Bezoutians, Euclidean algorithm, and orthogonal polynomials

Piotr Pragacz (Banach Institute, Warsaw)

Reference:

A. Lascoux, P. Pragacz "Bezoutians, Euclidean algorithm, and orthogonal polynomials", Annals of Combinatorics 9 (2005).

1 Euclidean algorithm and continued fractions

$f = f(x)$ and $\varphi = \varphi(x)$ be two polynomials in $\mathbb{C}[x]$

Euclidean algorithm for division (with *nonstandard* signs):

$$f = \mathcal{Q}_0\varphi - \mathcal{R}_1, \quad \varphi = \mathcal{Q}_1\mathcal{R}_1 - \mathcal{R}_2, \quad \mathcal{R}_1 = \mathcal{Q}_2\mathcal{R}_2 - \mathcal{R}_3 \ , ...$$

$$..., \mathcal{R}_{n-2} = \mathcal{Q}_{n-1}\mathcal{R}_{n-1} - \mathcal{R}_n, \quad \mathcal{R}_{n-1} = \mathcal{Q}_n\mathcal{R}_n \, .$$

The polynomial $\mathcal{R}_i$ is called the ith *remainder* and $\mathcal{Q}_j$ is called the jth *quotient*.

$\deg \mathcal{Q}_i = \deg \mathcal{R}_{i-1} - \deg \mathcal{R}_i$. In the generic case, $\deg \mathcal{Q}_i = 1$ and $\deg \mathcal{R}_i = n - i$ for $i = 1, ..., n$.

We say that (f, φ) is a *general pair* if $\deg \mathcal{Q}_i = 1$ for $i = 1, ..., n$.

We set $\mathcal{R}_{-1} := f$, $\mathcal{R}_0 := \varphi$, and $\mathcal{R}_{n+1} := 0$. Then the above sequence of equations is given compactly by the equations

$$\mathcal{R}_{i-1} = \mathcal{Q}_i\mathcal{R}_i - \mathcal{R}_{i+1} \ , \quad i = 0, \dots, n.$$

$$\frac{\varphi}{f} = \frac{1}{\mathcal{Q}_0 - \frac{\mathcal{R}_1}{\varphi}} \,.$$

Iterating:

$$\frac{\varphi}{f} = \cfrac{1}{\mathcal{Q}_0 - \cfrac{1}{\mathcal{Q}_1 - \cfrac{1}{\ddots \, \cfrac{1}{\mathcal{Q}_n}}}}$$

– a continued fraction with $n+1$ levels numbered from top to bottom with $k = 1, ..., n+1$.

Definition 1 *The intermediate fractions (stopping at the level $k = 1, ..., n$ instead of the last one) are rational functions*

$$\mathcal{N}_k/\mathcal{D}_k = \mathcal{N}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1})/\mathcal{D}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1})$$

Formal expression, NO cancellations !

$\mathcal{N}_k/\mathcal{D}_k = k$th *convergent*, $\mathcal{N}_k =$ *numerator* and $\mathcal{D}_k =$ *denominator* of the kth convergent.

$\mathcal{D}_k$ (resp. $\mathcal{N}_k$) is, in the generic case, of degree $m - n + k - 1$ and $k - 1$.

$$\frac{\mathcal{N}_1}{\mathcal{D}_1} = \frac{1}{\mathcal{Q}_0} \,, \quad \frac{\mathcal{N}_2}{\mathcal{D}_2} = \frac{\mathcal{Q}_1}{\mathcal{Q}_0 \mathcal{Q}_1 - 1} \,, \quad \frac{\mathcal{N}_3}{\mathcal{D}_3} = \frac{\mathcal{Q}_1 \mathcal{Q}_2 - 1}{\mathcal{Q}_0 \mathcal{Q}_1 \mathcal{Q}_2 - \mathcal{Q}_0 - \mathcal{Q}_2} \,.$$

2 Motivations

Use Euclidean algorithm for f and φ of degrees $(n{+}1, n)$.

$\mathbb{A}$ = (multi)set of roots of f. (Can think that $\mathbb{A}$ is generic.)

