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Universal Gysin formulas for flag bundles

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To the memory of Alain Lascoux

We give push-forward formulas for all flag bundles of types A, B, C, D. The formulas (and also the proofs) involve only Segre classes of the original vector bundles and characteristic classes of universal bundles. As an application, we provide new determinantal formulas.

Keywords: Push-forward; Segre classes; classical flag bundles; determinantal formulas.

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0. Introduction

A proper morphism $F\colon Y \to X$ of non-singular algebraic varieties over an algebraically closed field yields an additive map $F_*\colon A^\bullet Y \to A^\bullet X$ of Chow groups induced by push-forward cycles, often called the *Gysin map*. We will alternatively denote F_* by $\int_Y^X$. Note that the theory developed in Fulton's book [7] allows one to generalize the results of the present paper to singular varieties over a field and their Chow groups; moreover, for complex varieties, one can also use the cohomology rings with integral coefficients.

Push-forward formulas show how the classes of algebraic cycles on Y go via the Gysin map to classes of algebraic cycles on X. In the present paper, we are interested in push-forwards in flag bundles. We shall give formulas for the classical types A, B, C, D. These have a *universal* character in three aspects:

– these involve characteristic classes of universal vector bundles;
– these universally hold for *any* polynomial in such classes;
– these use in a universal way only the Segre classes of the original vector bundles.

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The starting point of our argument is a reformulation of the classical formula for push-forward of powers of the hyperplane class in a projective bundle, that we recall. Let $E \to X$ be a vector bundle of rank n on a variety X. Let $\mathbf{P}(E) \to X$ be the projective bundle of lines in E and let $\xi := c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ be the hyperplane class. For any i, the ith *Segre class of* E is

$$s_i(E) := \int_{\mathbf{P}(E)}^{X} \xi^{i+n-1}.$$

This is a definition in [7] and a lemma in preceding intersection theory (see e.g. [13, Lemma 1]).

Then, consider a polynomial $f(\xi) = \sum_i \alpha_i \xi^i$ with coefficients α_i in the Chow ring of X (here, we identify $A^\bullet X$ with a subring of $A^\bullet \mathbf{P}(E)$ and throughout the text, we will often omit pullback notation for vector bundles and algebraic cycles). Using the projection formula

$$\int_{\mathbf{P}(E)}^{X} f(\xi) = \sum_i \alpha_i s_{i-n+1}(E). \tag{1}$$

This formula can be transformed using the point of view of generating series. Before going further, let us introduce some notation. For a monomial m and a Laurent polynomial f, we will denote by $[m](f)$ the coefficient of m in f and we will call m the *extracted monomial*. It is clear that for any *shifting monomial* $\tilde{m}$

$$[\tilde{m}m](\tilde{m}f) = [m](f).$$

We will use this property repeatedly.

Coming back to formula (1), let

$$s_x(E) := 1 + xs_1(E) + \cdots + x^{\dim(X)} s_{\dim(X)}(E),$$

be the *Segre polynomial* of E. Thus, by definition

$$s_{i-n+1}(E) = [x^{i-n+1}](s_x(E)) = [x^{-n+1}](x^{-i}s_x(E)).$$

In order to have non-negative powers we will use the change of variables $t = 1/x$. We get

$$s_{i-n+1}(E) = [t^{n-1}](t^i s_{1/t}(E)).$$

Remark that in the latter expression, the extracted monomial does *not* depend on i, whence, by linearity, for any polynomial $f \in A^\bullet X[t]$, the push-forward formula (1) becomes

$$\int_{\mathbf{P}(E)}^{X} f(\xi) = [t^{n-1}]\left(\sum_i \alpha_i t^i s_{1/t}(E)\right) = [t^{n-1}](f(t)s_{1/t}(E)). \tag{2}$$

Notably, this expression (2) involves only the Segre polynomial, that behave better than individual Segre classes with respect to relations in the Grothendieck ring of the base variety. This will play a significant role in our induction strategy to generalize the fundamental formula (2) to all partial flag bundles of types A, B, C, D.

This is done in Theorems 1.1, 2.1 and 3.1. The looked at push-forwards are presented as suitable coefficients of some polynomials in the ring $A^\bullet X[t_1, \dots, t_d]$, where $\{t_i\}$ are some auxiliary variables and d is an integer determined by the flag data. The proof relies on the idea to iterate formula (2) on chains of projective bundles (for types B and D, we assume that there exist isotropic subbundles of maximal possible dimension, cf. [6, 8] for a discussion on that assumption). This allows us to give a formula for full flag bundles. The case of a general flag bundle is then obtained similarly as in Damon's paper [4], without using Grassmann bundles.

