COMPUTING WITH DIVIDED DIFFERENCES

FOR CLASSICAL GROUPS

[PR] P. Pragacz, J. Ratajski : Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach, Compositio Math., 107 (1997), 11–87.

[LP1] A. Lascoux, P. Pragacz : Operator calculus for $\widetilde{Q}$-polynomials and Schubert polynomials, Adv. in Math. 140 (1998), 1–43.

[LP2] A. Lascoux, P. Pragacz : Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-functions, and vertex operators, Michigan Math. J. 48, (2000), 417–441.

Let n be a fixed positive integer.

The symmetric group (i.e. the Weyl group of type A) $\mathfrak{S}_n$ is the group with generators $s_1, \dots, s_{n-1}$ subject to the relations

$$s_i^2 = 1, \qquad s_{i-1}\, s_i\, s_{i-1} = s_i s_{i-1}\, s_i$$

$$s_i s_j = s_j s_i \quad \forall i, j : |i-j| > 1\ .$$

The hyperoctahedral group (i.e. the Weyl group of type B or C) $\mathfrak{B}_n$ (or $\mathfrak{C}_n$) is an extension of $\mathfrak{S}_n$ by an element s_0 such that

$$s_0^2 = 1, \quad s_0\, s_1\, s_0\, s_1 = s_1\, s_0\, s_1\, s_0,$$
$$s_0\, s_i = s_i\, s_0 \quad \text{for} \quad i \geq 2.$$

The Weyl group $\mathfrak{D}_n$ of type D is the extension of $\mathfrak{S}_n$ by an element $s_\heartsuit$ such that

$$s_\heartsuit^2 = 1, \quad s_1 s_\heartsuit = s_\heartsuit\, s_1, \quad s_\heartsuit\, s_2\, s_\heartsuit = s_2 s_\heartsuit\, s_2,$$
$$s_\heartsuit\, s_i = s_i\, s_\heartsuit \quad \text{for} \quad i > 2.$$

$\mathfrak{S}_n$, $\mathfrak{B}_n$, $\mathfrak{C}_n$, and $\mathfrak{D}_n$ act on $\mathbb{Z}[x_1, \dots, x_n]$:

$$s_i(x_i) = x_{i+1}, \qquad s_0(x_1) = -x_1, \qquad s_\heartsuit(x_1) = -x_2$$

DIVIDED DIFFERENCES

$$f\,\partial_i := f - f(\dots, x_{i+1}, x_i, \dots)/(x_i - x_{i+1})\,,$$

$$f\,\partial_0 := f - f(-x_1, x_2, \dots)/(-x_1)\,,$$

$$f\,\partial_\heartsuit := f - f(-x_2, -x_1, x_3, \dots)/(-x_1 - x_2)\,.$$

The ∂_i, ∂_0, $\partial_\heartsuit$ satisfy the Coxeter relations, together with the relations $\partial_\heartsuit^2 = 0 = \partial_i^2 \quad \text{for} \quad 0 \le i < n$.

Therefore, to any element w of the group W, there corresponds a *divided difference* ∂_w. Any reduced decomposition

$$s_{i_1}\, s_{i_2} \cdots s_{i_\ell} = w$$

gives rise to a factorization

$$\partial_{i_1}\, \partial_{i_2} \cdots \partial_{i_\ell} \quad \text{of} \quad \partial_w$$

Relations between divided differences can be represented planarly. By definition, a planar display will be identified with its reading from left to right and top to bottom (*row–reading*). We shall also use *column–reading*, that is, reading successive columns downwards, from left to right.

For example, we will write

$$\begin{array}{cc} \partial_2 & \\ \partial_1 & \partial_2 \end{array} \equiv \begin{array}{cc} \partial_1 & \partial_2 \\ & \partial_1 \end{array}$$

for the following equality for simple divided differences:

$$\partial_2\, \partial_1\, \partial_2 = \partial_1\, \partial_2\, \partial_1 \ .$$

Suppose that a rectangle is filled row-wise from left to right, and column-wise from bottom to top with the ∂_i numbered by consecutive numbers from $\{1, \dots, n-1\}$. Then one easily checks that its ROW–READING and COLUMN--READING produce two EQUAL divided differences.