Brioschi's relations (1856): for a fixed i,

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{N}_i(a)\frac{\varphi(a)}{\prod_{b\neq a}(a-b)} = 0\,, \tag{1}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_{i-1}(a)\frac{\varphi(a)}{\prod_{b\neq a}(a-b)} = 0\,, \tag{2}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i^2(a)\mathcal{Q}_i(a)\frac{\varphi(a)}{\prod_{b\neq a}(a-b)} = 0\,. \tag{3}$$

Explain this conceptually!

f = unitary polynomial of degree $n{+}1$,

$(n{+}1)\varphi(x)$ is the derivative of $f(x)$.

Sylvester (1853): " I found that:

$$\left|\mathcal{D}_i(a_j)^2\right|_{0\leq i,j\leq n} = 0\ ,$$

where $a_0, \ldots, a_n$ are the roots of f. "

Give a proof of this vanishing!

3 A quadratic expression for the Bezoutian

If $g = g(x, y)$ is a polynomial in x,y,

$$\partial_{xy}(g) := \frac{g(x, y) - g(y, x)}{x - y} \, .$$

Definition 2 *(Bézout)*

$$\mathrm{Bez}(f, \varphi) := \partial_{xy}\big(f(x)\varphi(y)\big) \, ,$$

and call this bivariate polynomial (in x and y) the Bezoutian (of f and φ).

$$\mathrm{Bez}(f, \varphi) = -\,\mathrm{Bez}(\varphi, f) \, ,$$

$$\mathrm{Bez}(\alpha f + \beta g, \varphi) = \alpha\,\mathrm{Bez}(f, \varphi) + \beta\,\mathrm{Bez}(g, \varphi) \, ,$$

where α and β do not depend on x, y.

Also, the coefficients of the bivariate polynomial $\mathrm{Bez}(f, \varphi)$ are symmetric.

$f(x) = x^3 + x^2 + 2x + 1$, and $\varphi(x) = x^2 + x + 1$. Then

$$\mathrm{Bez}(f, \varphi) = 1 + x^2 + y^2 + x^2y + xy^2 + x^2y^2.$$

A formula missed by classics: $\deg f \geq n = \deg \varphi$,

Theorem 3

$$\mathrm{Bez}(f,\varphi) = p_0\varphi(x)\varphi(y) + \sum_{i=1}^{n} p_i \mathcal{R}_i(x)\mathcal{R}_i(y)\,,$$

where $p_i = p_i(x,y) = \partial_{xy}(\mathcal{Q}_i(x))$*.*

Proof: iterate the following

Lemma 4 *For* $i = 0, 1, \ldots, n$*,*

$$\mathrm{Bez}(\mathcal{R}_{i-1},\mathcal{R}_i) = p_i\mathcal{R}_i(x)\mathcal{R}_i(y) + \mathrm{Bez}(\mathcal{R}_i,\mathcal{R}_{i+1}) \quad (*)\,.$$

Proposition 5 *If* (f,φ) *is a general pair of polynomials of degrees* $(n{+}1, n)$*, then*

$$\mathrm{Bez}(f,\varphi) = \alpha_0\varphi(x)\varphi(y) + \sum_{i=1}^{n} \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y)\,,$$

where α_i *is the coefficient of* x *in the (linear form)* $\mathcal{Q}_i(x)$*.*

$n = 2$, $f(x) = x^3 + x^2 + 2x + 1$, and $\varphi(x) = x^2 + x + 1$

$\mathcal{R}_1 = -x - 1$, $\mathcal{R}_2 = -1$, $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_2 = 1$

$\mathrm{Bez}(f,\varphi) = (x^2 + x + 1)(y^2 + y + 1) - (x + 1)(y + 1) + 1$

Proposition 6 *If* (f,φ) *is a general pair of polynomials of the same degree* n*, then*

$$\mathrm{Bez}(f,\varphi) = \sum_{i=1}^{n} \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y)\,.$$

4 $\mathcal{R}_i \in (f, \varphi)$

Lemma 7 *For* $i = 1, \ldots, n$,

$$\mathcal{R}_i = \varphi \mathcal{D}_i - f \mathcal{N}_i \, .$$

Corollary 8

$$\mathcal{R}_i \equiv \varphi \mathcal{D}_i \mod f \, .$$

Alternatively:

$$\mathcal{D}_{i-1} = \mathcal{Q}_i \mathcal{D}_i - \mathcal{D}_{i+1} \ , \quad i = 0, \ldots, n \, ,$$

where we set $\mathcal{D}_{-1} := 0$ and $\mathcal{D}_0 := 1$.

