Diagonal subschemes and vector bundles

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References:

P. Pragacz, V. Srinivas, V. Pati, *Diagonal subschemes and vector bundles,* Preprint MPIM Bonn 2006-110, and math.AG/0609381.

O. Debarre, *Letter of 20.12.2006.*

Suppose that s is a section of a vector bundle $\mathcal{E} \to X$. Consider the *zero scheme* of s

$$Z(s) = \{x \in X : s(x) = 0\}.$$

We will say that a variety X has the

Weak point property (P) – if for a point $x \in X$ there exist a vector bundle $\mathcal{E}$ on X of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\{x\} = Z(s)$.

Diagonal property (D) – if there exist a vector bundle $\mathcal{E}$ on $X \times X$ of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\Delta = Z(s)$, where Δ denotes the diagonal in $X \times X$.

Notice that (D) implies (P) – in fact, for any point $x \in X$ – via restriction from $X \times X$ to $X \times \{x\}$.

Consider the ideal sheaf $\mathcal{J}_\Delta \subset \mathcal{O}_{X \times X}$. The diagonal property (D) implies that

$$\mathcal{E}^*_{|\Delta} = \mathcal{J}_\Delta / \mathcal{J}_\Delta^2 \cong \Omega^1_X$$

(the last isomorphism uses the isomorphism $\Delta \cong X$). Therefore, Ω^1_X is locally free of rank equal to dimension of X. Hence X is smooth.

Also, by the Grothendieck formula, we obtain

$$[\Delta] = c_{\dim X}(\mathcal{E}).$$

Let V be an n-dimensional vector space. Fix an increasing sequence of integers

$$d_\bullet : 0 < d_1 < d_2 < \ldots < d_{k-1} < d_k = n\,.$$

Then by a $d_\bullet$-*flag* we mean an increasing sequence of linear subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = V$$

of V such that $\dim V_i = d_i$ for $i = 1, \ldots, k$. The set of all $d_\bullet$-flags forms the flag variety $F\ell^{d_\bullet}$. For example, the sequence

$$d_1 = r < d_2 = n$$

gives rise to the Grassmannian $G_r(V)$, parametrizing r-dimensional linear subspaces of V.

In analogy to the Hopf bundle $\mathcal{O}(-1)$ on the projective space, we have a rank r vector bundle on $G_r(V)$. Consider the subbundle $\mathcal{S}$ of the trivial vector bundle $\mathcal{V}_{G_r(V)}$ of rank r for which the fiber $\mathcal{S}_g$ over $g \in G_r(V)$ is just the r-dimensional subspace corresponding to g. This bundle is called the rank r *tautological subbundle* and is denoted by $\mathcal{S}$. We have the tautological vector bundle sequence over $G_r(V)$:

$$0 \to \mathcal{S} \to \mathcal{V}_{G_r(V)} \to \mathcal{Q} \to 0\,,$$

where $\mathcal{Q}$ is the rank $(n-r)$ *tautological quotient.*

Flag bundles SL_n/P have (D)

Let G_1 and G_2 be two copies of $G_r(V)$. Let $\mathcal{S}_1$ be the tautological subbundle on G_1, and let $\mathcal{Q}_2$ be the tautological quotient bundle on G_2. Moreover, let

$$p_i : G_1 \times G_2 \to G_i$$

denote the projection for $i = 1, 2$. Then the composition

$$p_1^*\mathcal{S}_1 \hookrightarrow p_1^*(\mathcal{V}_{G_1}) = p_2^*(\mathcal{V}_{G_2}) \to p_2^*\mathcal{Q}_2$$

gives rise to the section of the vector bundle

$$\underline{\mathrm{Hom}}(p_1^*\mathcal{S}_1, p_2^*\mathcal{Q}_2)$$

over $G_1 \times G_2$ which vanishes precisely on the diagonal. We conclude that $G_r(V)$ has (D).

