(a \cup b) = A_\alpha(a) \cup b + s_\alpha(a) \cup A_\alpha(b)\,. \tag{3}$$

Iterations of this equation will play an important role in this lecture.

Key equation (BGG, Demazure): for $f \in S^k$, in $H^\bullet(G/B, \mathbf{Z})$,

$$c(f) = \sum_{l(w)=k} A_w(f)[X^w]\,. \tag{4}$$

This equation is closely related to the question of finding polynomial representatives of Schubert classes.

Lemma 3 *For $l(vw^{-1}) = l(v) - l(w)$, we have*

$$A_w([X^v]) = [X^{vw^{-1}}]\,, \tag{5}$$

and in the opposite case, $A_w([X^v]) = 0$.

Lemma 4 *For a simple root α and $w \in W$,*

$$s_\alpha([X^w]) = [X^w] \qquad \text{if } \; l(ws_\alpha) = l(w) + 1\,; \tag{6}$$

$$s_\alpha([X^w]) = -[X^w] - \sum (\beta^\vee, \alpha)[X^{ws_\alpha s_\beta}]\,, \tag{7}$$

if $l(ws_\alpha) = l(w) - 1$, where the sum is over all positive roots $\beta \neq \alpha$ such that $l(ws_\alpha s_\beta) = l(w)$.

By combining Equations (2) and (5), we express c^v_{wv} as follows:

$$c^u_{wv} = A_u([X^w] \cup [X^v])\,. \tag{8}$$

Let $l(w) = k$ and $l(v) = l$. Take a reduced decomposition of u:

$$u = s_{\alpha_1} \cdots s_{\alpha_{k+l}}\,.$$

Iterating (3) we obtain

$$\begin{aligned} c^u_{wv} &= A_{\alpha_1} \cdots A_{\alpha_{k+l}}([X^w] \cup [X^v]) \\ &= \sum A_I([X^w]) \cup A^I_\alpha([X^v])\,, \end{aligned}$$

the sum over all $I = (i_1 < \cdots < i_k) \subset \{1, 2, \ldots, k+l\}$,

$A_I := A_{\alpha_{i_1}} \cdots A_{\alpha_{i_k}}$,

A^I_α is obtained by replacing in $A_{\alpha_1} \cdots A_{\alpha_{k+l}}$ each A_{α_i} by s_{α_i} for $i \in I$.

Theorem 1 *With the above notation,*

$$c^u_{wv} = \sum A^I_\alpha([X^v])\,, \tag{9}$$

the sum over all I s.t. $s_{\alpha_{r_1}} \cdots s_{\alpha_{r_k}}$ is a reduced decomposition of w.

Applying successively to the summands in (9), Equations (5), (6), and (7), we get an expression for c^u_{wv}.

Theorem 2 *(Chevalley, 1958) For $w \in W$, and a simple root α, we have in $H^*(G/B, \mathbf{Z})$:*

$$[X^w] \cup [X^{s_\alpha}] = \sum (\beta^\vee, \omega_\alpha)[X^{ws_\beta}], \qquad (10)$$

the sum over positive roots β s.t. $l(ws_\beta) = l(w) + 1$ and

ω_α = fundamental weight of α.

Proof. $(\omega_\alpha, \gamma^\vee) = \delta_{\alpha\gamma}$, where $\gamma \in \Delta$. This implies that $A_\gamma(\omega_\alpha) = \delta_{\alpha\gamma}$, and using Key Equation (4) we get $c(\omega_\alpha) = [X^{s_\alpha}]$.

By Theorem 1, the coefficient of $[X^u]$ in (10) is evaluated as the sum (9) with $[X^v]$ replaced by ω_α.

"Exchange Condition": a reduced decomposition for w can be gotten from the one for u by omitting one simple reflection if $u = ws_\beta$ for some (positive) root β.

Conversely, if $w = s_{\alpha_1} \cdots s_{\alpha_{p-1}} s_{\alpha_{p+1}} \cdots s_{\alpha_h}$, then $w^{-1}u = s_{\alpha_h} \cdots s_{\alpha_p} \cdots s_{\alpha_h} = s_\beta$ for $\beta = s_{\alpha_h} \cdots s_{\alpha_{p+1}}(\alpha_p)$ that is positive: $s_{\alpha_h} \cdots s_{\alpha_1}$ is reduced!