Example

$$\begin{array}{ccc} \partial_3\partial_4\partial_5\partial_6 & & \\ & \partial_2\partial_3\partial_4\partial_5 & \\ & & \partial_1\partial_2\partial_3\partial_4 \end{array} = \begin{array}{ccccccc} \partial_3 & & \partial_4 & & \partial_5 & & \partial_6 \\ \partial_2 & \cdot & \partial_3 & \cdot & \partial_4 & \cdot & \partial_5 \\ \partial_1 & & \partial_2 & & \partial_3 & & \partial_4 \end{array}$$

Example $\partial_{[q+1,\dots,n,1,\dots,q]} =$

$= (\partial_q \cdots \partial_{n-1}) \cdots (\partial_2 \cdots \partial_{n-q+1})(\partial_1 \cdots \partial_{n-q})$

$$= \begin{array}{ccc} \partial_q & \cdots & \partial_{n-1} \\ \vdots & & \vdots \\ \partial_2 & \cdots & \partial_{n-q+1} \\ \partial_1 & \cdots & \partial_{n-q} \end{array}$$

$= (\partial_q \cdots \partial_2 \partial_1) \cdots (\partial_{q+1} \cdots \partial_3 \partial_2)(\partial_{n-1} \cdots \partial_{n-q+1} \partial_{n-q})$

More generally, the planar arrays that we shall use, will have the property that their ROW–READING and COLUMN--READING are EQUAL.

Example

$$\begin{array}{cccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \end{array}$$

encodes the equality

$$\partial_0\partial_1\partial_0\partial_2\partial_1\partial_0\partial_3\partial_2\partial_1 = \partial_0\partial_1\partial_2\partial_3\partial_0\partial_1\partial_2\partial_0\partial_1 \,.$$

SCHUBERT POLYNOMIALS OF LASCOUX & SCHUTZENBERGER

Define the vector $\rho := [n-1, \ldots, 1, 0]$.

One defines recursively *Schubert polynomials* Y_α, for any sequence $\alpha \in \mathbb{N}^n$, with $\alpha \subseteq \rho$, by

$$Y_\alpha \, \partial_i = Y_\beta \ , \quad \text{if} \ \ \alpha_i > \alpha_{i+1} \ \ ,$$

where

$$\beta = [\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i - 1, \alpha_{i+2}, \ldots, \alpha_n] \, ,$$

starting from $Y_\rho = x^\rho$.

If $\alpha \in \mathbb{N}^n$ is weakly decreasing, then Y_α is equal to the *monomial* x^α.

If $\alpha_1 \leq \cdots \leq \alpha_k$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$, for some $k \leq n$, then Y_α is the *Schur polynomial* $S_{(\alpha_1, \ldots, \alpha_k)}(x_1, \ldots, x_k)$.

The ring $\mathbb{Z}[X]$ is a free $Sym(X)$–module with a basis given by Schubert polynomials Y_α where $\alpha \in \mathbb{N}^n$ and $\alpha \subseteq \rho$.

$\widetilde{Q}$–POLYNOMIALS OF [PR]

We set $\widetilde{Q}_i := \Lambda_i = \Lambda_i(\mathcal{X})$, the i-th elementary symmetric polynomial in $\mathcal{X} = (x_1, \dots, x_n)$.

Given two nonnegative integers $i \geq j$, we put

$$\widetilde{Q}_{i,j} := \widetilde{Q}_i \widetilde{Q}_j + 2 \sum_{p=1}^{j} (-1)^p \widetilde{Q}_{i+p} \widetilde{Q}_{j-p} .$$

Given any partition $I = (i_1 \geq \ldots \geq i_k)$, where we can assume k to be even,we set

$$\widetilde{Q}_I := \text{Pfaffian}(M) ,$$

where $M = (m_{p,q})$ is the $k \times k$ skew-symmetric matrix with $m_{p,q} = \widetilde{Q}_{i_p, i_q}$ for $1 \leq p < q \leq k$.

Equivalently, $\widetilde{Q}_I$ is defined recurrently on $l = l(I)$ by putting for odd l,

$$\widetilde{Q}_I := \sum_{j=1}^{l} (-1)^{j-1} \widetilde{Q}_{i_j} \cdot \widetilde{Q}_{(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_l)}$$

and for even l,

$$\widetilde{Q}_I := \widetilde{Q}_I(X) = \sum_{j=2}^{l} (-1)^j \widetilde{Q}_{i_1, i_j} \cdot \widetilde{Q}_{(i_2, \dots, i_{j-1}, i_{j+1}, \dots, i_l)} .$$

Exercise $\widetilde{Q}_{i,i}(\mathcal{X}) = \Lambda_i(x_1^2, \dots, x_n^2)$.

The ring $Sym(n)$ is a free $Sym^C(n)$-module with a basis of the $\widetilde{Q}_I(X)$, where strict $I \subset \rho_n$.

