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$F: X^{ns} \to Y^{ns}$ proper $\rightsquigarrow$ $F_*: A^\cdot X \to A^\cdot Y$ additive map induced by push-forward of cycles.

residues, equivariant cohomology ... (with L. Darondeau)

We use elementary approach (1

$$\int_X^Y \quad \alpha \in A^\cdot X, \quad F_*(\alpha) = \int_X^Y \alpha$$

$E \to X$ vector bundle $\quad \pi: P(E) \to X$ projective bundle
$\operatorname{rk} E = n \quad$ of lines in E

$\mathcal{O}_{P(E)}(1)$ invertible sheaf of Serre on $P(E)$

$$\xi := c_1 \mathcal{O}_{P(E)}(1).$$

$$(*) \quad \int_{P(E)}^X \xi^i = s_{i-n+1}(E) \quad \text{Segre class}$$

Segre polynomial $s_x(E) = 1 + x s_1(E) + x^2 s_2(E) + \dots$

$$s_{i-n+1}(E) = [x^{i-n+1}](s_x(E)) = [x^{-n+1}](x^{-i} s_x(E))$$

$$x = 1/t, \quad s_{i-n+1}(E) = [t^{n-1}](t^i s_{1/t}(E))$$

$$(*) \quad \int_{P(E)}^X \xi^{i} = [t^{n-1}](t^{i} s_{1/t}(E))$$

$$f(t) \in A^\cdot X[t] \Rightarrow \int_{P(E)}^X f(\xi) = [t^{n-1}](f(t)\, s_{1/t}(E))$$

ff

Flag bundles:

(A) $1 \le d_1 < \dots < d_m \le n-1$; $\pi: F(d_1, \dots, d_m)(E) \to X$ flag bundle of subspaces of dim $d_1, \dots, d_m$ in E.

(C) $\operatorname{rk} E = 2n$, $\omega: E \otimes E \to L$ symplectic form, $1 \le d_1 < \dots < d_m \le n$, $\pi: F^\omega(d_1, \dots, d_m)(E) \to X$ flag bundle of isotropic subspaces of dim $d_1, \dots, d_m$ in E.

(B), (D) orthog. form, isotropic subspaces.

They all have universal flags $\mathcal{U}_{d_1} \subset \dots \subset \mathcal{U}_{d_m}$.

Examples. For flag bundle $F \to X$, the ξ_i are the Chern roots of the tautological subbundles on F.

$$\int_{F(d)(E)}^{X} f(\xi_1, \dots, \xi_d) = \left[\prod t_i^{n-i}\right] \left(f(t) \prod_{i<j} (t_i - t_j) \prod s_{1/t_i}(E) \right)$$

$$\int_{F(1,\dots,n-1)(E)} f(\xi) = \left[\prod t_i^{n-1}\right] \left(f(t) \prod_{i<j} (t_i - t_j) \prod s_{1/t_i}(E) \right)$$

$$\int_{F^{\omega}(d)(E)} f(\xi) = \left[\prod t_i^{2n-i}\right] \left(f(t) \prod_{i<j} (t_i^2 - t_j^2) \prod s_{1/t_i}(E) \right)$$

(A) in general $0 = d_0 < d_1 < \dots < d_m \leq n-1$, $d = d_m$

$$\xi_i = -c_1(\mathcal{U}_{d+1-i} / \mathcal{U}_{d-i})$$

Thm (D-P)
$$\int_{F(d_1, \dots, d_m)(E)} f(\xi_1, \dots, \xi_d) =$$

$$[t_1^{e_1} \cdots t_d^{e_d}] \left(f(t_1, \dots, t_d) \prod_{i<j} (t_i - t_j) \prod s_{1/t_i}(E) \right)$$

e_j? $\quad j \in \left] d-d_k, d-d_{k-1} \right] \quad \exists k$

j is on the i-th place $\Rightarrow e_j = n-i$.

Proof (1) for $F(1,2,\dots,d)(E)$ $\quad d \leq n-1$ (full flag)

We can iterate ff in the chain of projective bundles:

$$F(1,\dots,d)(E) \to F(1,\dots,d-1)(E) \to \dots \to F(1)(E) \to X$$

$$\int_{F(1,\dots,d)(E)}^{X} f(\xi_1, \dots, \xi_d) =$$

$$[t_1^{n-1} \cdots t_d^{n-d}] \left(f(t_1, \dots, t_n) \prod_{i<j} (t_i - t_j) \prod s_{1/t_i}(E) \right).$$

Commutative diagram of flag bundles of Damon;

$F := F(d_1, \dots, d_m)(E)$ $\quad d = d_m$

$Y := F(\mathcal{U}_1) \times_F F(\mathcal{U}_{d_2}/\mathcal{U}_{d_1}) \times_F \cdots \times_F F(\mathcal{U}_{d_m}/\mathcal{U}_{d_{m-1}})$

$\simeq F(1, \dots, d)(E)$ where $F(E)$ is the bundle of complete flags in E

$$\begin{tikzcd}
F(1,\dots,d)(E) \arrow[r, "\theta", "\simeq"'] \arrow[d, "\pi''"'] & Y \arrow[d, "\pi'"] \\
X & F \arrow[l, "\pi"']
\end{tikzcd}$$

Lemma π'_* is surjective.