– same recurrent relations for denominators and remainders, and $\mathcal{R}_0 = \varphi \mathcal{D}_0$.

Corollary 9 *If* (f, φ) *is a general pair of polynomials of degrees* $(n{+}1, n)$*, then the following congruence holds modulo the ideal* $(f(x), f(y))$*:*

$$\operatorname{Bez}(f, \varphi) \equiv \varphi(x)\varphi(y)\big(\alpha_0 + \sum_{i=1}^{n} \alpha_i \mathcal{D}_i(x)\mathcal{D}_i(y)\big) \, .$$

If we specialize x and y to roots of f, then this congruence is replaced by an *equality.*

5 Nullity of Sylvester's determinant

Brioschi generalized Sylvester's relation to any polynomial φ of degree n such that f and φ are relatively prime and the pair (f, φ) is general.

Proposition 10 *With these assumptions and previous notations, if we set*

$$\mathcal{D}_0(x)^2 := \alpha_0 - \frac{f'(x)}{\varphi(x)}\,,$$

then $\left|\mathcal{D}_i(a_j)^2\right|_{0\le i,j\le n}$ *vanishes.* (*)

6 Sylvester's formula for the Jacobian

$F = F(x, y)$, $\Phi = \Phi(x, y)$ homogeneous forms in x and y,
$N = \deg F = \deg \Phi$; set $f = F(x, 1)$, $\varphi = \Phi(x, 1)$;
let $\deg f \ge \deg \varphi = n$; perform Euclid's algorithm for f/φ.

Proposition 11

$$\frac{1}{N} J(F, \Phi)\Big|_{y=1} = \sum_{i=0}^{n} \mathcal{Q}_i' \mathcal{R}_i^2\,.$$

Proof. R.H.S $= W(f, \varphi) = \lim_{y\to x} \mathrm{Bez}(f, \varphi)$. $\square$

7 Bezoutians in terms of the roots of polynomials

Need: *multi-Schur functions.*

By an *alphabet* $\mathbb{A}$, we understand a (finite) multi-set of elements in a commutative ring.

We identify an alphabet $\mathbb{A}$ with the sum $\sum_{a\in\mathbb{A}} a$.

Definition 12 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}-\mathbb{B})$ of the difference of the alphabets are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \prod_{b\in\mathbb{B}}(1-bz)/\prod_{a\in\mathbb{A}}(1-az)\,.$$

For example, if $\mathbb{A}$ is of cardinality m, then

$$S_m(x-\mathbb{A}) = \prod_{a\in\mathbb{A}}(x-a)$$

is a monic polynomial with the multi-set of roots $\mathbb{A}$.

Definition 13 *Given* $I=(i_1,i_2,\ldots,i_r)\in\mathbb{N}^r$*, and alphabets* $\mathbb{A}_1,\ldots,\mathbb{A}_r$*,* $\mathbb{B}_1,\ldots,\mathbb{B}_r$*, the multi-Schur function* $S_I(\mathbb{A}_1-\mathbb{B}_1,\ldots,\mathbb{A}_r-\mathbb{B}_r)$ *is*

$$S_I(\mathbb{A}_1-\mathbb{B}_1,\ldots,\mathbb{A}_r-\mathbb{B}_r) := \left| S_{i_k+k-h}(\mathbb{A}_k-\mathbb{B}_k) \right|_{1\leq h,k\leq r} .$$

$$S_{5,7,2,0,3}(\mathbb{A}-\mathbb{B},\mathbb{C}-\mathbb{D},\mathbb{E}-\mathbb{F},\mathbb{G}-\mathbb{H},\mathbb{K}-\mathbb{L}) =$$