To give an insight, we give first three examples. All unexplained notation can be found in Secs. 1, 2 and 3. Let us however quickly mention that throughout this text, when we consider a certain flag bundle $F \to X$, the ξ_i are the Chern roots of (the dual of) the tautological subbundles on F; the letter f denotes a polynomial in the indicated number of variables with coefficients in $A^\bullet X$ (recall that we identify $A^\bullet X$ with subrings of the Chow rings of flag bundles); and by tacit assumption $f(\xi_1, \dots, \xi_d) \in A^\bullet F$, i.e. the polynomial f has the appropriate symmetries. These examples are:

– for the Grassmann bundle $\mathbf{F}(d)(E) \to X$ of a rank n vector bundle,

$$\int_{\mathbf{F}(d)(E)}^{X} f(\xi_1, \dots, \xi_d) = \left[\prod_{i=1}^{d} t_i^{(n-i)}\right] \left(f(t_1, \dots, t_d) \prod_{1 \le i < j \le d} (t_i - t_j) \prod_{1 \le i \le d} s_{1/t_i}(E) \right);$$

– for the complete flag bundle $\mathbf{F}(E) \to X$ of a rank n vector bundle,

$$\int_{\mathbf{F}(E)}^{X} f(\xi_1, \dots, \xi_{n-1}) = \left[\prod_{i=1}^{n-1} t_i^{(n-1)}\right] \left(f(t_1, \dots, t_{n-1}) \prod_{1 \le i < j \le n-1} (t_i - t_j) \prod_{1 \le i \le n-1} s_{1/t_i}(E) \right);$$

– for the symplectic Grassmann bundle $\mathbf{F}^\omega(d)(E) \to X$ of a rank $2n$ symplectic vector bundle (assuming here that the symplectic form ω has values in $\mathcal{O}_X$),

$$\int_{\mathbf{F}^\omega(d)(E)}^{X} f(\xi_1, \dots, \xi_d) = \left[\prod_{i=1}^{d} t_i^{(2n-i)}\right] \left(f(t_1, \dots, t_d) \prod_{1 \le i < j \le d} (t_i^2 - t_j^2) \prod_{1 \le i \le d} s_{1/t_i}(E) \right).$$

The reformulation (2) and the idea to iterate the obtained formula on chains (or "towers") of projective bundles originally appeared in the paper [5] by the first author. The idea of generalizing this formula to flag bundles was signaled by Bérczi, and it became clear that this suggestion was relevant as we recovered a formula for type A of Ilori from [12]. After the first version of the paper was completed, Manivel informed us that, independently, in their recent paper [14], Kaji and Terasoma prove a formula for type A (in the particular case of full flag bundles).

There exist various approaches to push-forward formulas for flag bundles:

— using Grothendieck residues (Akyildiz–Carrell [1], Damon [4], Quillen [22]);
— using localization and residues at infinity (Bérczi–Szenes [2], Tu [23], Zielenkiewicz [24, 25]);
— using symmetrizing operators (Brion [3], the second author, e.g. [18, 19] and Ratajski [21]);
— using Schur functions and Grassmann extensions (Józefiak, Lascoux and the second author [13]; for supersymmetric functions see [9, 18]);
— using residues and Grassmann extensions (Kazarian [15, 16]), which leads to formulas showing similarities with ours.

Also, in the recent paper [20] by the second author, a deformation of a push-forward formula from [13] is shown for Hall–Littlewood polynomials.

In the present paper, we use only elementary linear algebra of polynomials, whose coefficients yield the sought push-forwards. A use of Segre polynomials $s_x(E)$, specialized with $x = 1/t$, leads to remarkable compact expressions. Moreover, it allows us to generalize the formula for type A uniformly to other classical types B, C, D, for which, to the best of our knowledge, no general Gysin formula was known.

To give a first illustration of the usefulness of our formulas, in Sec. 4, we provide new determinantal formulas for the push-forward of monomials and of Schur classes. In a forthcoming paper, we will also apply our formulas in order to compute the fundamental classes of Schubert varieties.

1. Universal Push-Forward Formulas for Ordinary Flag Bundles

We first consider type A.

1.1. *Definition of partial flag bundles of type A*

Let $E \to X$ be a rank n vector bundle. Let $1 \leq d_1 < \cdots < d_m \leq n-1$ be a sequence of integers. We denote by $\pi\colon \mathbf{F}(d_1,\ldots,d_m)(E) \to X$ the bundle of flags of subspaces of dimensions $d_1,\ldots,d_m$ in E. On $\mathbf{F}(d_1,\ldots,d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of π^*E, where $\operatorname{rank}(U_{d_k}) = d_k$ (the fiber of U_{d_k} over the point $(V_{d_1} \subsetneq \cdots \subsetneq V_{d_m} \subset E_x)$, where $x \in X$, is equal to V_{d_k}). For a foundational account on flag bundles, see [11].