First sign that the $\widetilde{Q}_I$'s are well suited to divided differences:

$$\nabla := (\partial_0\partial_1 \cdots \partial_{n-1}) \cdots (\partial_0\partial_1)\partial_0$$

Denote by $\qquad \langle\ ,\ \rangle : Sym(n) \times Sym(n) \to Sym^C(n)$

the scalar product defined for $f, g \in Sym(n)$ by

$$\langle f, g\rangle := fg\,\nabla\,.$$

Define a strict partition $\qquad \rho(k) := (k, k-1, \ldots, 1).$

Orthogonality of $\widetilde{Q}$-polynomials: For strict $I, J \subseteq \rho(n)$,

$$\langle \widetilde{Q}_I, \widetilde{Q}_{\rho(n)\smallsetminus J}\rangle = \pm\delta_{IJ},$$

where $\rho(n) \smallsetminus J$ is the strict partition whose parts complement the parts of J in $\{n, n-1, \ldots, 1\}$.

Define $\qquad \widetilde{Q}(\mathcal{X}, \mathcal{Y}) := \sum \widetilde{Q}_I(\mathcal{X})\,\widetilde{Q}_{\rho(n)\smallsetminus I}(\mathcal{Y})\ ,$

where the summation is over all strict partitions $I \subseteq \rho(n)$.

$\widetilde{Q}(\mathcal{X}, \mathcal{Y})$ is a reproducing kernel for $\langle\ ,\ \rangle$:

$$\langle f(\mathcal{X})\,, \widetilde{Q}(\mathcal{X}, \mathcal{Y})\rangle = \pm f(\mathcal{Y})\ .$$

Exercise The determinant

$$F(\mathcal{X},\mathcal{Y}) := \ |\widetilde{Q}_{n+1+j-2i}(\mathcal{X}) + \widetilde{Q}_{n+1+j-2i}(\mathcal{Y})|_{1\le i,j\le n}$$

is also a reproducing kernel for this scalar product:

$$\langle f(\mathcal{X})\,, F(\mathcal{X},\mathcal{Y})\rangle \equiv \pm f(\mathcal{Y})\ .$$

$\equiv$ means: "congruent modulo the ideal generated by $f(x_1^2,\ldots,x_n^2) - f(y_1^2,\ldots,y_n^2)$, where $f \in Sym(n)$ ".

How ∇ acts on $\widetilde{Q}_I Y_\alpha$?

Exercise (i) For $\alpha \in \mathbb{N}^n$, $\alpha \subseteq \rho$,

$$Y_\alpha\,\widetilde{Q}_{\rho(n)}\,\nabla = \pm Y_\alpha^\omega \qquad \text{where} \quad \omega = [n, n-1, \ldots, 1]$$

(ii) For strict $I \subsetneq \rho(n)$, and $\alpha \subseteq \rho$,

$$Y_\alpha\,\widetilde{Q}_I\,\nabla = 0\ .$$

Calculations by ACE:

$I = (i_1 > i_2 > \cdots > i_\ell > 0)$ a strict partition

$$\widetilde{Q}_I\,\partial_0^C\partial_1\cdots\partial_{i_1-1} = \pm\widetilde{Q}_{(i_2,i_3,\ldots)}$$

For $v(I) := [i_1,\ldots,i_\ell,\overline{j_1},\ldots,\overline{j_h}]\,,$

$$x^\rho\widetilde{Q}_{\rho(n)}\,\partial_{v(I)} = \pm\widetilde{Q}_I.$$

How to see this conceptually?

Key operators:

For $k \leq n$,

$$\nabla_k^C(n) := (\partial_0^C \partial_1 \cdots \partial_{n-1}) \cdots (\partial_0^C \partial_1 \cdots \partial_{n-k})$$
$$\nabla_k^B(n) := (\partial_0^B \partial_1 \cdots \partial_{n-1}) \cdots (\partial_0^B \partial_1 \cdots \partial_{n-k})$$

For $k \leq n/2$,

$$\nabla_k^D(n) := (\partial_\heartsuit \partial_2 \cdots \partial_{n-1} \partial_1 \partial_2 \cdots \partial_{n-2}) \cdots$$
$$\cdots (\partial_\heartsuit \partial_2 \cdots \partial_{n-2k+1} \partial_1 \partial_2 \cdots \partial_{n-2k})$$

How to compute with $\nabla_k^C(n)$?