Since the omitted simple reflection is unique, the looked at sum (9) has exactly one summand which is equal

$$s_{\alpha_1} \cdots s_{\alpha_{p-1}} A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha) = A_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_h}(\omega_\alpha)$$
$$= (\beta^\vee, \omega_\alpha).$$

Same method: for all G/P, P - parabolic.

θ – a subset of Δ.

W_θ – the subgroup of W generated by $\{s_\alpha\}_{\alpha\in\theta}$.

$P_\theta := BW_\theta B$.

$$W^\theta := \{w \in W :\ l(ws_\alpha) = l(w) + 1 \quad \forall\alpha \in \theta\}.$$

- the set of *minimal length left coset representatives* of W_θ in W.

$G/B \to G/P_\theta$ induces

$$H^\bullet(G/P_\theta, \mathbf{Z}) \hookrightarrow H^\bullet(G/B, \mathbf{Z})\,,$$

additively identifies $H^\bullet(G/P_\theta, \mathbf{Z})$ with $\bigoplus_{w\in W^\theta} \mathbf{Z}[X^w]$.

Multiplicatively, $H^\bullet(G/P_\theta, \mathbf{Q})$ is identified with the ring of invariants $H^\bullet(G/B, \mathbf{Q})^{W_\theta}$.

The restriction $c : S^{W_\theta} \to H^\bullet(G/P_\theta, \mathbf{Z})$ of the Borel characteristic map satisfies, for any W_θ-invariant f from S^k:

$$c(f) = \sum_{\substack{w\in W^\theta\\ l(w)=k}} A_w(f)[X^w]\,. \tag{11}$$

For maximal parabolics in the symplectic and orthogonal groups: combinatorial expressions for the structure constants in the product of an arbitrary Schubert class by a “special” one (Pragacz-Ratajski).

3 A combinatorial proof of the Pieri formula

for the Grassmannian $\mathrm{Gr}(n,m)$ of n-dimensional subspaces in $\mathbf{C}^m$. In fact, there are two Pieri formulas:

- for multiplication by the Chern classes of the tautological subbundle on $\mathrm{Gr}(n,m)$,

- and for multiplication by the Chern classes of the tautological quotient bundle on $\mathrm{Gr}(n,m)$.

The latter version appears more often – the Chern classes of the tautological quotient enjoy a simple interpretation in terms of the classical "Schubert conditions": the kth Chern class is represented by the locus of all n-planes which have positive dimensional intersection with a fixed $(m-n-k+1)$-plane.

By passing to the dual Grassmannian, we see that both formulas are, in fact, equivalent.

There are several (really) different proofs of the Pieri formula. The proof that appears most often in monographs is based on studying the *triple intersection of general translates of Schubert varieties.* This is a proof due to Hodge (1942).

We set $q := m - n$.

I, J will denote *strict* partitions contained in the partition $(m, m-1, \ldots, q+1)$ with exactly n parts.

Note that such partitions contain the "upper-left triangle"

$$\delta = (n, n-1, \ldots, 1)\,.$$

λ, μ will denote "ordinary" partitions in (q^n). In fact, there is a bijection between these two sets: with I, we associate λ defined by $\lambda_p = i_p - n + p - 1$ for $p = 1, \ldots, n$.

We associate with I the following permutation w_I in the symmetric group S_m:

$$w_I = \cdots (s_{q-\lambda_3+3} \cdots s_{q+1} s_{q+2})$$

$$(s_{q-\lambda_2+2} \cdots s_q s_{q+1})(s_{q-\lambda_1+1} \cdots s_{q-1} s_q)\,.$$

– a reduced decomposition of w_I.

$m = 7$, $n = 3$, and $I = (6, 4, 3)$;

then $\lambda = (3, 2, 2)$ and $w_I = s_5 s_6 s_4 s_5 s_2 s_3 s_4$ which is the permutation $[1, 3, 6, 7, 2, 4, 5]$.