$$= \begin{vmatrix} S_5(\mathbb{A}-\mathbb{B}) & S_8(\mathbb{C}-\mathbb{D}) & S_4(\mathbb{E}-\mathbb{F}) & S_3(\mathbb{G}-\mathbb{H}) & S_7(\mathbb{K}-\mathbb{L}) \\ S_4(\mathbb{A}-\mathbb{B}) & S_7(\mathbb{C}-\mathbb{D}) & S_3(\mathbb{E}-\mathbb{F}) & S_2(\mathbb{G}-\mathbb{H}) & S_6(\mathbb{K}-\mathbb{L}) \\ S_3(\mathbb{A}-\mathbb{B}) & S_6(\mathbb{C}-\mathbb{D}) & S_2(\mathbb{E}-\mathbb{F}) & S_1(\mathbb{G}-\mathbb{H}) & S_5(\mathbb{K}-\mathbb{L}) \\ S_2(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{C}-\mathbb{D}) & S_1(\mathbb{E}-\mathbb{F}) & 1 & S_4(\mathbb{K}-\mathbb{L}) \\ S_1(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{C}-\mathbb{D}) & 1 & 0 & S_3(\mathbb{K}-\mathbb{L}) \end{vmatrix} .$$

If

$$\mathbb{A}_1-\mathbb{B}_1=\cdots=\mathbb{A}_r-\mathbb{B}_r=\mathbb{A}-\mathbb{B} ,$$

we write $S_I(\mathbb{A}-\mathbb{B})$ and call it a *(super) Schur function.*

$$i_1 = \cdots = i_p = i\,, \quad i_{p+1} = \cdots = i_r = j\,,$$

$$\mathbb{A}_1 - \mathbb{B}_1 = \cdots = \mathbb{A}_p - \mathbb{B}_p = \mathbb{A} - \mathbb{B}\,,$$

$$\mathbb{A}_{p+1} - \mathbb{B}_{p+1} = \cdots = \mathbb{A}_r - \mathbb{B}_r = \mathbb{C} - \mathbb{D}\,.$$

We write:

$$S_{i^p\,;\,j^{r-p}}(\mathbb{A}-\mathbb{B}\,;\,\mathbb{C}-\mathbb{D})\,.$$

$$S_{1,1,1,1,5,5,5}(\mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{C}-\mathbb{D}, \mathbb{C}-\mathbb{D}, \mathbb{C}-\mathbb{D})$$

$$= S_{1^4;5^3}(\mathbb{A}-\mathbb{B}; \mathbb{C}-\mathbb{D})\,.$$

For a partition I, $S_I(\mathbb{A}-\mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B}-\mathbb{A})$, where J=conjugate partition of I.

Given two alphabets $\mathbb{A}$, $\mathbb{B}$, we set

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a \in \mathbb{A},\ b \in \mathbb{B}} (a - b)\,.$$

We have $R(\mathbb{A}, \mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})$, where $m = \mathrm{card}(\mathbb{A})$ and $n = \mathrm{card}(\mathbb{B})$.

Polynomial $S_n(x-\mathbb{B}) = R(x, \mathbb{B})$,

$R(\mathbb{A}, \mathbb{B})$=resultant of $R(x, \mathbb{A})$ and $R(x, \mathbb{B})$.

$\mathbb{A}$ and $\mathbb{B}$ general alphabets of cardinalities $n{+}1$ and n.

Proposition 14

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) = R(x{+}y,\mathbb{B})$$

$$+\sum_{i=1}^{n}(-1)^i \frac{S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})S_{1^{n-i};(i+1)^i}(\mathbb{B}-y;\mathbb{B}-\mathbb{A})}{S_{i^{i-1}}(\mathbb{B}-\mathbb{A})S_{(i+1)^i}(\mathbb{B}-\mathbb{A})}\,.$$

Proof.

We use our quadratic expression for $\mathrm{Bez}(\mathbb{A},\mathbb{B})$.

$$\mathcal{R}_i(x) = S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,,$$

$$\alpha_i = \frac{(-1)^i}{S_{i^{i-1}}(\mathbb{B}-\mathbb{A})S_{(i+1)^i}(\mathbb{B}-\mathbb{A})}\,.$$

Combine these three facts. $\Box$

The polynomial $S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$ is, up to sign, a multi-Schur presentation of the ith *subresultant.*

$(\mathbb{A},\mathbb{B})$ general alphabets with the same cardinality n.

