Thom polynomials

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Let $k \geq 0$ be a fixed integer. By a *(stable) singularity* we shall mean an equivalence class of stable germs

$$(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)\,,$$

where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x,v) \mapsto (\kappa(x), v)$).

We recall that the *Thom polynomial* $\mathcal{T}^\eta$ of a singularity η is a polynomial in the formal variables $c_1, c_2, \ldots$ that after the substitution

$$c_i = c_i(f^*TY - TX) = [c(f^*TY)/c(TX)]_i\,,$$

for a general map $f : X \to Y$ between complex analytic manifolds, evaluates the Poincaré dual of $[V^\eta(f)]$, where $V^\eta(f)$ is the cycle carried by the closure of the set

$$\{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}\,.$$

(This is the Thom-Damon theorem.)

Difficult to compute! Computations use the "desingularization method" or the "method of restriction equations".

By *codimension of a singularity* η, $\operatorname{codim}(\eta)$, we shall mean $\operatorname{codim}_X(V^\eta(f))$ for such an f.

Method of restriction equations (Rimanyi et al.)

Let $k \geq 0$ be a fixed integer, and let $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ be a singularity with a prototype $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\phi, \psi) \in \operatorname{Diff}(\mathbf{C}^n, 0) \times \operatorname{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \phi^{-1} = \kappa\}$$

of κ will be denoted by G_η. Even if $\operatorname{Aut}\kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way. In fact, G_η can be chosen so that the images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$ will be denoted by

$$\lambda_1(\eta) \quad \text{and} \quad \lambda_2(\eta)\,.$$

The vector bundles associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called

$$E'_\eta \quad \text{and} \quad E_\eta\,.$$

The *total Chern class of the singularity* η is defined in $H^*(BG_\eta, \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}\,.$$

The *Euler class* of η is defined in $H^{2\,\mathrm{codim}(\eta)}(BG_\eta, \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta)\,.$$

Theorem 1 *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\mathrm{codim}(\eta)$*, are not zero-divisors.(This is the so-called "Euler condition"). Then*

(i) if $\xi \neq \eta$ *and* $\mathrm{codim}(\xi) \leq \mathrm{codim}(\eta)$*, then* $\mathcal{T}^\eta(c(\xi)) = 0$*;*

(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$*.*

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

1. To use this method of determining the Thom polynomials of singularities, one needs their classification.

2. The Euler condition holds for the singularities discussed in this talk.

• Computations of Thom polynomials for $k = 0, 1$ in the basis of the monomials in Chern classes: $c_{i_1} c_{i_2} \cdots$ (slides).

Note that signs occur.

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional $\mathbf{C}$-algebras.

– A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$,

$$G_\eta = U(1) \times U(k),$$

Writing x and $y_1, \ldots, y_k$ for the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$,

$$c(A_i) = \frac{1+(i+1)x}{1+x} \prod_{j=1}^{k} (1+y_j),$$

$$e(A_i) = i!\; x^i \prod_{j=1}^{k} (y_j - ix) \cdots (y_j - 2x)(y_j - x);$$

– $I_{2,2}$ (of T-B type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^2+y^2)$,

$$G_\eta = U(1) \times U(1) \times U(k) \quad \text{(almost)},$$

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k} (1+y_j),$$

$$e(I_{2,2}) = x_1x_2(2x_1-x_2)(2x_2-x_1) \prod_{j=1}^{k} (y_j-x_1)(y_j-x_2)(y_j-x_1-x_2);$$

– $III_{2,2}$ (of T-B type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy,x^2,y^2)$, (here $k\geq 1$),

$$G_\eta = U(2)\times U(k{-}1),$$

$$c(III_{2,2}) = \frac{(1{+}2x_1)(1{+}2x_2)(1{+}x_1{+}x_2)}{(1{+}x_1)(1{+}x_2)}\prod_{j=1}^{k-1}(1+y_j).$$

Let η be a singularity. Consider a (homogeneous) ideal in the polynomial ring $R=\mathbf{Z}[c_1,c_2,\ldots]$:

$\mathcal{P}^\eta$ = ideal of polynomials which – after the substitution $c_i=c_i(f^*TY-TX)$ – are supported on $V^\eta(f)$, where $f:X\to Y$ is a general map between complex analytic manifolds.