In general, for $I = (m \geq i_1 > \cdots > i_n \geq 1)$, we have in S_m,

$$w_I = [j_1 < \cdots < j_q\,, m+1-i_n < \cdots < m+1-i_1]\,,$$

where $j_1, \ldots, j_q$ are uniquely determined by I.

Back to geometry:

$B \subset SL_m(\mathbf{C})$ - Borel group of lower triangular matrices.

$P = P_\theta$, where θ is obtained by omitting the simple root $\varepsilon_n - \varepsilon_{n+1}$ in the basis $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m$ of the root system of type (A_{m-1}):

$$\{\varepsilon_i - \varepsilon_j \mid i \neq j\} \subset \oplus_{i=1}^m \mathbf{R}\varepsilon_i \,.$$

$SL_m(\mathbf{C})/P = \mathrm{Gr}(n, m)$.

We set $X^I := \overline{Bw_0 w_I P/P}$, where $w_0 = [m, m-1, \ldots, 1]$, and $X^\lambda := X^I$ for λ associated with I as above.

$[X^\lambda] \in H^{2|\lambda|}(\mathrm{Gr}(n, m), \mathbf{Z})$,

$(k+)$ = strict partition $(k+n, n-1, \ldots, 1)$,

its associated λ is a one-row partition (k).

Goal: compute the coefficients c_J in:

$$[X^I] \cup [X^{(k+)}] = \sum_J c_J [X^J] \,.$$

$x_i := -\varepsilon_{m+1-i}\,$;

$c(x_1), \ldots, c(x_q)$ - Chern roots of the tautological quotient bundle on $\mathrm{Gr}(n,m)$.

The Borel characteristic map: $H^\bullet(\mathrm{Gr}(n,m), \mathbf{Z})$ is a quotient of the ring S' of polynomials symmetric in $x_1, \ldots, x_q$ and in $x_{q+1}, \ldots, x_m$.

– goes back to Giambelli (1902).

s_α and A_α indexed by the simple roots corresponding to P are:

– s_i interchanges x_i with x_{i+1}, leaving other variables invariant.

– A_i is the ith simple (Newton's) divided difference ∂_i: for $f \in S'$,

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}\,.$$

A_w on S', $w \in S_m$, will be denoted by ∂_w.

$e_k = e_k(x_1, \ldots, x_q) =$ the kth elementary symmetric polynomial in $x_1, \ldots, x_q$.

Lemma 5 *For any* $k = 1, \ldots, q$, $\;c(e_k) = [X^{(k)}]\,$.

Proof. Sufficient:

$$\partial_w(e_k) = 0 \quad \text{unless} \quad w = w_{(k+)}\,, \quad \text{and} \quad \partial_{w_{(k+)}}(e_k) = 1\,.$$

$w_{(k+)} = s_{q-k+1} \cdots s_{q-1} s_q$.

By induction on the number of variables, invoking:

$$\partial_i(e_r(x_1, \ldots, x_j)) \neq 0 \quad \text{only if} \quad j = i\,,$$

$$\partial_i(e_r(x_1, \ldots, x_i)) = e_{r-1}(x_1, \ldots, x_{i-1})\,.$$

Lemma says: $X^{(k)}$ represents the kth Chern class of the tautological quotient bundle on $\mathrm{Gr}(n, m)$.

Let J^* be the effect of subtracting the triangle δ from J.

D will denote a subset of J^*.

We introduce an operator ∂_J^D and word r_D.

$m = 8$, $n = 3$, and $J = (8, 6, 5)$;

$$\begin{array}{ccccccccc}
 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
1 & \times & \times & \times & \bullet & \bullet & \bullet & \bullet & \bullet \\
2 & \times & \times & \bullet & \circ & \circ & \circ & & \\
3 & \times & \bullet & \circ & \bullet & \bullet & & &
\end{array}$$

$$\partial_J^D = s_4 s_5 \partial_6 s_7 \partial_3 \partial_4 \partial_5 s_6 s_1 s_2 s_3 s_4 s_5$$

$$r_D = s_4 s_5 s_7 s_6 s_1 s_2 s_3 s_4 s_5\,.$$

If r_D is a reduced decomposition of w_I, then D is a disjoint union of the following "p-ribbons":

For fixed $p = 1, \ldots, n$, the *p-ribbon* consists of all boxes of D giving rise to those s_i (in r_D) which "transport" the item "$m + 1 - i_p$" from its position in $[1, 2, \ldots, m]$ to its position in the sequence w_I.