Proposition 15

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) = \sum_{i=1}^{n}(-1)^i \frac{S_{1^{n-i};i^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})S_{1^{n-i};i^i}(\mathbb{B}-y;\mathbb{B}-\mathbb{A})}{S_{(i-1)^{i-1}}(\mathbb{B}-\mathbb{A})S_{i^i}(\mathbb{B}-\mathbb{A})}\,.$$

8 First congruences

$\mathbb{A}$ and $\mathbb{B}$ are alphabets of cardinalities m and n with $m \geq n$,

Lemma 16 $i = 1, \ldots, n$,

$\mathcal{R}_i(x) = \pm S_{(m-n+i)^{i-1}}(\mathbb{B}-\mathbb{A}-x) f(x) + S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\varphi(x)\,.$

Corollary 17

$$\mathcal{R}_i(x) \equiv S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\,\varphi(x) \quad \mod \; f(x)\ .$$

$(\mathbb{A}, \mathbb{B})$ general alphabets of cardinalities $n+1$ and n.

By the corollary, modulo the ideal $(f(x), f(y))$,

$$\mathrm{Bez}(\mathbb{A}, \mathbb{B}) \equiv R(x+y, \mathbb{B})\Big(1 + \sum_{i=1}^{n} \alpha_i S_{i^i}(\mathbb{A}-\mathbb{B}-x) S_{i^i}(\mathbb{A}-\mathbb{B}-y)\Big)\,.$$

Theorem 18

$$\mathrm{Bez}(\mathbb{A}, \mathbb{B}) \equiv R(x+y, \mathbb{B})\Big(1 + \sum_{i=1}^{n} (-1)^i \frac{S_{i^i}(\mathbb{A}-\mathbb{B}-x) S_{i^i}(\mathbb{A}-\mathbb{B}-y)}{S_{(i-1)^i}(\mathbb{A}-\mathbb{B}) S_{i^{i+1}}(\mathbb{A}-\mathbb{B})}\Big)\,.$$

Theorem 19 *For a general pair of alphabets* $(\mathbb{A}, \mathbb{B})$ *of the same cardinality* n, *modulo the ideal* $(f(x), f(y))$,

$$\mathrm{Bez}(\mathbb{A}, \mathbb{B}) \equiv R(x+y, \mathbb{B})\Big(\frac{1}{S_1(\mathbb{A}-\mathbb{B})} +$$

$$+ \sum_{i=1}^{n-1} (-1)^i \frac{S_{(i+1)^i}(\mathbb{A}-\mathbb{B}-x) S_{(i+1)^i}(\mathbb{A}-\mathbb{B}-y)}{S_{i^i}(\mathbb{A}-\mathbb{B}) S_{(i+1)^{i+1}}(\mathbb{A}-\mathbb{B})}\Big)\,.$$

9 Euclidean remainders versus orthogonal polynomials

$(\mathbb{A}, \mathbb{B})$ – alphabets of cardinalities $n+1$ and n.

Brioschi's identities make use of:

$$\mu : g(x) \mapsto \sum_{a \in \mathbb{A}} g(a) \frac{\varphi(a)}{R(a, \mathbb{A}-a)} \ .$$

μ is an incarnation of the Lagrange interpolation in the points $a \in \mathbb{A}$. It is characterized by: for $i \in \mathbb{N}$,

$$\mu(x^i) = S_i(\mathbb{A}-\mathbb{B})(= \text{“}moments\text{”})\ .$$

Given a linear functional, it is known how to write an orthogonal basis in terms of moments.

Rewriting, in terms of Schur functions, of the classical expressions of orthogonal polynomials in terms of Hankel determinants involving moments:

Proposition 20 *The Schur polynomials* $P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)\ ,\ k = 0, 1, \ldots, n$, *form a unique (up to normalization) orthogonal basis with respect to the functional* μ, *of polynomials of respective degrees* $0, 1, \ldots, n$.

Since "$\mathcal{D}_i \longleftrightarrow P_i$", we get the basic relations:

Proposition 21 *For any fixed* $0 \le i < j \le n$*,*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{D}_j(a) \frac{\varphi(a)}{R(a, \mathbb{A}{-}a)} = 0\,, \tag{4}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_j(a) \frac{\varphi(a)}{R(a, \mathbb{A}{-}a)} = 0\,. \tag{5}$$

Brioschi's relations follow:

(2) a particular case of (5) with consecutive i and j.