For $k = 1$:
Suppose $n \geq p > 0$. Let $IpJ \subseteq \rho(n)$ be a strict partition. Then

$$x_1^{n-p} \widetilde{Q}_{IpJ}\, \partial_0^C \partial_1 \cdots \partial_{n-1} = \pm \widetilde{Q}_{IJ}\,.$$

If $H \subseteq \rho(n)$ a strict partition not containing p, then

$$x_1^{n-p} \widetilde{Q}_H\, \partial_0^C \partial_1 \cdots \partial_{n-1} = 0.$$

Theorem 1. Let $k \leq n$ and let $\alpha = [\alpha_1 \leq \cdots \leq \alpha_k] \in \mathbb{N}^k$ with $\alpha_k \leq n - k$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I\, Y_\alpha$ under $\nabla_k^C(n)$ is 0 unless $n - 0 - \alpha_1, \ldots, n - (k-1) - \alpha_k$ are parts of I. In this case, the image is $\pm\widetilde{Q}_J$, where J is the strict partition with parts $\{i_1, \ldots, i_{\ell(I)}\} \setminus \{n - 0 - \alpha_1, \ldots, n - (k-1) - \alpha_k\}$.

Example For $n = 7$ and $k = 2$,

$$\widetilde{Q}_{(5,4,3,2,1)}\, Y_{[2,5]}\, \nabla_2^C(7)$$
$$= \widetilde{Q}_{(5,4,3,2,1)}\, Y_{[2,5]}\, (\partial_0^C \partial_1\partial_2\partial_3\partial_4\partial_5\partial_6)(\partial_0^C \partial_1\partial_2\partial_3\partial_4\partial_5)$$
$$= \pm\widetilde{Q}_{(4,3,2)}.$$

and for $k = 3$,

$$\widetilde{Q}_{(7,5,4,3,1)}\, Y_{[2,3,4]} \nabla_3^C(7) = \pm\widetilde{Q}_{(7,4)}.$$

Sketch of proof

Work in Sym , the ring of symmetric functions in countable number of indeterminates. Let (,) be the standard scalar product on Sym.

Foulkes' derivative Df for $f \in Sym$: $\quad (g \cdot D_f, h) = (g, f \cdot h)$

Lemma $s_0 = 1 - 2D_{P_1} x_1 + 2D_{P_2} x_1^2 - 2D_{P_3} x_1^3 + \cdots$.

P_i is the sum of all hook Schur functions of degree i.

So $\qquad \partial_0^C = D_{P_1} - D_{P_2} x_1 + D_{P_3} x_1^2 - \cdots$.

Using this equality and $x_1^p \partial_1 \cdots \partial_{n-1} = S_{p-n+1}(x_1, \ldots, x_n)$, we compute for $0 < a \leq n$,

$$x_1^{n-a} \partial_0^C \partial_1 \cdots \partial_{n-1} = D_{P_a} - D_{P_{a+1}} S_1 + D_{P_{a+2}} S_2 - \cdots.$$

(after restriction to the first n variables)

Define

$U_a^S := D_{P_a} - D_{P_{a+1}} S_1 + D_{P_{a+2}} S_2 - \cdots$

$U_a^\Lambda := D_{P_a} - D_{P_{a+1}} \Lambda_1 + D_{P_{a+2}} \Lambda_2 - \cdots .$

To show: $\quad \widetilde{Q}_{IaJ}\ U^S = \pm \widetilde{Q}_{IJ} \quad$ or $\quad 0.$

U_a^S is dual to U_a^Λ for the involution: $S_I \to S_I$.
So if we define Q'_I as duals to $\widetilde{Q}_I$, then to show is

$$Q'_{IaJ}\ U_a^\Lambda = \pm Q'_{IJ} \quad \text{or} \quad 0.$$

Littlewood: The basis $\{Q'_J\}$, where J runs over strict partitions, is conjugate to $\{P_J\}$, the basis of Schur's P-functions.

Recall that for $J \in \mathbb{N}^{*m}$ with odd m,

$P_J = P_{j_1} \cdot P_{j_2,\ldots,j_m} - P_{j_2} \cdot P_{j_1,j_3,\ldots,j_m} + \ldots + P_{j_m} \cdot P_{j_1,\ldots,j_{m-1}}$
and with even m, $P_J =$

$P_{j_1,j_2} \cdot P_{j_3,\ldots,j_m} - P_{j_1,j_3} \cdot P_{j_2,j_4,\ldots,j_m} + \ldots + P_{j_1,j_m} \cdot P_{j_2,\ldots,j_{m-1}}$.

Here:

$$P_{i,j} = P_i \cdot P_j + 2\sum_{k=1}^{j-1} (-1)^k P_{i+k} P_{j-k} + (-1)^j P_{i+j}.$$

Define : $\quad V_a^\Lambda := P_a - D_{\Lambda_1} P_{a+1} + D_{\Lambda_2} P_{a+2} - \cdots .$

To show: $\qquad P_J\ V_a^\Lambda = P_{aJ}$.