The component of minimal degree of $\mathcal{P}^\eta$ is generated by $\mathcal{T}^\eta$. The $\mathcal{P}$–ideal of the singularity $\Sigma^i:(\mathbf{C}^m,0)\to(\mathbf{C}^n,0)$ was studied by the author in the 80's in the context of *enumerative geometry of degeneracy loci.*

We use a "shifted" parameter $r:=k+1$, write $\eta(r)$ for the singularity $\eta:(\mathbf{C}^\bullet,0)\to(\mathbf{C}^{\bullet+r-1},0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}^\eta_r$.

We have $c_i(f^*TY-TX)=S_i(TX^*-f^*TY^*)$, where S_i means the *Segre class.* We shall follow of the notation on the RHS and use, more generally, Schur functions $S_{(i_1,i_2,\ldots,i_h)}$.

Alphabet $\mathbb{A}$: a (finite) multi-set of elements in a commutative ring

For $m \in \mathbf{N}$, by "$\mathbb{A}_m$" we shall mean an alphabet $\mathbb{A} = (a_1, \ldots, a_m)$ (of cardinality m).

Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the *complete functions* $S_i(\mathbb{A}-\mathbb{B})$ are defined by:

$$\sum S_i(\mathbb{A}-\mathbb{B}) z^i = \prod_{b \in \mathbb{B}} (1-bz) / \prod_{a \in \mathbb{A}} (1-az)\,.$$

We shall identify an alphabet $\mathbb{A} = \{a_1, \ldots, a_m\}$ with the sum $a_1 + \cdots + a_m$ and perform usual algebraic operations on such elements.

Given $I = (i_1, i_2, \ldots, i_h) \in \mathbf{Z}^h$, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the *Schur function* $S_I(\mathbb{A}-\mathbb{B})$ is

$$S_I(\mathbb{A}-\mathbb{B}) := \Big| S_{i_q+q-p}(\mathbb{A}-\mathbb{B}) \Big|_{1 \le p,q \le h}\,.$$

We shall mostly use the case when I is a partition

$$I = (0 \le i_1 \le \cdots \le i_h)\,.$$

$$S_{33344}(\mathbb{A}-\mathbb{B}) = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix},$$

where $S_i = S_i(\mathbb{A}-\mathbb{B})$.

"Duality property": $S_I(\mathbb{A}-\mathbb{B}) = S_J(\mathbb{B}^*-\mathbb{A}^*)$,

where J is the *conjugate* partition of I and
$\mathbb{A}^* = \{-a_1, -a_2, \ldots\}$.

Symmetric function: a $\mathbf{Z}$-linear combination of $S_I(-)$

To allow one specializing a letter to an integer r inside a symmetric function (without introducing additional variables), we write $\boxed{r}$ for this specialization. Boxes have to be treated as single variables.

$S_i(2) = i + 1$ but $S_i(\boxed{2}) = 2^i$

$S_2(\mathbb{X}_2) = x_1^2+x_1x_2+x_2^2$ but $S_2(\boxed{x_1+x_2}) = x_1^2+2x_1x_2+x_2^2$

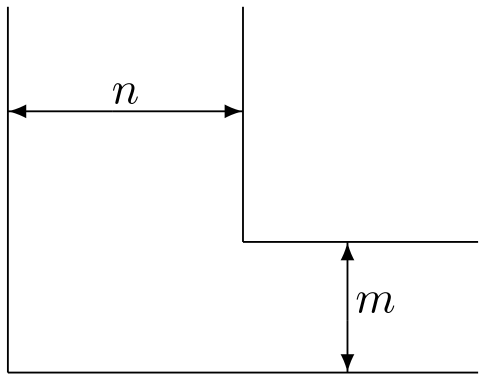
$S_{11}(\mathbb{X}_2) = x_1x_2$ but $S_{11}(\boxed{x_1+x_2}) = 0$ etc.