1.2. *Step-by-step construction of full flag bundles*

In order to use formula (2), we recall the construction of the flag bundles $\mathbf{F}(1,2,\ldots,d)(E)$ for $d = 1,2,\ldots,n-1$ as the chains of projective bundles of lines [10] (see also [9, §2.2]). We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = E,$$

such that for each i, the dimension of V_i is i. For V_1, we can take any line in E. It follows that $\mathbf{F}(1)(E) \simeq \mathbf{P}(E)$. Next, we consider $\mathbf{F}(1,2)(E) \to \mathbf{F}(1)(E)$ above V_1. In order to get a two-dimensional subspace $V_1 \subsetneq V_2 \subsetneq E$, it suffices to pick one more line in $\mathbf{P}(E/V_1)$. We iterate this construction,

$$\begin{array}{cc} \mathbf{F}(1,\ldots,i,i+1)(E) & \\ \downarrow & (\text{fiber} = \mathbf{P}(E/V_i)) \\ \mathbf{F}(1,\ldots,i)(E) & \end{array}$$

picking one line in $\mathbf{P}(E/V_i)$ at each step, until E/V_{n-1} is one-dimensional, so $\mathbf{P}(E/V_{n-1})$ is a point.

Globalizing this construction over X, we obtain a chain of projective bundles of lines

$$\mathbf{F}(E) \coloneqq \mathbf{F}(1,\ldots,n-1)(E) \to \mathbf{F}(1,\ldots,n-2)(E)$$
$$\to \cdots \to \mathbf{F}(1,2)(E) \to \mathbf{F}(1)(E) \to X,$$

which is the same as

$$\mathbf{F}(E) \coloneqq \mathbf{P}(E/U_{n-1}) \to \mathbf{P}(E/U_{n-2}) \to \cdots \to \mathbf{P}(E/U_1) \to \mathbf{P}(E) \to X.$$

In this paper, the flag bundles $\mathbf{F}(1,\ldots,d)(E)$, for $d = 1,\ldots,n-1$, are termed *full*, since these involve all first consecutive integers up to a certain d and we call *complete* flag bundle, denoted $\mathbf{F}(E)$, the full flag bundle if $d = n-1$. Note that this terminology may vary in the literature.

1.3. *Universal push-forward formula for flag bundles*

Let $E \to X$ be a vector bundle of rank n. Given a sequence of integers $0 = d_0 < d_1 < \cdots < d_m \le n-1$ as in Sec. 1.1, we set $d \coloneqq d_m$ and write for $k = 1,\ldots,m$

$$r_k \coloneqq d_k - d_{k-1} = \operatorname{rank}(U_{d_k}/U_{d_{k-1}}).$$

For $i = 1,\ldots,d$, we denote ξ_i the hyperplane class on $\mathbf{P}(E/U_{d-i})$

$$\xi_i \coloneqq -c_1(U_{d+1-i}/U_{d-i}).$$

The classes $\xi_1,\ldots,\xi_d$ are Chern roots of $U_d^\vee$ and generate the cohomology of the fibers of

$$\mathbf{F}(1,\ldots,d)(E) \overset{\overset{\xi_1}{\downarrow}}{\to} \mathbf{F}(1,\ldots,d-1)(E) \overset{\overset{\xi_2}{\downarrow}}{\to} \cdots\cdots \overset{\overset{\xi_{d-1}}{\downarrow}}{\to} \mathbf{F}(1)(E) \overset{\overset{\xi_d}{\downarrow}}{\to} X \tag{3}$$

in the above chain of projective bundles.

The following Gysin formula holds for the partial flag bundle $\mathbf{F}(d_1,\ldots,d_m)(E) \to X$.

Theorem 1.1. *For any rational equivalence class* $f(\xi_1, \dots, \xi_d) \in A^\bullet(\mathbf{F}(d_1, \dots, d_m)(E))$, *one has*

$$\int_{\mathbf{F}(d_1,\dots,d_m)(E)}^X f(\xi_1,\dots,\xi_d) = [t_1{}^{e_1} \cdots t_d{}^{e_d}] \left(f(t_1,\dots,t_d) \prod_{1\le i<j\le d} (t_i - t_j) \prod_{1\le i\le d} s_{1/t_i}(E) \right),$$

where for $j = d - d_k + i$ *with* $i = 1, \dots, r_k$, *we denote* $e_j \coloneqq n - i$.

Proof. The idea of the proof is to iterate formula (2) on the above chain of projective bundles of lines.

We first consider full flags. The case of $\mathbf{F}(1)(E) = \mathbf{P}(E) \to X$ was already treated in the introduction. We now generalize formula (2) to the case of full flag bundles $\mathbf{F}(1,\dots,d)(E) \to X$, by induction on $d \ge 1$.

By construction, the flag bundle $\mathbf{F}(1,\dots,d)(E)$ is the total space of the projective bundle of lines of the tautological quotient bundle $E/U_{d-1} \to \mathbf{F}(1,\dots,d-1)(E)$. This quotient bundle having rank $(n-d+1)$, a plain application of formula (2) yields

$$\int_{\mathbf{F}(1,\dots,d)(E)}^X f(\xi_1,\dots,\xi_d) = [t_1{}^{n-d}] \left(\int_{\mathbf{F}(1,\dots,d-1)(E)}^X f(t_1,\xi_2,\dots,\xi_d) s_{1/t_1}(E/U_{d-1}) \right).$$