Lemma $P_{aJ} = \sum_{j\geq 0} (-1)^j\ P_J\ D_{\Lambda_j}\ P_{a+j}$.

There is a complement to the theorem: erasing the “zero part”.

$$\widetilde{Q}_I\, x_1^n\, \partial_0^C \partial_1 \cdots \partial_{n-1} = \begin{cases} -\widetilde{Q}_I, & n+\ell(I) \quad \text{odd} \\ 0, & n+\ell(I) \quad \text{even}\,. \end{cases}$$

Example For $n = 5$ and $k = 1$,

$$x_1^5 \widetilde{Q}_{(5,3,2,1)} \partial_0^C \partial_1 \partial_2 \partial_3 \partial_4 = -\widetilde{Q}_{(5,3,2,1)}$$

$$\text{and} \quad x_1^5 \widetilde{Q}_{(5,2,1)} \partial_0^C \partial_1 \partial_2 \partial_3 \partial_4 = 0.$$

Theorem 2. Let $k \le n$ and let $\alpha = [\alpha_1 \le \cdots \le \alpha_k] \in \mathbb{N}^k$ with $\alpha_k = n - k + 1$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I\, Y_\alpha$ under $\nabla_k^C(n)$ is 0 unless $\ell(I) \not\equiv n \pmod 2$ and $n - 0 - \alpha_1, \ldots, n - (k-2) - \alpha_{k-1}$ are parts of I. In this case, the image is $\pm\widetilde{Q}_J$, where J is the strict partition with parts

$$\{i_1, \ldots, i_{\ell(I)}\} \setminus \{n - 0 - \alpha_1, \ldots, n - (k-2) - \alpha_{k-1}\}\,.$$

Example For $n = 7$ and $k = 4$,

$$\widetilde{Q}_{(7,6,4,3,2,1)}\, Y_{[1,2,2,4]}\, \nabla_4^C(7) = \pm\widetilde{Q}_{(7,2,1)}$$

and

$$\widetilde{Q}_{(7,6,4,3,2,1)}\, Y_{[1,3,4,4]}\, \nabla_4^C(7) = \pm\widetilde{Q}_{(7,4,2)}.$$

$\partial_\heartsuit$ as a vertex operator:

$$\partial_\heartsuit(f) = f + 2\sum_{i\geq 1}(-1)^i D_{P_i}(x_1^i + x_2^i)$$
$$+4\sum_{i\geq j\geq 1}(-1)^{i+j} D_{P_i\cdot P_j}(x_1^i x_2^j + x_1^j x_2^i)$$

The group $\mathfrak{D}_n$ is a subgroup of $\mathfrak{B}_n$ by sending $s_\heartsuit$ to $s_0\, s_1\, s_0$.

Example As operators on $Sym(2)$,

$$\nabla_1^D(2) = \partial_\heartsuit = x_1x_2\partial_0\partial_1\partial_0 + x_1\partial_0\partial_1 .$$

Proposition As operators on $Sym(2k)$,

$$\nabla_k^D(2k) = x_1\cdots x_{2k}\,\nabla_{2k}^B(2k) + x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(2k)\,.$$

For example, as operators on $Sym(4)$,

$$\nabla_2^D(4) = x_1x_2x_3x_4 \begin{array}{cccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \\ & & & \partial_0 \end{array} + x_1x_2x_3 \begin{array}{cccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \end{array}$$

The Proposition is proved by comparing the values of the $\widetilde{Q}_I(\mathcal{X})$ under the operators:

$$\nabla^D_k(2k)\,, \quad x_1\cdots x_{2k}\,\nabla^B_{2k}(2k), \quad \text{and} \quad x_1\cdots x_{2k-1}\,\nabla^B_{2k-1}(2k),$$

Lemma ("Linearity formula") For any strict partition I,

$$\widetilde{Q}_I(x_1,\dots,x_m) = \sum_{j=0}^{\ell(I)} x_m^j \Big(\sum_{|I|-|J|=j} \widetilde{Q}_J(x_1,\dots,x_{m-1})\Big),$$

where the sum is over all (i.e. not necessarily strict) partitions $J \subseteq I$ such that I/J has at most one box in every row.