Resultant of two alphabets $\mathbb{A}, \mathbb{B}$:

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b)$$

Fix two positive integers m and n.

A partition *is contained in* the (m,n)-hook if its Young diagram is contained in the "tickened" hook:



"Vanishing property": Given alphabets $\mathbb{A}$ and $\mathbb{B}$ of cardinalities m and n, if a partition I is *not* contained in the (m,n)-hook, then:

$$S_I(\mathbb{A}-\mathbb{B}) = 0\,.$$

For example,

$$S_{4569}(\mathbb{A}_2-\mathbb{B}_4) = S_{4569}(a_1{+}a_2{-}b_1{-}b_2{-}b_3{-}b_4) = 0$$

because 4569 is not contained in the $(2,4)$-hook.

Theorem 2 *If $\mathbb{A}_m$ and $\mathbb{B}_n$ are variables, then the functions $S_I(\mathbb{A}_m-\mathbb{B}_n)$, for I running over partitions contained in the (m,n)-hook, are* **Z***-linearly independent, and form a* **Z***-basis of the group of the so-called "supersymmetric functions".*

$A_1(1)$: S_1

$A_2(1)$: $2S_2 + S_{1,1}$

$A_3(1)$: $6S_3 + 5S_{12} + S_{111}$

$A_4(1)$: $24S_4 + 26S_{13} + 10S_{22} + 9S_{112} + S_{1111}$

$I_{2,2}(1)$: S_{22}

$A_5(1)$:
$120S_5 + 154S_{14} + 92S_{23} + 71S_{113} + 14S_{1112} + 35S_{122} + S_{11111}$

$I_{2,3}(1)$: $4S_{23} + 2S_{122}$

$A_6(1)$: $720S_6 + 1044S_{15} + 770S_{24} + 266S_{33} + 580S_{114} + 455S_{123} + 70S_{222} + 155S_{1113} + 84S_{1122} + 20S_{11112} + S_{111111}$

$I_{2,4}(1)$: $16S_{24} + 4S_{33} + 12S_{123} + 5S_{222} + 2S_{1122}$

$I_{3,3}(1)$: $2S_{24} + 6S_{33} + 3S_{123} + S_{1122}$

$A_1(2)$: S_2

$A_2(2)$: $4S_4 + 2S_{13} + S_{22}$

$A_3(2)$: $36S_6 + 30S_{15} + 19S_{24} + 5S_{33} + 6S_{114} + 5S_{123} + S_{222}$

$A_4(2)$: $507S_8 + 555S_{17} + 391S_{26} + 240S_{35} + 76S_{44} + 216S_{116} + 210S_{125} + 104S_{134} + 55S_{224} + 21S_{233} + 24S_{1115} + 26S_{1124} + 10S_{1133} + 9S_{1223} + S_{2222}$

$III_{2,2}(2)$: S_{33}

$I_{2,2}(2)$: $3S_{34} + S_{133}$

$III_{2,3}(2)$: $8S_{35} + 4S_{134} + 2S_{233}$

Schur functions give better insight into the structure of Thom polynomials. In 2004, it was stated

Positivity conjecture (Feher-Komuves, Pragacz):

Conjecture 3 *Let η be a (stable) singularity. Then for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}_r^\eta = \sum \alpha_I S_I(TX^* - f^*TY^*)\,,$$

is nonnegative.

Theorem 4 *Let a singularity η be of Thom-Boardman type Σ^i. Then all summands in the Schur function expansion of $\mathcal{T}_r^\eta$ are indexed by partitions containing the rectangle partition $(r{+}i{-}1, \ldots, r{+}i{-}1)$ (i times).*

This follows from the structure of the $\mathcal{P}$-ideal of the singularity Σ^i (P. Pragacz, Ann.ENS, 1988):

$$\mathcal{P}^{\Sigma^i} = \mathbf{Z} \cdot S_{(r+i-1)^i} + \text{terms of higher degree.}$$

1. Any $\mathbf{Z}$-combination of Schur functions indexed by partitions containing $(r+i-1)^i$ belongs to $\mathcal{P}^{\Sigma_i(r)}$.