In order to use induction, we need to express the total Segre class $s(E/U_{d-1})$ in terms of the remaining classes $\xi_2, \dots, \xi_d$, which is easily done by use of the Whitney sum formula for the following relation in the Grothendieck group of $\mathbf{F}(1,\dots,d-1)(E)$

$$[E/U_{d-1}] = [E] - ([U_{d-1}/U_{d-2}] + \cdots + [U_1/U_0])$$

which yields

$$s(E/U_{d-1}) = \prod_{j=2}^d (1-\xi_j) s(E).$$

The reformulation of this formula in terms of generating series is

$$s_{1/t_1}(E/U_{d-1}) = \prod_{j=2}^d \frac{(t_1 - \xi_j)}{t_1} s_{1/t_1}(E).$$

Hence, the above formula becomes

$$\int_{\mathbf{F}(1,\dots,d)(E)}^X f(\xi_1,\xi_2,\dots,\xi_d) = [t_1{}^{n-d}] \left(\int_{\mathbf{F}(1,\dots,d-1)(E)}^X f(t_1,\xi_2,\dots,\xi_d)(1/t_1)^{d-1} \prod_{1<j\le d} (t_1 - \xi_j) s_{1/t_1}(E) \right).$$

It is good to reshape this expression by multiplication of both the extracted monomial and the series by the shifting monomial ${t_1}^{d-1}$

$$\int_{\mathbf{F}(1,\dots,d)(E)}^{X} f(\xi_1,\xi_2,\dots,\xi_d) = [{t_1}^{n-1}]\left(\int_{\mathbf{F}(1,\dots,d-1)(E)}^{X} f(t_1,\xi_2,\dots,\xi_d)\prod_{1<j\le d}(t_1-\xi_j)s_{1/t_1}(E)\right).$$

After the first step, we have to integrate the polynomial in the variables $\xi_2,\dots,\xi_d$

$$f(t_1,\xi_2,\dots,\xi_d)\prod_{1<j\le d}(t_1-\xi_j)s_{1/t_1}(E)$$

along the fiber of the projective bundle of lines $\mathbf{F}(1,\dots,d-1)(E)\to\mathbf{F}(1,\dots,d-2)(E)$.

Iterating the same reasoning, until we have replaced all classes ξ_i by some formal variables t_i, we obtain the announced expression

$$\int_{\mathbf{F}(1,\dots,d)(E)}^{X} f(\xi_1,\dots,\xi_d) = [t_1^{n-1}\cdots t_d^{n-1}]\left(f(t_1,\dots,t_d)\prod_{1\le i<j\le d}(t_i-t_j)\prod_{1\le i\le d}s_{1/t_i}(E)\right).$$

A useful particular case

In the particular case of the complete flag bundle $\mathbf{F}(E)\to X$ of a rank r vector bundle $E\to X$ on a variety X, and for the polynomial

$$g_r(t_1,\dots,t_{r-1}) := {t_1}^1{t_2}^2\cdots{t_{r-1}}^{r-1},$$

the formula reads

$$\int_{\mathbf{F}(E)}^{X} g_r(\xi_1,\dots,\xi_{r-1}) = [X]. \tag{4}$$

Indeed, in the formula of Theorem 1.1,

$$\int_{\mathbf{F}(E)}^{X} g_r(\xi_1,\dots,\xi_{r-1}) = [t_1^{r-1}\cdots t_{r-1}^{r-1}]\left(\prod_{1\le i\le r-1}{t_i}^i\prod_{1\le i<j\le r-1}(t_i-t_j)\prod_{1\le i\le r-1}s_{1/t_i}(E)\right),$$

we have chosen the degree of g_r so that the homogeneous degree of the product of the first two factors is exactly the homogeneous degree of the extracted monomial. As a consequence, only the constant term of the third factor $\prod s_{1/t_i}(E)$ can contribute. This coefficient is $1=[X]\in A^0X$, whence

$$\int_{\mathbf{F}(E)}^{X} g_r(\xi_1,\dots,\xi_{r-1}) = [t_1^{r-1}\cdots t_{r-1}^{r-1}]\left(\prod_{1\le i\le r-1}{t_i}^i\prod_{1\le i<j\le r-1}(t_i-t_j)\right)[X].$$

For $r = 2$, the coefficient of $[X]$ is obviously 1. For $r > 2$, after shifting both the extracted monomial and the series by the monomial ${t_{r-1}}^{r-1}$, one is led to extract the part of homogeneous degree 0 with respect to t_{r-1}. It amounts to replacing $\prod_{i=1}^{r-2}(t_i - t_{r-1})$ by $\prod_{i=1}^{r-2} t_i$. Then, shifting by this monomial one gets

$$[t_1^{r-1} \cdots t_{r-1}^{r-1}] \left(\prod_{1 \le i \le r-1} {t_i}^i \prod_{1 \le i < j \le r-1} (t_i - t_j) \right)$$
$$= [t_1^{r-2} \cdots t_{r-2}^{r-2}] \left(\prod_{1 \le i \le r-2} {t_i}^i \prod_{1 \le i < j \le r-2} (t_i - t_j) \right).$$

This is the adequate induction formula.