Moreover, given a partition $I' = (\dots, i, j, j, k, \dots)$ and denoting $I = (\dots, i, k, \dots)$, one has the *factorization property*:

$$\widetilde{Q}_{I'} = \widetilde{Q}_{j,j}\widetilde{Q}_I .$$

Corollary Let $k \le n/2$. As operators on $Sym(2k\,|\,n-2k)$,

$$\nabla^D_k(n) = x_1\cdots x_{2k}\,\nabla^B_{2k}(n) + x_1\cdots x_{2k-1}\,\nabla^B_{2k-1}(n)\,\partial_1\cdots\partial_{n-2k}\,.$$

This property is obtained by composing the expression of $\nabla^D_k(2k)$ from the Proposition with the divided difference

$$\begin{matrix} \partial_{2k} & \cdots & \partial_{n-1} \\ \vdots & & \vdots \\ \partial_2 & \cdots & \partial_{n-2k+1} \\ \partial_1 & \cdots & \partial_{n-2k} \end{matrix}$$

Define: $$\widetilde{P}_I := 2^{-\ell(I)}\,\widetilde{Q}_I\,.$$

Theorem 3. Let $k \le n/2$. Suppose that $I \subseteq \rho(n-1)$ is a strict partition. Let $\alpha = [\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2k}] \in \mathbb{N}^{2k}$ with $\alpha_{2k} \le n-2k$. Then the image of $\widetilde{P}_I\, Y_\alpha$ under $\nabla_k^D(n)$ is 0 unless all the integers:
$n-1-\alpha_1, \ldots, n-2k-\alpha_{2k}$ belong to $\{i_1, \ldots, i_{\ell(I)}, 0\}$.
In this case, the image is $\pm \widetilde{P}_J$, where J is the strict partition with parts

$$\{i_1, \ldots, i_{\ell(I)}\} \smallsetminus \{n-1-\alpha_1, \ldots, n-2k-\alpha_{2k}\}\,.$$

Example (i) For $n=7$ and $k=1$,

$$\widetilde{P}_{(5,4,3,2,1,0)}\, Y_{[1,3]} \nabla_1^D(7)$$

$$= \widetilde{P}_{(5,4,3,2,1,0)}\, Y_{[1,3]}\, \partial_\heartsuit \partial_2 \partial_3 \partial_4 \partial_5 \partial_6 \partial_1 \partial_2 \partial_3 \partial_4 \partial_5 = \pm \widetilde{P}_{(4,3,1)}$$

and

$$\widetilde{P}_{(6,4,3,2,1,0)}\, Y_{[2,5]}\, \nabla_1^D(7) = \pm \widetilde{P}_{(6,3,2,1)}.$$

(ii) For $n=7$ and $k=2$,

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,1,2]}\, \nabla_2^D(7) = \pm \widetilde{P}_{(6,2)}$$

and

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,1,3]}\, \nabla_2^D(7) = \pm \widetilde{P}_{(6,2,1)}\,.$$

Sketch of proof

To compute is the action of $\nabla_k^D(n) = \Omega_1 + \Omega_2$, where

$$\Omega_1 := x_1 \cdots x_{2k} \, \nabla_{2k}^B(n)$$

and

$$\Omega_2 := x_1 \cdots x_{2k-1} \, \nabla_{2k-1}^B(n) \, \partial_1 \cdots \partial_{n-2k} \, .$$

Lemma For $\alpha = [\alpha_1, \dots, \alpha_k] \in \mathbb{N}^k$,

$$Y_\alpha x_1 \cdots x_k = Y_{[\alpha_1+1, \dots, \alpha_k+1]} \, .$$

The image of $\widetilde{Q}_I(\mathcal{X}) \, Y_\alpha$ under Ω_1 is given by Theorem 1 combined with the Lemma if $\alpha_{2k} < n - 2k$ and by Theorem 2 combined with the Lemma in the case $\alpha_{2k} = n - 2k$.

Since x_{2k} appears in Y_α, the same results, however, *do not* directly furnish the value of $\widetilde{Q}_I(X_n) \, Y_\alpha$ under Ω_2. To end this computation, we proceed as follows. For simplicity of indices, let us take temporarily $n = 7$ and $k = 2$. Suppose that $\alpha = [\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4] \in \mathbb{N}^4$ is s.t. $\alpha_4 \le 3$.

To compute:

$$\widetilde{Q}_I \, Y_\alpha \, x_1 x_2 x_3 \quad \begin{array}{ccccccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & & & & \partial_1 & \partial_2 & \partial_3 \end{array}$$

One has, for any integers a, b, c, d, k :

$1 \leq a < b, \quad c < d \leq n, \quad a+d = b+c, \quad k < d-b,$

$$\begin{array}{ccccc} \partial_{b+1} & \cdots & \partial_{b+k} & & \\ \partial_b & \cdots & \cdots & \cdots & \partial_d \\ \vdots & & & & \vdots \\ \partial_a & \cdots & \cdots & \cdots & \partial_c \end{array} \equiv \begin{array}{ccccc} \partial_b & \cdots & \cdots & \cdots & \partial_d \\ \vdots & & & & \vdots \\ \partial_a & \cdots & \cdots & \cdots & \partial_c \\ & & \partial_{c-k} & \cdots & \partial_{c-1} \end{array}$$