2. No $\mathbf{Z}$-combination $\sum_I \alpha_I S_I$, where all $I \not\supset (r+i-1)^i$, belongs to $\mathcal{P}^{\Sigma_i(r)}$.

Singularity $I_{2,2}(r)$, $\operatorname{codim}(I_{2,2}(r)) = 3r+1$.

$$\mathcal{T}_r := \mathcal{T}_r^{I_{2,2}}\,.$$

$\mathcal{T}_1 = S_{22} = S_{22}(TX^* - f^*TY^*)$.

Assume $r \geq 2$.

Equations in P characterizing the Thom polynomial:

$$P(-\mathbb{B}_{r-1}) = P(x - \boxed{2x} - \mathbb{B}_{r-1}) = P(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0\,,$$

$$P(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}) =$$

$$= x_1x_2(x_1-2x_2)(x_2-2x_1)\ R(\mathbb{X}_2 + \boxed{x_1+x_2}, \mathbb{B}_{r-1})\,.$$

$$\mathbb{D} := \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1+x_2}\,.$$

The vanishing imposed by $III_{2,2}$:

$$P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,.$$

(The singularities $\neq I_{2,2}$ with codimension $\leq \operatorname{codim}(I_{2,2})$ are: A_0, A_1, A_2, $III_{2,2}$.)

Goal: give a presentation of $\mathcal{T}_r$ as a $\mathbf{Z}$-linear combination of Schur functions with explicit algebraic expressions of the coefficients.

Lemma 5 *A partition appearing in the Schur function expansion of $\mathcal{T}_r$ contains $(r+1, r+1)$ and has at most three parts.*

Recursive description of $\mathcal{T}_r$.

Φ – the linear endomorphism on the **Z**-module spanned by S_I with $l(I) \leq 3$, that sends S_{i_1,i_2,i_3} to S_{i_1+1,i_2+1,i_3+1}.

Let $\overline{\mathcal{T}_r}$ denote the sum of terms in $\mathcal{T}_r$ which are indexed by partitions of length ≤ 2.

$\overline{\mathcal{T}_1} = \mathcal{T}_1 = S_{22}$

Lemma 6 *For $r \geq 2$,* $\quad \mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\mathcal{T}_{r-1})$.

Proposition 7 *For $r \geq 1$, $\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1x_2)^{r+1}\ S_{r-1}(\mathbb{D})$.*

$S_{r-1}(\mathbb{D})$=$s_{r-1}(\mathrm{Sym}^2(E))$, the *Segre class*, $\ \mathrm{rank}(E) = 2$.

Complete quadrics: Schubert, Giambelli, Laksov, Lascoux, Thorup, Pragacz: this Segre class is:

$$\sum_{p\leq q,\ p+q=r-1} \left[\binom{r}{p+1} + \binom{r}{p+2} + \cdots + \binom{r}{q+1}\right]\ S_{p,q}(E)\,.$$

$$\overline{\mathcal{T}}_2 = 3S_{34}$$

$$\overline{\mathcal{T}}_3 = 7S_{46} + 3S_{55}$$

...

$\mathcal{T}_2 = \Phi(\mathcal{T}_1)+\overline{\mathcal{T}}_2 = \Phi(S_{22})+3S_{34} = S_{133}+3S_{34}$

$\mathcal{T}_3 = \Phi(\mathcal{T}_2)+\overline{\mathcal{T}}_3 = \Phi(S_{133}+3S_{34})+\overline{\mathcal{T}}_3 = S_{244}+3S_{145}+7S_{46}+3S_{55}$

...

(“r” appears in conditions for the partitions, and in binomial numbers in the coefficients.)