From full flags to partial flags

We follow here [4]. Let $0 = d_0 < d_1 < \cdots < d_m = d$ be an increasing sequence of integers. Recall that on $F \coloneqq \mathbf{F}(d_1, \ldots, d_m)(E)$, there is the universal flag of vector bundles

$$0 \subsetneq U_{d_1} \subsetneq \cdots \subsetneq U_{d_m} \subsetneq E,$$

where $\operatorname{rank}(U_{d_i}) = d_i$. The fiber product

$$\mathbf{Y} \coloneqq \mathbf{F}(U_{d_1}) \times_F \mathbf{F}(U_{d_2}/U_{d_1}) \times_F \cdots \times_F \mathbf{F}(U_{d_m}/U_{d_{m-1}})$$

is isomorphic to $\mathbf{F}(1, \ldots, d)(E)$ with the natural projection map $\mathbf{F}(1, \ldots, d)(E) \to F$, and we get a commutative diagram

$$\begin{array}{ccc} \mathbf{F}(1, \ldots, d)(E) & \xrightarrow[\simeq]{\theta} & \mathbf{Y} \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi'} \\ X & \xleftarrow{\pi} & F \end{array} . \qquad (5)$$

The fiber of π' over the point $(V_{d_1} \subsetneq V_{d_2} \subsetneq \cdots \subsetneq V_{d_m} \subsetneq E_x) \in F$ is the product of complete flag varieties

$$\mathbf{F}(V_{d_1}) \times \mathbf{F}(V_{d_2}/V_{d_1}) \times \cdots \times \mathbf{F}(V_{d_m}/V_{d_{m-1}}).$$

For $k = 1, \ldots, m$ and for $i = 1, \ldots, r_k - 1$, let $\eta_{(d+1-d_k)+i}$ denote the pullback to $A^\bullet(\mathbf{F}(U_{d_k}/U_{d_{k-1}}))$ of the hyperplane classes of the projective bundles

$$\mathbf{F}(1, \ldots, r_k - i)(U_{d_k}/U_{d_{k-1}}) \to \mathbf{F}(1, \ldots, r_k - i - 1)(U_{d_k}/U_{d_{k-1}}),$$

cf. (3) with $E = U_{d_k}/U_{d_{k-1}}$ and $d = r_k - 1$.

Applying the splitting principle (cf. [10]) to each graded piece $U_{d_k}/U_{d_{k-1}} (k = 1, 2, \ldots, m)$ of the "universal" filtration, we infer that the pullbacks $\theta^*\eta_{d-d_k+2}, \ldots, \theta^*\eta_{d-d_{k-1}}$ of these classes are the corresponding classes $\xi_{d-d_k+2}, \ldots, \xi_{d-d_{k-1}}$ in $A^\bullet(\mathbf{F}(1, \ldots, d)(E))$. Note that we do not define classes η_{d-d_k+1}, but it will not play any role in the sequel of the proof.

Our goal is to compute the push-forward $\pi_* = \int_F^X$. Since π'_* is surjective (this follows from (4), see (6) below), it is enough to understand $\pi_* \circ \pi'_*$, which is equal to $\pi''_* \circ (\theta^{-1})_*$. But from the full flag case, we know π'_* and also π''_*, so we can proceed.

We are now in position to prove the generalization of (2) to the case of partial flag bundles. Let

$$g(t_1,\dots,t_d) \coloneqq \prod_{1\le k\le m}\prod_{1\le i\le r_k} {t_{(d-d_k)+i}}^{i-1} = \prod_{1\le k\le m} g_{r_k}(t_{d-d_k+2},\dots,t_{d-d_{k-1}}).$$

It follows from (4) that $\pi'_* g(\eta_1,\dots,\eta_d) = [F]$, whence, by the projection formula for π', we have

$$\pi'_*(f(\xi_1,\dots,\xi_d)g(\eta_1,\dots,\eta_d)) = f(\xi_1,\dots,\xi_d). \tag{6}$$

This implies

$$\pi_* f(\xi_1,\dots,\xi_d) = \pi_*\pi'_*(f(\xi_1,\dots,\xi_d)g(\eta_1,\dots,\eta_d)).$$

Now, from the commutativity of (5), we get

$$\begin{aligned}\pi_*\pi'_*(f(\xi_1,\dots,\xi_d)g(\eta_1,\dots,\eta_d)) &= \pi''_*(\theta^{-1})_*(f(\xi_1,\dots,\xi_d)g(\eta_1,\dots,\eta_d))\\ &= \pi''_*(f(\xi_1,\dots,\xi_d)g(\xi_1,\dots,\xi_d)).\end{aligned}$$

This last expression, using the formula for full flag bundles, is equal to

$$[t_1^{n-1}\cdots t_d^{n-1}]\left(f(t_1,\dots,t_d)g(t_1,\dots,t_d)\prod_{1\le i\le d} s_{1/t_i}(E)\prod_{1\le i<j\le d}(t_i-t_j)\right).$$