Thanks to these relations,

$$\begin{array}{ccccccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & & & & \partial_1 & \partial_2 & \partial_3 \end{array}$$

$$= \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} \begin{pmatrix} \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & \partial_1 & \partial_2 & \partial_3 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} \begin{pmatrix} \partial_4 & \partial_5 & \partial_6 & \\ \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} (\partial_4 \partial_5 \partial_6) \begin{pmatrix} \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{pmatrix}$$

Since $\partial_4\partial_5\partial_6$ commutes with the divided differences on its left, the last expression is rewritten as

$$\partial_4\partial_5\partial_6 \quad \begin{matrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{matrix} = \partial_4\partial_5\partial_6 \, \nabla_3^B(7)$$

Since $\partial_4\partial_5\partial_6$ commutes with $x_1x_2x_3$ and $\widetilde{Q}_I = \widetilde{Q}_I(x_1, \ldots, x_7)$, the polynomial to be computed is equal to

$$Y_\alpha \, \partial_4\partial_5\partial_6 \, \widetilde{Q}_I \, x_1x_2x_3 \, \nabla_3^B(7) \, .$$

However, by the definition, the image of $Y_{[\alpha_1,\alpha_2,\alpha_3,\alpha_4]}$ under $\partial_4\partial_5\partial_6$ is

$$Y_{[\alpha_1,\alpha_2,\alpha_3,0,0,0,\alpha_4-3]} = Y_{[\alpha_1,\alpha_2,\alpha_3]}$$

if $\alpha_4 = 3$, and 0 otherwise. Hence, by the Lemma, the polynomial to be computed is equal to

$$Y_{[\alpha_1+1,\alpha_2+1,\alpha_3+1]} \widetilde{Q}_I \nabla_3^B(7) \, .$$

Since $\alpha_3 + 1 \leq n - (2k-1) = 7 - 3$, Theorem 1 provides the end of the computation with Ω_2.

Example (i) For $n = 5$, $k = 1$, $\quad \widetilde{P}_{(3,2)} \, Y_{[1,3]} \, \nabla_1^D(5) = \widetilde{P}_2$

Ω_1 contributes (by Theorem 2):

$$\widetilde{Q}_{(3,2)} \, Y_{[1,3]} \, x_1x_2 \nabla_2^B(5) = \widetilde{Q}_{(3,2)} \, Y_{[2,4]} \, \nabla_2^B(5) = 4\widetilde{Q}_2$$

Ω_2 contributes (by Theorem 1):

$$\widetilde{Q}_{(3,2)} \, Y_{[1,3]} \, x_1 \nabla_1^B(5) \, \partial_1\partial_2\partial_3 = \widetilde{Q}_{(3,2)} \, Y_{[2]} \, \nabla_1^B(5) = -2\widetilde{Q}_2 \, .$$

Example For $n = 7$, $k = 2$,

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,2,3]}\, \nabla_2^D(7) = -\widetilde{P}_{(6,3,1)}$$

and the contribution comes from both operators Ω_1, Ω_2:

$$\begin{aligned}&\widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,1,2,3]}\, x_1x_2x_3x_4\, \nabla_4^B(7)\\ &= \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[2,2,3,4]}\, \nabla_4^B(7) = -16\widetilde{Q}_{(6,3,1)}\end{aligned}$$

by Theorem 2, and

$$\begin{aligned}&\widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,1,2,3]}\, x_1x_2x_3\, \nabla_3^B(7)\, \partial_1\partial_2\partial_3\\ &= \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[2,2,3]}\, \nabla_3^B(7) = 8\widetilde{Q}_{(6,3,1)}\,.\end{aligned}$$

by Theorem 1.

For $n = 7$, $k = 2$,

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[0,1,2,2]}\, \nabla_2^D(7) = -\widetilde{P}_{(5,3)}$$

and only the operator Ω_1 gives the contribution:

$$\begin{aligned}&Q_{(6,5,4,3,2,1)}\, Y_{[0,1,2,2]}\, x_1x_2x_3x_4\nabla_4^B(7)\\ &= \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,2,3,3]}\, \nabla_4^B(7) = -16\widetilde{Q}_{(5,3)}\end{aligned}$$

by Theorem 1.