Morin singularities $A_i(r)$.

$$F_r^{(i)}(-) := \sum_J S_J(\boxed{2}+\boxed{3}+\cdots+\boxed{i})S_{r-j_{i-1},\ldots,r-j_1,r+|J|}(-)\,,$$

where the sum is over partitions $J \subset (r^{i-1})$, and
$F_r^{(1)}(-) = S_r(-)$.

Proposition 8 *We have*

$$F_r^{(i)}(x - \mathbb{B}_r) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}\,, \mathbb{B}_r)\,.$$

Singularities A_1, A_2.

$$P(-\mathbb{B}_{r-1}) = 0, \quad P(x - \mathbb{B}_{r-1} - \boxed{2x}) = R(x, \mathbb{B}_{r-1}+\boxed{2x})$$

characterize $\mathcal{T}_r^{A_1}$.

$$P(-\mathbb{B}_{r-1}) = P(x - \mathbb{B}_{r-1} - \boxed{2x}) = 0,$$

$$P(x-\mathbb{B}_{r-1} - \boxed{3x}) = R(x+\boxed{2x}, \mathbb{B}_{r-1}+\boxed{3x})$$

characterize $\mathcal{T}_r^{A_2}$.

Theorem 9 $F_r^{(1)} = S_r$ *and* $F_r^{(2)} = \sum_{j\le r} 2^j S_{r-j,r+j}$ *are the Thom polynomials of* $A_1(r)$ *and* $A_2(r)$.

Theorem 10 *Suppose that* $\Sigma^j(f) = \emptyset$ *for* $j \ge 2$. *(This says that the kernel of* $df : TX \to f^*TY$ *is a line bundle.) Then, for any* $r \ge 1$, *we have*

$$\mathcal{T}_r^{A_i} = F_r^{(i)}\,.$$

Gaffney's result on $A_4(1)$ revisited :

$$F_1^{(4)} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4\,; \qquad \mathcal{T}_1^{A_4} = F_1^{(4)} + 10S_{22}$$

A_0, A_1, A_2, A_3 impose:

$$F_1^{(4)}(0) = F_1^{(4)}(x - \boxed{2x}) = F_1^{(4)}(x - \boxed{3x}) = F_1^{(4)}(x - \boxed{4x}) = 0\,,$$

A_4 imposes:

$$F_1^{(4)}(x - \boxed{5x}) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \boxed{5x})\,.$$

Not satisfied nullity imposed by $I_{2,2}$:

$$F_1^{(4)}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = (-10)x_1x_2(x_1 - 2x_2)(x_2 - 2x_1)\,;$$

To see it, we use

$$F_1^{(4)}(x - \mathbb{B}_1) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \mathbb{B}_1)\,;$$

substituting e.g. $x_1 = 2x_2$

$$\begin{aligned} F_1^{(4)}(x_2 - \boxed{2x_1}) &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{2x_1}) \\ &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{4x_2}) = 0\,; \end{aligned}$$

the coefficient -10 results from specialization $x_1 = x_2 = 1$.

$$S_{22}(0) = S_{22}(x - \boxed{2x}) = \cdots = S_{22}(x - \boxed{5x}) = 0$$

(22 is not contained in the $(1,1)$-hook.)

$$S_{22}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = x_1x_2(x_1 - 2x_2)(x_2 - 2x_2)$$

Singularity $A_3(r)$.

$$F_r^{(3)} = \sum_{j_1 \le j_2 \le r} S_{j_1,j_2}(\boxed{2} + \boxed{3}) S_{r-j_2,r-j_1,r+j_1+j_2}\,.$$

Singularities $\neq A_3$, whose codimension is $\le \operatorname{codim}(A_3)$ are: A_0, A_1, A_2 and, for $r \ge 2$, $III_{2,2}$.