In order to get the announced formula, it suffices to shift both the series and the extracted monomial by the monomial $g(t_1,\dots,t_d)$. Since

$$\frac{t_1^{n-1}\cdots t_d^{n-1}}{g(t_1,\dots,t_d)} = \prod_{1\le k\le m}\prod_{1\le i\le r_k} {t_{(d-d_k)+i}}^{n-i} = \prod_{j=1}^{d} {t_j}^{e_j},$$

this concludes the proof. □

As noticed by Ilori [12, pp. 629–630], it can be useful to have a formula with all the Chern roots of E. In this context, we denote ξ_i the hyperplane class of $\mathbf{P}(E/U_{n-i})$, independently of d. One has

$$\xi_i = -c_1(U_{n-i+1}/U_{n-i}),$$

and the Chern roots of $U_d^\vee$ are now denoted $\xi_{n-d+1},\dots,\xi_n$.

Given a sequence of integers $0 = d_0 < d_1 < \cdots < d_m \le n-1$ as in Sec. 1.1, beside the above notation, we set $d_{m+1} \coloneqq n$ and accordingly $r_{m+1} \coloneqq n - d_m$.

Proposition 1.2. *With the above notation, one has*

$$\begin{aligned}&\int_{\mathbf{F}(d_1,\dots,d_m)(E)}^X f(\xi_1,\dots,\xi_n)\\ &\quad = [{t_1}^{e_1}\cdots {t_n}^{e_n}]\left(f(t_1,\dots,t_n)\prod_{1\le i<j\le n}(t_i-t_j)\prod_{1\le i\le n} s_{1/t_i}(E)\right),\end{aligned}$$

where for $j = n - d_k + i$ with $i = 1,\dots,r_k$, we denote $e_j \coloneqq n - i$.

Proof. Firstly, we treat the complete flag bundle $\mathbf{F}(E) \to X$. In the Grothendieck group of $\mathbf{F}(E)$,

$$[E/U_{n-1}] = [E] - \sum_{i=1}^{n-1} [U_i/U_{i-1}],$$

which yields

$$s(E/U_{n-1}) = \sum_{j \geq 0} \xi_1^j = \prod_{j=2}^{n} (1 - \xi_j) s(E).$$

As a consequence (we skip the detail)

$$\xi_1^j = [t_1^{n-1}] \left(t_1^j \prod_{1<j\leq n} (t_1 - \xi_j) s_{1/t_1}(E) \right).$$

By linearity, one gets a polynomial expression in the $n-1$ remaining classes $\xi_2, \dots, \xi_n$ — that are the Chern roots of the universal subbundle on $\mathbf{F}(E)$ —

$$f(\xi_1, \xi_2 \dots, \xi_n) = [t_1^{n-1}] \left(f(t_1, \xi_2, \dots, \xi_n) \prod_{1<j\leq n} (t_1 - \xi_j) s_{1/t_1}(E) \right),$$

and an application of the formula proved in Theorem 1.1 yields a similar formula with one more formal variable

$$\int_{\mathbf{F}(E)}^{X} f(\xi_1, \dots, \xi_n) = [t_1^{n-1} \cdots t_n^{n-1}] \left(f(t_1, \dots, t_n) \prod_{1\leq i\leq n} s_{1/t_i}(E) \prod_{1\leq i<j\leq n} (t_i - t_j) \right).$$

To get the formula for partial flag bundles, we adopt the same strategy as above, but consider the projection $\mathbf{F}(E) \to \mathbf{F}(d_1, \dots, d_m)(E)$. The fiber over a point $(V_{d_1} \subsetneq V_{d_2} \subsetneq \cdots \subsetneq V_{d_m} \subseteq E_x)$ is the product of complete flag varieties

$$\mathbf{F}(V_{d_1}) \times \mathbf{F}(V_{d_2}/V_{d_1}) \times \cdots \times \mathbf{F}(E_x/V_{d_m}).$$

One infers the stated formula, with shifted exponents. □

2. Universal Push-Forward Formulas for Symplectic Flag Bundles

Let now us deal with the symplectic setting. It can be regarded as the most simple case after the A case, as any line is isotropic for the symplectic form. This last fact will no longer be true in the orthogonal setting.

2.1. *Definition of partial isotropic flag bundles of type C*

Let $E \to X$ be a rank $2n$ vector bundle equipped with a non-degenerate symplectic form $\omega \colon E \otimes E \to L$ (with values in a certain line bundle $L \to X$). We say that a subbundle S of E is isotropic if S is a subbundle of its symplectic complement S^ω, where

$$S^\omega := \{w \in E \mid \forall\, v \in S \colon \omega(w, v) = 0\}.$$

The maximal rank of an isotropic subbundle is n.

Let $1 \leq d_1 < \cdots < d_m \leq n$ be a sequence of integers. We denote by $\pi\colon \mathbf{F}^\omega(d_1,\ldots,d_m)(E) \to X$ the bundle of flags of isotropic subspaces of dimensions $d_1,\ldots,d_m$ in E. On $\mathbf{F}^\omega(d_1,\ldots,d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of π^*E, where $\operatorname{rank}(U_{d_k}) = d_k$.