In the case of arbitary $d_\bullet$-flags, the tautological sequence takes the form:

$$\mathcal{S}_1 \hookrightarrow \mathcal{S}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{S}_{k-1} \hookrightarrow \mathcal{S}_k = \mathcal{V} \overset{q_1}{\twoheadrightarrow} \mathcal{Q}_1 \overset{q_2}{\twoheadrightarrow} \mathcal{Q}_2 \overset{q_3}{\twoheadrightarrow} \cdots \overset{q_k}{\twoheadrightarrow} \mathcal{Q}_k\,,$$

where $\mathrm{rank}(\mathcal{S}_i) = d_i$ for $i = 1, \ldots, k$, and $\mathcal{Q}_i$ is the quotient of $\mathcal{V}$ by $\mathcal{S}_i$, so that $\mathrm{rank}(\mathcal{Q}_i) = n - d_i$.

Let F_1 and F_2 be two copies of $F\ell^{d_\bullet}$ and

$$p_i : F_1 \times F_2 \to F_i$$

denote the projection for $i = 1, 2$. Consider the map

$$\varphi : \bigoplus_{i=1}^{k-1} \underline{\mathrm{Hom}}(p_1^*\mathcal{S}_i, p_2^*\mathcal{Q}_i) \to \bigoplus_{i=1}^{k-2} \underline{\mathrm{Hom}}(p_1^*\mathcal{S}_i, p_2^*\mathcal{Q}_{i+1})$$

defined by

$$\varphi(\sum_{i=1}^{k-1} f_i) = (\sum_{i=1}^{k-2} (f_{i+1}|_{\mathcal{S}_i} - q_i \circ f_i)) \, .$$

One checks that φ is surjective. Set $\mathcal{K} = \mathrm{Ker}\, \varphi$. Then the compositions

$$p_1^*\mathcal{S}_i \hookrightarrow p_1^*(\mathcal{V}_{F_1}) = p_2^*(\mathcal{V}_{F_2}) \to p_2^*\mathcal{Q}_i$$

for $i = 1, \ldots, k$, give rise to a section s of $\mathcal{K}$ such that $\Delta = Z(s)$.

Note that

$$\mathrm{rank}\, \mathcal{K} = \sum_{i=1}^{k-1} (n - d_i)(d_i - d_{i-1}) = \dim F\ell^{d_\bullet} \, .$$

So the flag variety $F\ell^{d_\bullet}$ has (D).

Cohomologically trivial line bundles

A line bundle $\mathcal{L}$ over X is called *cohomologically trivial* (we shall write "c.t.") if $H^i(X, \mathcal{L}) = 0$ for all $i \geq 0$.

Example 1 Any smooth projective curve supports a c.t. line bundle.

Any abelian variety supports a c.t. line bundle.

Theorem 2 *(i) Let $p_1, p_2 : X \times X \to X$ be the two projections. If X has (D), and moreover, the following isomorphism $(*)$ holds:*

$$\operatorname{Pic}(X \times X) \cong p_1^* \operatorname{Pic}(X) \oplus p_2^* \operatorname{Pic}(X),$$

then there exists a c.t. line bundle $\mathcal{L}$ over X such that

$$\det(\mathcal{E}) = p_1^* \mathcal{L}^{-1} \otimes p_2^* (\mathcal{L} \otimes \omega_X^{-1}) \, .$$

(ii) If $\dim X = 2$ and there exists a c.t. line bundle on X, then X has (D).

Apply to the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta \to 0 \, .$$

and any line bundle $\mathcal{L}$ on $X \times X$, the functor $\operatorname{Hom}(-, \mathcal{L})$ (and its derived functors) to get the sequence of global $\operatorname{Ext}_{X \times X}$'s $(**)$:

$$\operatorname{Ext}^{n-1}(\mathcal{I}_\Delta, \mathcal{L}) \to \operatorname{Ext}^n(\mathcal{O}_\Delta, \mathcal{L}) \xrightarrow{\alpha} H^n(X \times X, \mathcal{L}) \, .$$