Equations characterizing $\mathcal{T}_r^{A_3}$:

$$P(-\mathbb{B}_{r-1}) = P(x - \mathbb{B}_{r-1} - \boxed{2x}) = P(x - \mathbb{B}_{r-1} - \boxed{3x}) = 0\,,$$

$$P(x - \mathbb{B}_{r-1} - \boxed{4x}) = R(x + \boxed{2x} + \boxed{3x}, \mathbb{B}_{r-1} + \boxed{4x})$$

$$P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,.$$

The first four equations are satisfied by $F_r^{(3)}$.

$F_1^{(3)} = S_{111} + 5S_{12} + 6S_3$ is $\mathcal{T}_1$.

For $r \ge 2$, $F_r^{(3)}$ does not satisfy the last vanishing, imposed by $III_{2,2}$. $\mathcal{T}_2$ is

$$S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33} = F_2^{(3)} + 5S_{33}\,.$$

Correction term H_r (with partitions containing $(r+2, r+2)$):

$$H_1 = 5S_{33}$$

$$H_2 = 5S_{144} + 24S_{45}$$

$$H_3 = 5S_{255} + 24S_{156} + 24S_{66} + 89S_{57}$$

$$H_4 = 5S_{366} + 24S_{267} + 24S_{177} + 89S_{168} + 113S_{78} + 300S_{69}$$

Theorem 11 $\mathcal{T}_r^{A_3} = F_r^{(3)} + H_r\,.$

Fix $m, n, k \in \mathbf{N}$.

Aut_n:= group of k-jets of automorphisms of $(\mathbf{C}^n, 0)$.

$\mathcal{J} = \mathcal{J}(m,n)$:= space of k-jets of functions $(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$.

$$G := \mathrm{Aut}_m \times \mathrm{Aut}_n .$$

Consider the classifying principal G-bundle $EG \to BG$, i.e. a contractible space EG with a free action of the group G.

$$\widetilde{\mathcal{J}} := \widetilde{\mathcal{J}}(m,n) = EG \times_G \mathcal{J} \, .$$

Let $\Sigma \subset \mathcal{J}$ be an analytic closed G-invariant subset – "class of singularities".

$$\widetilde{\Sigma} := EG \times_G \Sigma \subset \widetilde{\mathcal{J}} .$$

Let $\mathcal{T}^{\Sigma} \in H^{2\,\mathrm{codim}(\Sigma)}(\widetilde{\mathcal{J}}, \mathbf{Z})$ be the dual class of $[\widetilde{\Sigma}]$. Since

$$H^{\bullet}(\widetilde{\mathcal{J}}, \mathbf{Z}) \cong H^{\bullet}(BG, \mathbf{Z}) \cong H^{\bullet}(BGL_m \times BGL_n, \mathbf{Z}) \, ,$$

$\mathcal{T}^{\Sigma}$ is identified with a polynomial in $c_1, \ldots, c_m$ and $c'_1, \ldots, c'_n$ which are the Chern classes of universal bundles R_m and R_n on BGL_m and BGL_n:

$$\mathcal{T}^{\Sigma} = \mathcal{T}^{\Sigma}(c_1, \ldots, c_m, c'_1, \ldots, c'_n) = \sum_I \alpha_I S_I(R_m^* - R_n^*) \, .$$

This is a classical Thom polynomial. Our goal is to show that all $\alpha_I \geq 0$.

Pull back the bundle $\widetilde{\mathcal{J}}$ from BG to $BGL_m \times BGL_n$ via the map induced by

$$GL_m \times GL_n \hookrightarrow \mathrm{Aut}_m \times \mathrm{Aut}_n \,.$$

Since $GL_m \times GL_n$ acts linearly on $\mathcal{J}$, the obtained pullback bundle is now the vector bundle on $BGL_m \times BGL_n$ associated with the representation of $Gl_m \times GL_n$ on $\mathcal{J}$:

$$\mathcal{J}(R_m, R_n) := \left(\oplus_{i=1}^{k} \mathrm{Sym}^i(R_m^*)\right) \otimes R_n \,.$$

$\mathcal{J}(R_m, R_n)$ contains the preimage of $\widetilde{\Sigma} \subset \widetilde{\mathcal{J}}$, denoted by $\Sigma(R_m, R_n)$, whose dual class is the Thom polynomial.