In the above example, for $I = (7, 5, 2)$, 1-ribbon = the dots in the first row, 2-ribbon = $\{(3,4), (3,5), (2,6)\}$, and 3-ribbon = $\{(3,7)\}$.

Suppose that p is such that the p-ribbon is not empty (this is equivalent to the fact that the box $(p, n + p - 1)$ belongs to the p-ribbon).

Then the column-numbers of boxes in the p-ribbon are: $m + 1 - i_p, \ldots, n + p - 2, n + p - 1$, and their row-numbers weakly increase while reading D from left to right and from top to bottom.

By Theorem 1 and Lemma 5, we have

$$c_J = \sum \partial_J^D (e_k) \,, \tag{12}$$

sum over all $D \subset J^*$ s.t. r_D is a reduced decomposition of w_I.

Lemma 6 *If there are boxes* (i,j) *and* $(i-1,j-1)$ *in* $J^* \setminus D$*, then* $\partial_J^D(e_k) = 0$.

Proof. We set $E := \prod_{i=1}^{n}(1+x_i)$ and we shall prove that $\partial_J^D(E) = 0$.

To compute with compositions of the s_i's and ∂_i's in ∂_J^D, it is handy to introduce the following more general functions.

For $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \{0,1\}^m$, we set $E_{\mathbf{a}} := \prod_{i=1}^{m}(1+a_i x_i)$, so that $E = E_{(1,\ldots,1,0,\ldots,0)}$ with q 1's.

We have:

$$s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'} \tag{13}$$

where $\mathbf{a}' = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_m)$.

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \quad \text{if } a_{i+1} = a_i + d\,, \tag{14}$$

where $\mathbf{a}' = (a_1, \ldots, 0, 0, \ldots, a_n)$ is $\mathbf{a}$ with a_i, a_{i+1} replaced by zeros.

Using (13) and (14), we see that the operator ∂_j in ∂_J^D, corresponding to box (i,j) "kills" the function $E_{\mathbf{a}}$ that has been obtained by applying the previous operators s_r and ∂_r (in ∂_J^D) to E.

Lemma implies: – there is *at most one* $D \subset J^*$ such that r_D is a reduced decomposition of w_I and $\partial_I^D(e_k) \neq 0$, namely $D = I^*$.

(p-ribbon must exactly = pth row of I^* .)

– $D = I^*$ gives a non-zero contribution to the sum in (12) iff $J \setminus I$ is a *horizontal strip with pairwise separated rows.*

In this case, using (13) and (14), we obtain $\partial_J^{I^*}(e_k) = 1$.

Theorem 3 *(Pieri, 1893) (i) For any partition $\lambda \subset (q^n)$ and $k = 1, \ldots, q$,*

$$[X^\lambda] \cup [X^{(k)}] = \sum_\mu [X^\mu] \,, \tag{15}$$

where $|\mu| = |\lambda| + k$ and $\mu \setminus \lambda$ is a horizontal strip.

(ii) For any partition $\lambda \subset (q^n)$ and $p = 1, \ldots, n$,

$$[X^\lambda] \cup [X^{(1,\ldots,1)}] = \sum_\mu [X^\mu] \,, \tag{16}$$

where 1 appears p times, $|\mu| = |\lambda| + p$ and $\mu \setminus \lambda$ is a vertical strip.

In $H^\bullet(\mathrm{Gr}(3,8), \mathbf{Z})$:

$$[X^{(4,2)}] \cup [X^{(3)}] = [X^{(5,4)}] + [X^{(5,3,1)}] + [X^{(5,2,2)}]$$
$$+[X^{(4,4,1)}] + [X^{(4,3,2)}] \,.$$