Applications for type B

Proposition For a strict partition $I = (i_1, i_2, \ldots)$,

$$\widetilde{P}_I\, \partial_0 \partial_1 \cdots \partial_{i_1 - 1} = \pm \widetilde{P}_{(i_2, \ldots)}\,.$$

We associate with every strict partition $I = (i_1, \ldots, i_\ell > 0)$

$$v(I) := [i_1, \ldots, i_\ell, \overline{j_1}, \ldots, \overline{j_h}]\,,$$

where $j_1 < \cdots < j_h$.

Theorem For every strict partition $I \subseteq \rho(n)$,

$$x^\rho\, \widetilde{P}_{\rho(n)}\, \partial_{v(I)} = \pm \widetilde{P}_I.$$

This leads to the following characterization of $\widetilde{P}$-polynomials via divided differences:

Corollary For any strict partition I, let $w(I) := v(I)^{-1} w_0^B$, that is $w(I) = [\overline{i_1}, \ldots, \overline{i_\ell}, j_1, \ldots, j_h]^{-1}$. Then $w = w(I)$ is the unique element of $\mathfrak{B}_n$ such that $\ell(w) = |I|$ and $\widetilde{P}_I\, \partial_w \neq 0$.

More generally, consider, for any $w \in \mathfrak{B}_n$, the *orthogonal Schubert polynomial*

$$X_w^B = X_w^B(n) = x^\rho\, \widetilde{P}_{\rho(n)} \partial_{w_0^B w}$$

of degree $\ell(w)$. In particular,

$$X_{w_0}^B = x^\rho\, \widetilde{P}_{\rho(n)}.$$

These Schubert polynomials have the *stability property*:
for $w \in \mathfrak{B}_n \subset \mathfrak{B}_{n+1}$,

$$X^B_w(n+1)|_{x_{n+1}=0} = X^B_w(n).$$

Also,

$$X^B_{[\overline{i_1},\dots,\overline{i_\ell},j_1,\dots,j_h]} = \pm \widetilde{P}_I\,,$$

a property that has some geometric meaning.

Applications for type D

With every strict partition $I = (i_1, \dots, i_\ell > 0)\,$,
we associate the following element $v(I) \in \mathfrak{D}_n\,$.
If $n - \ell$ is even, we set

$$v(I) := [i_1+1, i_2+1, \dots, i_\ell+1, \overline{j_1}, \dots, \overline{j_h}]\,,$$

and if $n - \ell$ is odd,

$$v(I) := [i_1+1, i_2+1, \dots, i_\ell+1, 1, \overline{j_1}, \dots, \overline{j_h}]\,.$$

Theorem For a strict partition $I \subseteq \rho(n-1)$,

$$x^\rho \widetilde{P}_{\rho(n-1)}\, \partial_{v(I)} = \pm \widetilde{P}_I.$$

Corollary For a strict partition $I \subseteq \rho(n-1)$, we set $w(I) := v(I)^{-1} w_0^D$. More explicitly,
for even ℓ, $w(I) = [\overline{i_1+1}, \ldots, \overline{i_\ell+1}, j_1, \ldots, j_h]^{-1}$,
and for odd ℓ, $w(I) = [\overline{i_1+1}, \ldots, \overline{i_\ell+1}, \overline{1}, j_1, \ldots, j_h]^{-1}$.
Then $w = w(I)$ is the unique element of $\mathfrak{D}_n$ such that $\ell(w) = |I|$ and $\widetilde{P}_I \, \partial_w \neq 0$.

More generally, consider, for any $w \in \mathfrak{D}_n$, the *orthogonal Schubert polynomial*

$$X_w^D = X_w^D(n) = x^\rho \widetilde{P}_{\rho(n-1)} \partial_{w_0^D w}$$

of degree $\ell(w)$. These Schubert polynomials have the *stability property*: for $w \in \mathfrak{D}_n \subset \mathfrak{D}_{n+1}$,

$$X_w^D(n+1)|_{x_{n+1}=0} = X_w^D(n).$$

Also, for even ℓ,

$$X^D_{[\overline{i_1+1}, \ldots, \overline{i_\ell+1}, j_1, \ldots, j_h]} = \pm \widetilde{P}_I \,,$$

and for odd ℓ,

$$X^D_{[\overline{i_1+1}, \ldots, \overline{i_\ell+1}, \overline{1}, j_1, \ldots, j_h]} = \pm \widetilde{P}_I \,.$$

Proposition For a strict partition $I = (i_1, i_2, i_3, i_4, \ldots) \subseteq \rho(n-1)$,

$$\widetilde{P}_I \, \partial_\heartsuit \partial_2 \cdots \partial_{i_1} \partial_1 \partial_2 \cdots \partial_{i_2} = \pm \widetilde{P}_{(i_3, i_4, \ldots)}.$$