Let E and F be vector bundles of ranks m and n on a variety X.

$$\mathcal{J}(E, F) := \left(\oplus_{i=1}^{k} \mathrm{Sym}^i(E^*)\right) \otimes F \,.$$

In fact, the pair of bundles (E, F) corresponds to a principal $GL_m \times GL_n$-bundle $B(E, F)$ and

$$\mathcal{J}(E, F) = B(E, F) \times_{GL_m \times GL_n} \mathcal{J}$$

is the bundle associated with the representation.

$$\Sigma(E, F) := B(E, F) \times_{GL_m \times GL_n} \Sigma \subset \mathcal{J}(E, F) \,.$$

Lemma 12 *The dual class of* $[\Sigma(E, F)]$ *in*

$$H^{2\,\mathrm{codim}(\Sigma)}(\mathcal{J}(E, F), \mathbf{Z}) \cong H^{2\,\mathrm{codim}(\Sigma)}(X, \mathbf{Z})$$

is equal to $\sum_I \alpha_I S_I(E^* - F^*)$.

Cones in ample vector bundles (Fulton-Lazarsfeld)

E – vector bundle of rank e on X.

Cone C in E: a subvariety of E which is stable under the natural $\mathbb{G}_{\mathrm{m}}$-action on E; assume that C is of pure dimension d.

$$z(C,E) := s_E^*([C]) \in A_{d-e}(X)\,,$$

where $s_E : X \to E$ is the zero section.

Theorem 13 *(F-L I) Suppose that E is a vector bundle of rank e on a projective variety X, and let $C \subset E$ be a cone of pure dimension e. If some symmetric power of E is globally generated (resp. E is ample), then*

$$\int_X z(C,E) \geq 0 \quad \textit{(resp.} \quad \int_X z(C,E) > 0)\,.$$

(Also, for $z(C,E) \in H_0(X,\mathbf{Z})$, $\deg_X z(C,E) \geq 0$ or > 0.)

Let P be a symmetric polynomial in e variables and of degree n. P is *numerically positive for ample vector bundles* if for every projective variety X of dimension n, and every ample vector bundle E of rank e on X, $\int_X P(E) > 0$.

$P(E) = P(\text{Chern roots of } E)$.

Theorem 14 *(F-L II) A homogeneous polynomial $\sum_I \beta_I \mathcal{S}_I$, where $\beta_I \in \mathbf{Z}$, is numerically positive for ample vector bundles iff for any partition I we have $\beta_I \geq 0$, and additionally $\sum_I \beta_I > 0$.*

Proof of the Positivity Conjecture (P. Pragacz and A. Weber (2006)):

Let $c = \mathrm{codim}(\Sigma)$. Take X – a projective variety of dimension c, F – an ample vector bundle on X with $\mathrm{rank}(F) = n' = n + r \geq c$, and $E = \mathbf{1}^{m'}$ – a trivial bundle on X of rank $m' = m + r$ (for some $r \geq 0$).

$\Sigma(\mathbf{1}^{m'}, F)$ is a cone in $\mathcal{J}(\mathbf{1}^{m'}, F)$ because $\mathbb{G}_{\mathrm{m}} \subset \mathrm{Aut}_{n'}$.

The vector bundle $\mathcal{J}(\mathbf{1}^{m'}, F) = F^{\oplus N}$ is ample.

The cone class $z\big(\Sigma(E,F), \mathcal{J}(E,F)\big) \in H_0(X, \mathbf{Z})$

is dual to the universal expression:

$$\sum_I \alpha_I S_I(E^* - F^*) = \sum_I \alpha_I S_I(-F^*) = \sum_I \alpha_I S_{I^\sim}(F) =: P(F).$$

F-L I tell us:

$$\int_X P(F) = \deg_X\big(z(\Sigma(\mathbf{1}^{m'}, F), F^{\oplus N})\big) > 0\,,$$

that is, the polynomial P is numerically positive for ample vector bundles. So F-L II tell us: all $\alpha_I \geq 0$. Q.E.D.