Our goal is to compute: $\pi_* = \int_F^X$. Since π'_* is surj., enough to understand

$$\pi_* \pi'_* = \pi''_* \circ (\theta^{-1})_* .$$

From full flag case we know π''_*, so we can proceed.

(C) $$\int_{F^{\omega}(d_1,\dots,d_m)(E)} f(\xi) = \left[\prod t_j^{e_j}\right]\left(f(t) \prod (t_i - t_j)(t_i + t_j + c_1 L) \times \prod s_{\nu/t_i}(E)\right)$$

$e_j = 2n - i$

(B), (D) $$\int_{F^{Q}(d_1,\dots,d_m)(E)} f(\xi) = \left[\prod t_j^{e_j}\right]\Big(f(t) \prod (2t_i + c_1 L)$$
$$\times \prod_{i<j} (t_i - t_j)(t_i + t_j + c_1 L) \prod s_{\nu/t_i}(E)\Big)$$

Formulas via Schur functions

$\lambda = (\lambda_1, \dots, \lambda_d)$ partition $\rightsquigarrow$ $s_\lambda(t_1, \dots, t_d) = \det\left(t_j^{\lambda_i + d - i}\right) \Big/ \prod_{i<j}(t_i - t_j)$

Schur function

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Jacobi-Trudi $s_\lambda(t_1,\dots,t_d) = \det\left(s_{\lambda_i - i+j}(t_1,\dots,t_d)\right)$

E v.b. $\lambda \in \mathbb{Z}^d$, $s_\lambda(E) = \det\left((s_{\lambda_i - i+j})(E)\right)$.

<u>Prop</u> $E \to X$ v.b. of rank n. For $d=1,\dots,n-1$, and any partition λ

$$\int_{F(d)(E)}^{X} s_\lambda(\xi_1,\dots,\xi_d) = s_{\lambda-(n-d)^d}(E).$$

<u>Pf</u> To compute

$$[t_1^{e_1}\cdots t_d^{e_d}]\Big(\underbrace{s_\lambda(t_1,\dots,t_d)\prod_{i<j}(t_i-t_j)}_{\det(t_j^{\lambda_i+d-i})}\prod s_{1/t_i}(E)\Big)$$

$e_j = n-j$ $\quad (*) = \det\left([t_j^{e_j}]\left(t_j^{\lambda_i+d-i}\, s_{1/t_j}(E)\right)\right)$

$$= \det\left(s_{\lambda_i+d-i-e_j}(E)\right)$$

$\lambda_i + d - i - e_j = \lambda_i - (n-d) + (j-i)$. $\square$

quadratic Schur: $s_\lambda^{(2)}(E) = \det\left(s_{\lambda_i+\underline{2}(j-i)}(E)\right)$.

<u>Prop</u> $E \to X$ sympl. v.b. of rank $2n$. For any $d=1,\dots,n$ and any partition λ

$$\int_{F^\omega(d)(E)}^{X} s_\lambda(\xi_1,\dots,\xi_d) = s^{(2)}_{\lambda-\mu}(E),$$

where $\mu_i = 2n-d+1-i$.

(*) <u>Lemma</u> For any $f_{ij} \in A^\cdot X[t_j, t_j^{-1}]$, $1 \le i,j \le d$, and any exponents $e_1,\dots,e_d \in \mathbb{Z}$

$$[t_1^{e_1}\cdots t_d^{e_d}]\left(\det(f_{ij})\right) = \det\left([t_j^{e_j}](f_{ij})\right).$$

Kempf-Laksov bundles

$E \to X$ v.b. $\operatorname{rk} E = n$
(resp. $\operatorname{rk} E = 2n$)

$E_\bullet$ filtered v.b.
(resp. $E_\bullet$ flag of isotropic subbundles of E and their symplectic complements)
μ - strict partition with $\ell(\mu) = d \leq n$.

K-L bundle parametrizes (in both cases)

$$\{\underset{\text{v.b. on } X}{V_1 \subset \cdots \subset V_d} : \operatorname{rk} V_i = i,\ V_i \subset E_{\mu_{d+1-i}}\} \xrightarrow[\theta]{} X$$

Thm (D-P) $\theta_* f(\xi_1, \dots, \xi_d) =$

$$\left[\prod_i t_i^{\mu_i - 1}\right] \left(f(t_1, \dots, t_d) \prod_{i<j} (t_i - t_j) \overset{\text{sympl.}}{\prod_{\substack{i<j \\ \mu_i + \mu_j \geq 2n+1}} (c_1 L + t_i + t_j)} \prod s_{1/t_i}(E_{\mu_i}) \right)$$