2.2. *Step-by-step construction of isotropic full flag bundles*

Let us recall the construction of $\mathbf{F}^\omega(1,2,\ldots,d)(E)$ for $d = 1,2,\ldots,n$ as chains of projective bundles of lines [9, §6.1]. We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq E,$$

of E such that for each i, the dimension of V_i is i and V_i is isotropic. In particular, $V_n = V_n^\omega$ is a maximal isotropic subspace of (E,ω). For V_1, we can take any line in E, since ω is skew-symmetric. It follows that $\mathbf{F}^\omega(1)(E) \simeq \mathbf{P}(E)$. Next, we consider $\mathbf{F}^\omega(1,2)(E) \to \mathbf{F}^\omega(1)(E)$ above V_1. In order to get an isotropic subspace $V_1 \subsetneq V_2 \subsetneq V_1^\omega$, it suffices to pick one more line in $\mathbf{P}(V_1^\omega/V_1)$. Iterating this construction,

$$\begin{array}{c} \mathbf{F}^\omega(1,\ldots,i,i+1)(E) \\ \downarrow \\ \mathbf{F}^\omega(1,\ldots,i)(E) \end{array} \quad (\text{fiber} = \mathbf{P}(V_i^\omega/V_i)),$$

picking one line in $\mathbf{P}(V_i^\omega/V_i)$ at each step, one ends up with V_n^ω/V_n, which is zero dimensional, thus obtaining a maximal isotropic subspace V_n.

Globalizing this construction over X, we obtain a chain of projective bundles of lines

$$\mathbf{F}^\omega(1,\ldots,n)(E) \to \mathbf{F}^\omega(1,\ldots,n-1)(E) \to \cdots \to \mathbf{F}^\omega(1,2)(E) \to \mathbf{F}^\omega(1)(E) \to X,$$

which is the same as

$$\mathbf{P}(U_{n-1}^\omega/U_{n-1}) \to \mathbf{P}(U_{n-2}^\omega/U_{n-2}) \to \cdots \to \mathbf{P}(U_1^\omega/U_1) \to \mathbf{P}(E) \to X.$$

2.3. *Useful relations in the Grothendieck group*

Since ω is everywhere non-degenerate, one can consider the isomorphism $\iota_\omega\colon E \ni v \mapsto \omega(v,\cdot) \in \operatorname{Hom}(E,L)$. Restricting the map $\omega(v,\cdot)$ to U_1, one obtains an isomorphism

$$\iota_\omega\colon E/U_1^\omega \ni v \mapsto \omega(v,\cdot) \in \operatorname{Hom}(U_1,L),$$

which yields the relation in the Grothendieck group of $\mathbf{F}^\omega(1)(E)$

$$[U_1^\omega] = [E] - [U_1^\vee \otimes L].$$

It follows that

$$[U_1^\omega/U_1] = [E] - [U_1] - [U_1^\vee \otimes L].$$

For $i = 2, \ldots, n$, doing all the same reasoning on U_{i-1}^{ω} instead of E, one obtains, in the Grothendieck group of $\mathbf{F}^{\omega}(1, \ldots, i)(E)$

$$[U_i^{\omega}/U_i] = [U_{i-1}^{\omega}/U_{i-1}] - [U_i/U_{i-1}] - [(U_i/U_{i-1})^{\vee} \otimes L].$$

Note that this formula also applies to $i = 1$ with the natural convention $U_0 = 0$. An induction yields

$$[U_j^{\omega}/U_j] = [E] - \sum_{i=1}^{j}([U_i/U_{i-1}] + [(U_i/U_{i-1})^{\vee} \otimes L]). \tag{7}$$

2.4. *Universal push-forward formula for the symplectic case*

Let $(E, \omega) \to X$ be a vector bundle having even rank $2n$, equipped with a everywhere non-degenerate symplectic form $\omega\colon E \otimes E \to L$, for a certain line bundle $L \to X$.

Given a sequence of integers $0 = d_0 < d_1 < \cdots < d_m \leq n$ as in Sec. 2.1, we set $d \coloneqq d_m$ and for $k = 1, \ldots, m$ we write $r_k \coloneqq d_k - d_{k-1}$. We also set for $i = 1, \ldots, d$

$$\xi_i \coloneqq -c_1(U_{d+1-i}/U_{d-i}).$$

The following Gysin formula holds for the isotropic partial flag bundle $\mathbf{F}^{\omega}(d_1, \ldots, d_m)(E) \to X$.

Theorem 2.1. *For any rational equivalence class* $f(\xi_1, \ldots, \xi_d) \in A^{\bullet}(\mathbf{F}^{\omega}(d_1, \ldots, d_m)(E))$, *one has*

$$\int_{\mathbf{F}^{\omega}(d_1,\ldots,d_m)(E)}^{X} f(\xi_1, \ldots, \xi_d)$$
$$= [{t_1}^{e_1} \cdots {t_d}^{e_d}] \left(f(t_1, \ldots, t_d) \prod_{1 \leq i < j \leq d} (t_i - t_j)(t_i + t_j + c_1(L)) \prod_{1 \leq i \leq d} s_{1/t_i}(E) \right),$$

where for $j = d - d_k + i$ *with* $i = 1, \ldots, r_k$, *we denote* $e_j \coloneqq 2n - i$.