Proposition 3 *Let $\mathcal{L}$ be a line bundle on $X \times X$ whose restriction to Δ coincides with ω_Δ. Assume that there exist a rank n vector bundle $\mathcal{E}$ on $X \times X$ with $\det(\mathcal{E}^*) = \mathcal{L}$, and $s \in \Gamma(X \times X, \mathcal{E})$ satisfying $\mathrm{Im}(s^*) = \mathcal{I}_\Delta$. Then α in the exact sequence (**) vanishes. The converse holds if $n = \dim(X) = 2$.*

Proof of the theorem Suppose that there exists a vector bundle $\mathcal{E}$ on $X \times X$ of rank n such that the diagonal is the zero scheme of its section s. Let $\mathcal{L} = \det(\mathcal{E}^*)$, and form the corresponding exact sequence (**). From the proposition, we have $\alpha = 0$. Consider the dual linear map to α:

$$\alpha^* : H^n(X \times X, \mathcal{L})^* \to \mathrm{Ext}^n(\mathcal{O}_\Delta, \mathcal{L})^* \cong H^n(\Delta, \omega_\Delta) = k\,.$$

Using (*) choose $\mathcal{M} \in \mathrm{Pic}(X)$ such that

$$\mathcal{L} = \det(\mathcal{E}^*) \cong p_1^*(\mathcal{M}) \otimes p_2^*(\mathcal{M}^{-1} \otimes \omega_X)\,.$$

By Serre duality on $X \times X$, we get that

$$H^n(X \times X, \mathcal{L})^* \cong H^n(X \times X, \mathcal{L}^{-1} \otimes \omega_{X \times X})$$

$$\cong H^n(X \times X, p_1^*(\mathcal{M}^{-1} \otimes \omega_X) \otimes p_2^*\mathcal{M}).$$

From the Künneth formula, we have

$$H^n(X \times X, p_1^*(\mathcal{M}^{-1} \otimes \omega_X) \otimes p_2^*(\mathcal{M}))$$

$$= \oplus_{i=0}^n H^i(X, \mathcal{M}^{-1} \otimes \omega_X) \otimes H^{n-i}(X, \mathcal{M})\,.$$

Further, on any summand on the right, the induced map

$$H^i(X, \mathcal{M}^{-1} \otimes \omega_X) \otimes H^{n-i}(X, \mathcal{M}) \hookrightarrow$$

$$H^n(X \times X, p_1^*(\mathcal{M}^{-1} \otimes \omega_X) \otimes p_2^*(\mathcal{M})) \stackrel{\alpha^*}{\to} H^n(\Delta, \omega_\Delta) = k$$

coincides with the *Serre duality pairing* on cohomology of X, and is hence a non-degenerate bilinear form, for each $0 \leq i \leq n$.

Thus, α^* vanishes if and only if all the summands in the Kïneth formula vanish, which says that $\mathcal{M}$ is c.t.

Conversely, if $\mathcal{M}$ is c.t., then in the exact sequence (**) determined by

$$\mathcal{L} = p_1^*\mathcal{M} \otimes p_2^*(\mathcal{M}^{-1} \otimes \omega_X),$$

the map α is the zero map, by reversing the above argument. Hence, if $n = 2$, we deduce that $X \times X$ supports a vector bundle $\mathcal{E}$ of rank 2 and a section s with zero scheme Δ, by the $n = 2$ case of the proposition.

Corollary 4 *Suppose that the isomorphism* $(*)$ *holds for* X*, and* X *supports no c.t. bundle. Then* X *has not (D).*

Let X be a smooth proper variety over an algebraic closed field. The isomorphism $(*)$ holds for X if and only if $\operatorname{Pic} X$ is a finitely generated abelian group. Also, if $H^1(X, \mathcal{O}_X) = 0$ then $(*)$ holds for X.

When a smooth projective surface has (D)?

We remind that for a surface X to have (D) is almost equivalent to the existence of a c.t. line bundle on X.