For non-stable singularities, one can still expand the Thom polynomials in the basis

$$\{S_I(R_m^*) \cdot S_J(R_n^*)\}$$

of products of Schur functions, instead of $\{S_I(R_m^* - R_n^*)\}$.

Thom polynomials for invariant cones

Data: $(n_1, n_2, \ldots, n_p) \in \mathbf{N}^p$ and

V a representation of $G = \prod_{i=1}^p GL_{n_i}$.

V gives rise to a *functor* ϕ defined for a collection of bundles on a variety X:

$$E_1, E_2, \ldots, E_p \mapsto \phi(E_1, E_2, \ldots, E_p)\,,$$

with $\dim E_i = n_i$. By passing to the dual bundles, we may assume that the functor ϕ is covariant in each variable.

$P(E_\bullet) = P(E_1, E_2, \ldots, E_p)$ – principal G-bundle associated with the vector bundles $E_1, E_2, \ldots, E_p$. V gives rise to a new vector bundle:

$$V(E_\bullet) := V(E_1, E_2, \ldots, E_p) := P(E_\bullet) \times_G V\,.$$

Consider $\Sigma \subset V$ – an invariant cone. It gives rise to

$$\Sigma(E_\bullet) = \Sigma(E_1, E_2, \ldots, E_p) := P(E_\bullet) \times_G \Sigma \subset V(E_\bullet)\,.$$

"Thom polynomial":

$$\mathcal{T}^\Sigma \in H^*(\prod_{i=1}^p BGL_{n_i}, \mathbf{Z})$$

evaluates the dual class of $[\Sigma(R^{(1)}, R^{(2)}, \ldots, R^{(p)})]$, where $R^{(i)}$ is the pullback of the tautological vector bundle from BGL_{n_i} to $\prod_{i=1}^p BGL_{n_i}$. It depends on the Chern classes of the $R^{(i)}$'s.

Let $\mathcal{T}^{\Sigma}(E_1, \ldots, E_p)$ be $\mathcal{T}^{\Sigma}$ with $c_j(R^{(i)})$ replaced by $c_j(E_i)$.

Lemma 15 *For any vector bundles $E_1, E_2, \ldots, E_p$ on a variety X, the dual class of $[\Sigma(E_\bullet)]$ in*

$$H^{2\,\mathrm{codim}(\Sigma)}(V(E_\bullet), \mathbf{Z}) = H^{2\,\mathrm{codim}(\Sigma)}(X, \mathbf{Z})$$

is equal to $\mathcal{T}^{\Sigma}(E_1, \ldots, E_p)$.

The functor ϕ *preserves global generateness* if for g.g. bundles $E_1, E_2, \ldots, E_p$, the bundle $\phi(E_1, E_2, \ldots, E_p)$ is g.g. Polynomial functors preserve g.g. but $\mathrm{Hom}(-, E)$ or $\mathrm{Hom}(-,-)$ do not.

Expand:

$$\mathcal{T}^{\Sigma} = \sum \alpha_{I_1,\ldots,I_p}\ S_{I_1}(R^{(1)})\ S_{I_2}(R^{(2)})\ \cdots\ S_{I_p}(R^{(p)})\,,$$

where $\alpha_{I_1,\ldots,I_p} \in \mathbf{Z}$.

Theorem 16 *(P. Pragacz and A. Weber, 2006) (1) If ϕ preserves g.g., then the coefficients $\alpha_{I_1,\ldots,I_p}$ are nonnegative.*

(2) If ϕ is polynomial, then $\alpha_{I_1,\ldots,I_p}$ are nonnegative, and their sum is positive. (The same holds true under some weaker, but more technical assumption.)

For $p = 2$, we get the nonnegativity of the coefficients in the expansions of Thom polynomials in the basis $\{S_I(R_m^*) \cdot S_J(R_n)\}$ (for all singularities).