We can replace in (1) the polynomial $\mathcal{N}_i$, which is of degree $i-1$, by any polynomial of degree $< i$, because $\{\mathcal{D}_i\}$ forms an orthogonal basis.

Since $\mathcal{Q}_i\mathcal{D}_i = \mathcal{D}_{i-1} + \mathcal{D}_{i+1}$, (3) results from the orthogonality of $\mathcal{D}_i$ with $\mathcal{D}_{i+1}$, and with $\mathcal{D}_{i-1}$.

Corollary 22 *For any fixed* $0 \le i < j \le n$*,*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{R}_j(a) \frac{1}{\varphi(a)R(a, \mathbb{A}{-}a)} = 0\,,$$

$$\sum_{a\in\mathbb{A}} \mathcal{R}_i(a)\mathcal{R}_j(a) \frac{1}{\varphi(a)R(a, \mathbb{A}{-}a)} = 0\,.$$

Definition 23 *Given a family of orthogonal polynomials $P_0(x), \ldots, P_n(x)$ associated with a linear functional μ, then the Christoffel-Darboux kernel is:*

$$K(x,y) := \sum_{i=0}^{n} P_i(x)P_i(y)/\mu\big(P_i(x)^2\big)\,.$$

$(\mathbb{A},\mathbb{B})$ alphabets of cardinalities $(n{+}1,n)$. Take again the Lagrange functional.

For this functional,

$$\mu\big(P_i(x)^2\big) = (-1)^i S_{(i-1)^i}(\mathbb{A}{-}\mathbb{B})S_{i^{i+1}}(\mathbb{A}{-}\mathbb{B})\,.$$

Now $(\mathbb{A},\mathbb{B})$ is general. Modulo the ideal $(f(x), f(y))$:

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(x)\varphi(y)\,K(x,y)\,.$$

Proposition 24 *Given a polynomial $g(x)$,*

$$\mu\big(g(x)\ \mathrm{Bez}(\mathbb{A},\mathbb{B})\big) \equiv \varphi(y)^2\,g(y) \quad \mod f(y)\,.$$

Proof. $\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(y)\partial_{xy}f(x)$ mod $f(y)$, and

$$\partial_{xy}f(x) = S_n(x+y-\mathbb{A})\,.$$

Modulo $f(y)$: $\mu\big(\varphi(y)\cdot x^i S_n(x+y-\mathbb{A})\big) =$

$$= \varphi(y)[S_{i+n}(\mathbb{A}-\mathbb{B}) + \cdots + S_i(\mathbb{A}-\mathbb{B})S_n(y-\mathbb{A})] \equiv$$

$$\equiv \varphi(y)S_{i+n}(\mathbb{A}-\mathbb{B}+y-\mathbb{A}) = \varphi(y)^2y^i\,. \ \square$$

$(\mathbb{A},\mathbb{B})$ of the same cardinality n.

Lemma 25 *For $i = 1,\dots,n$,*

$$\mathcal{R}_i(x) \equiv S_{i^{i-1}}(\mathbb{A}-\mathbb{B}-x)\,\varphi(x) \quad \mod f(x)\ .$$

Proposition 26 *The Schur functions* $S_{k^{k-1}}(\mathbb{A}-\mathbb{B}-x)$*, $k = 1,\dots,n$, form a unique (up to normalization) orthogonal basis of polynomials of respective degrees $0,1,\dots,n-1$, with respect to the following functional ν:*

$$\nu : x^i \mapsto S_{i+1}(\mathbb{A}-\mathbb{B})\,, \qquad i \in \mathbb{N}\,.$$

Proposition 27 *For any polynomial $g(x)$, the following congruence holds:*

$$\nu(g(x)\,\mathrm{Bez}(\mathbb{A},\mathbb{B})) \equiv \varphi(y)^2\, g(y) \quad \mod f(y)\,.$$

$* \quad * \quad * \quad * \quad *$

We may safely associate to the

Christoffel-Darboux kernel

the names of Bézout, Cayley, Sylvester, Hermite, and Brioschi (much before Christoffel and Darboux) and recognize that these mathematicians treated the case of a finite discrete measure, with the help of the Euclidean algorithm.