1. There exists a surface Y having (D), and a birational proper map $f : Y \to X$.
2. If $f : Y \to X$ is a birational map, X has (D) and $\operatorname{Pic} X$ is finitely generated, then Y has (D).
3. If X is birational to one of the following: a ruled or an abelian surface or a K3 surface with 2 disjoint smooth rational curves or an elliptic surface with a section or a complex Enriques or hyperelliptic surface, then X has (D).
4. If $\operatorname{Pic} X = \mathbb{Z}$, $\Gamma(X, \mathcal{O}_X(1)) \neq 0$ and X has (D), then $X = \mathbb{P}^2$.

More precisely, the first item says that any surface – after blowing up sufficiently many points – becomes a surface having (D).

The last item implies that (D) fails for general K3 surfaces or general hypersurfaces in $\mathbb{P}^3$ of degree greater than or equal to 4.

When a higher dimensional variety has (D)?

We now look at varieties of dimension ≥ 3.

We first consider the varieties with Picard group $\mathbb{Z}$.

Proposition 5 *Let X be a smooth projective variety of dimension $d \geq 3$ over a field with* $\operatorname{Pic} X = \mathbb{Z}$. *If X has (D) and $H^0(X, \mathcal{O}_X(1)) \neq 0$ then X is a Fano variety and $\omega_X \cong \mathcal{O}_X(-n)$ for some $n \geq 2$.*

This result has two useful consequences.

Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of multidegree $(d_1, \ldots, d_r)$ such that $r < n - 3$ and $\sum d_i \geq n$. Then X has not (D).

In particular, the non-Fano hypersurfaces in the projective spaces have not (D).

Also, let X be a smooth projective Fano variety such that $b_2(X) = 1$ and $\omega_X = \mathcal{O}_X(-1)$ (i.e. X is of index 1). Then X has not (D).

Let X be a scheme and $\mathcal{L}$ be a line bundle over X. We say that X has the

$\mathcal{L}$-point property if for every (closed) point $x \in X$ there exists a vector bundle $\mathcal{F}$ over X such that $d = \operatorname{rank} \mathcal{F} = \dim X$, $\det(\mathcal{F}) = \mathcal{L}$, and there exists a section s of $\mathcal{F}$ such that $\{x\} = Z(s)$.

Note that in this case $c_1(\mathcal{F}) = c_1(\mathcal{L})$ and $c_d(\mathcal{F}) = [x]$.

Theorem 6 *Let X be a smooth proper variety over an algebraically closed field. If* $\operatorname{Pic} X$ *is finitely generated and X has (D), then there exists a c.t. line bundle $\mathcal{L}$ on X such that*

(i) X has the $\mathcal{L}^{-1}$-point property, and

(ii) X has the $\mathcal{L} \otimes \omega_X^{-1}$-point property.

Corollary 7 *Let X be a scheme with finitely generated* $\operatorname{Pic} X$. *If for any c.t. line bundle $\mathcal{L}$, either $\mathcal{L}^{-1}$-point property fails or $\mathcal{L} \otimes \omega_X^{-1}$-point property fails then X has not (D).*

For example, if X is a smooth complex projective quadric of dimension 3, then $\mathcal{O}_X(-1)$ and $\mathcal{O}_X(-2)$ are the unique c.t. line bundles on X. One checks – with the help of the corollary – that X has not (D).

Sometimes, (D) boils down to (P):

Proposition 8 *Let X be a group variety over an algebraically closed field. Then X has (D) iff it has (P).*

Indeed, assume that X has (P). Let $\mathcal{E}$ be a vector bundle on X s.t. $\operatorname{rank} \mathcal{E} = \dim X$, and let $s \in \Gamma(X, \mathcal{E})$ be s.t. $Z(s) = \{x\}$ for some $x \in X$. Since X is a group variety, we have the morphisms $\mu : X \times X \to X$ of multiplication and $i : X \to X$ of inverse. Consider the morphism $f : X \times X \to X$, defined by

$$f(u, v) = \mu(\mu(u, i(v)), x) \, .$$

Since $f^{-1}(x) = \Delta$, the vector bundle $f^* \mathcal{E}$, together with section $f^* s$, implies (D) for X.

In particular, an abelian variety has (D) iff it has (P). Quite recently, O. Debarre has proved:

(i) Any Jacobian of a smooth projective connected curve has (P).

(ii) There exist non-principally polarized abelian varieties in dimension greater than 2, which fail to have (P).

Moreover, Debarre suggests:

(P) may characterize Jacobians among all principally polarized abelian varieties.

Affine case

Let k be an algebraically closed field and A be a finitely generated k-algebra. Let $X = \operatorname{Spec} A$. If $\dim X = 2$, then X has (D) by Serre's construction. What about higher dimensions?

Theorem 9 *An affine algebraic group over an algebraically closed field has (D).*

Indeed, M. Kumar and P. Murthy proved that an affine algebraic group over an algebraically closed field has (P). It suffices then to invoke Proposition 8.

Conjecture 10 *There exists smooth complex varieties of any dimension greater than 2 for which (D) fails.*

This conjecture leads to the following question:

Let k be an algebraically closed field and A be a regular k-algebra. Let K be an extension field of k (not necessarily algebraically closed), $A_K := A \otimes_k K$ and let $M \subset A_K$ be a maximal ideal with residue field K. Does there exist a projective A_K-module P of rank $n = \dim A$ such that there is a surjection $P \twoheadrightarrow M$?

The question has a positive answer by P. Murthy when K is an *algebraically closed* extension of k. If $X = \operatorname{Spec} A$ has (D), then the question has a positive answer for any field extension K. So, a negative answer to the question would produce counterexamples to (D).

Diagonal property in topology

We now use mostly the notation used by topologists.

Let M be a compact connected oriented smooth manifold of real dimension n, and Δ be the diagonal submanifold of $M \times M$.

We say that M has property (D_r) if there exist a smooth *real* vector bundle $\mathcal{E}$ over $M \times M$ with $\mathrm{rank}(\mathcal{E}) = n$ and a smooth section s of $\mathcal{E}$ s.t. s is transverse to the zero section $0_{\mathcal{E}}$ of $\mathcal{E}$ and $s^{-1}(0_{\mathcal{E}}) = \Delta$.

If the vector bundle $\mathcal{E}$ is *orientable*, then we say that M has the property (D_o).

If $\dim_{\mathbb{R}} M = 2m$ and the vector bundle $\mathcal{E}$ is a smooth *complex* vector bundle of $rank_{\mathbb{C}}(\mathcal{E}) = m$ then we say that M has the property (D_c).

We have the following relations between these properties:

$$(D) \Rightarrow (D_c) \Rightarrow (D_o) \Rightarrow (D_r).$$

Take a Riemannian metric on M. It induces a Riemannian metric on $M \times M$, on the tangent bundle τ_M, and on all its subbundles. Let U be a closed ϵ-tubular neighborhood of Δ in $M \times M$. Then, by the tubular neighborhood theorem, we have a diffeomorphism $\phi : (U, \partial U) \xrightarrow{\sim} (D(\nu), S(\nu))$, where $D(\nu)$ is the ϵ-disc bundle of the normal bundle $\rho : \nu \to \Delta$ of Δ in $M \times M$ and $S(\nu)$ is the ϵ-sphere bundle. Then $r := \rho \circ \phi : U \to \Delta$ is a strong deformation retraction of U to Δ. We have the following bundle diagram:

$$\begin{array}{ccc} r^*(\nu) & \longrightarrow & \rho^*(\nu) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ U & \xrightarrow{\phi} & D(\nu). \end{array}$$

The bundle $\rho^*(\nu)_{|S(\nu)} \to S(\nu)$ has a tautological section s given by $v \mapsto v$, which satisfies $\|s(v)\| = \epsilon$ for all $v \in S(\nu)$. So

$$\rho^*(\nu)_{|S(\nu)} \to S(\nu) = \xi \oplus \mathcal{L},$$

where $\mathcal{L}$ is the trivial line subbundle spanned by s and ξ is its orthogonal complement. Under this identification, $\nu \to \Delta$ is isomorphic to $\tau \to M$. Since M is orientable, so is ξ, and thus it is isomorphic to the quotient bundle $\rho^*(\tau_M)/\mathcal{L} \to S(\tau_M)$. Let $\mathcal{F} := \phi^*(\xi)$ – this is a rank $(n-1)$ subbundle of $r^*(\nu)_{|\partial U}$, and is isomorphic to $\rho^*(\tau_M)/\mathcal{L} \to S(\tau_M)$. Moreover, $\mathcal{F}_{|\partial U}$ is orientable.

Note that the restriction of ξ to each fiber $S(\nu_x)$ of

$$\rho : S(\nu) \to M$$

is the tangent bundle τ_{n-1} of the sphere $S(\nu_x)$.

Consequently, the bundle $\mathcal{F}$ when restricted to the fiber $r^{-1}(x)$ of the bundle

$$r : \partial U \to \Delta$$

is isomorphic to τ_{n-1}.

The following result is *key* tool in analyzing the topological diagonal properties:

Lemma 11 *Let M, Δ and U be as above. Set*

$$X := M \times M \setminus Int(U),$$

Then M has (D_r) if and only if the rank $(n-1)$ bundle $\mathcal{F} \to \partial U$ is isomorphic to the restriction to $\partial U = \partial X$ of a smooth rank $(n-1)$ bundle $\mathcal{G}$ on X.

Moreover, M has (D_o) if and only if the bundle G can be chosen to be orientable.

1. S^n has (D_r) if and only if $n = 1, 2, 4$ or 8 (all except the first have (D_o)).

2. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 2$. Then M has (D_c). (This is in contrast with the algebraic situation.)

3. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 3$. Assume that $H^1(M, \mathbb{Z}) = 0$ and $H^2(M, \mathbb{Z}) = \mathbb{Z}$. Then if M satisfies (D_c), the second Stiefel-Whitney class $w_2(M)$ vanishes (i.e. M is spin).

4. Let $M \subset \mathbb{CP}^N$ be a smooth projective variety of $\dim_{\mathbb{C}} M = 3$. Assume that M is a strict complete intersection or a set-theoretically complete intersection with $H_1(M, \mathbb{Z}) = 0$. Then M has (D_c) if and only if M is spin.

5. Let M be a smooth strict complete intersection of $\dim_{\mathbb{C}} M = 3$ in $\mathbb{CP}^n$ with $M = X_1 \cap \cdots \cap X_{n-3}$ where X_i is a smooth hypersurfaces of degree d_i. Then M has (D_c) only if $(n + 1 - \sum d_i)$ is even. In particular, a smooth hypersurface M in $\mathbb{CP}^4$ has (D_c) only if it is of odd degree.

6. Let $m \geq 2$. Then a smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}$ has not (D_c).

From the last item one can deduce that a smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}$ over *any algebraically closed field*, has not (D).

Conjecture 12 *Let Q_n be a smooth quadric hypersurface in $\mathbb{P}^{n+1}$ (over an algebraically closed field). Then Q_n has (D) if and only if $n = 1, 2$ or 4.*

Three other conjectures:

1. From the results on surfaces, it follows that any smooth projective toric surface has (D). We conjecture that any smooth toric variety has (D).

2. The Grassmannian of Lagrangian 2-planes in $\mathbb{C}^4$ is identified with the quadric Q_3, which has not (D). We conjecture that, in general, the homogeneous spaces Sp_n/P and $SO(n)/P$ (P being any parabolic subgroup) have not (D). (The question: "Do the flag varieties for the symplectic and orthogonal groups have (D)?" arose in discussions of the author with W. Fulton at the University of Chicago in 1996.)

3. We conjecture that (D) fails for a general cubic threefold X, though X has (P) and even the $\mathcal{O}_X(1)$-point property ($\mathcal{O}_X(-1)$ is the unique c.t. line bundle on X).