Proof. We prove the formula for type C in the very same way as we already did for type A.

We first consider full flags. We will prove the formula for $\mathbf{F}^{\omega}(1, \ldots, d)(E) \to X$ by induction on $d = 1, \ldots, n$. Note that for $d = 1$ the sought formula is exactly the same as formula (2). Thus, the result holds. Then for $1 < d \leq n$, we consider the projection $\mathbf{F}^{\omega}(1, \ldots, d)(E) \to \mathbf{F}^{\omega}(1, \ldots, d-1)(E)$. Using the formula (2), one states

$$\int_{\mathbf{F}^{\omega}(1,\ldots,d)(E)}^{\mathbf{F}^{\omega}(1,\ldots,d-1)(E)} f(\xi_1, \ldots, \xi_d) = [t_1^{2(n-d)+1}](f(t_1, \xi_2, \ldots, \xi_d) s_{1/t_1}(U_{d-1}^{\omega}/U_{d-1})).$$

In order to use induction, it remains to express the total Segre class $s(U_{d-1}^{\omega}/U_{d-1})$ as a polynomial in $\xi_2, \ldots, \xi_d$. In that aim, we use the relation (7) in the Grothendieck group of $\mathbf{F}^{\omega}(1,\ldots,d)(E)$

$$[U^{\omega}_{d-1}/U_{d-1}] = [E] - \sum_{j=2}^{d}([U_{d+1-j}/U_{d-j}] + [(U_{d+1-j}/U_{d-j})^{\vee} \otimes L]),$$

which yields

$$s_{1/t_1}(U^{\omega}_{d-1}/U_{d-1}) = \prod_{j=2}^{d}(t_1 - \xi_j)(t_1 + \xi_j + c_1(L))s_{1/t_1}(E)(1/t_1)^{2(d-1)}.$$

Using this expression of the Segre class in the above formula, and shifting by $t_1^{2(d-1)}$, one gets

$$\int_{\mathbf{F}^{\omega}(1,\ldots,d)(E)}^{\mathbf{F}^{\omega}(1,\ldots,d-1)(E)} f(\xi_1,\ldots,\xi_d)$$
$$= [t_1^{2n-1}]\left(f(t_1,\xi_2,\ldots,\xi_d)\prod_{1<j\leq d}(t_1 - \xi_j)(t_1 + \xi_j + c_1(L))s_{1/t_1}(E)\right).$$

One infers, using induction, that

$$\int_{\mathbf{F}^{\omega}(1,\ldots,d)(E)}^{X} f(\xi_1,\ldots,\xi_d)$$
$$= [t_1^{2n-1}\cdots t_d^{2n-1}]\left(f(t_1,\ldots,t_d)\prod_{1\leq i<j\leq d}(t_i - t_j)(t_i + t_j + c_1(L))\prod_{1\leq j\leq d}s_{1/t_j}(E)\right).$$

From full flags to partial flags

Clearly, any flag inside an isotropic subbundle is an isotropic flag. Let $0 = d_0 < d_1 < \cdots < d_m = d$ be an increasing sequence of integers. Recall that on $F \coloneqq \mathbf{F}^{\omega}(d_1,\ldots,d_m)(E)$, there is the universal flag of vector bundles

$$0 \subsetneq U_{d_1} \subsetneq \cdots \subsetneq U_{d_m} \subsetneq E,$$

where $\operatorname{rank}(U_{d_k}) = d_k$. The fiber product

$$\mathbf{Y} \coloneqq \mathbf{F}(U_{d_1}) \times_F \mathbf{F}(U_{d_2}/U_{d_1}) \times_F \cdots \times_F \mathbf{F}(U_{d_m}/U_{d_{m-1}})$$

is isomorphic to $\mathbf{F}^{\omega}(1,\ldots,d)(E)$ with the natural projection map $\mathbf{F}^{\omega}(1,\ldots,d)(E) \to F$, and we get a commutative diagram

$$\begin{array}{ccc} \mathbf{F}^{\omega}(1,\ldots,d)(E) & \xrightarrow[\simeq]{\theta} & \mathbf{Y} \\ \big\downarrow{\scriptstyle \pi''} & & \big\downarrow{\scriptstyle \pi'} \\ X & \xleftarrow{\pi} & F \end{array}.$$

The arguments developed in case A still apply, resulting in the same shift of the extracted monomial in the general formula. This concludes the proof. $\square$

3. Universal Push-Forward Formulas for Orthogonal Flag Bundles

We will need only few modifications to deal with the orthogonal setting. We will do almost the same reasoning, replacing the projective bundle of lines $\mathbf{P}(E)$ by the quadric bundle of isotropic lines $\mathbf{Q}(E)$.

3.1. *Definition of partial isotropic flag bundles of types B and D*

Let $E \to X$ be a vector bundle of rank $2n$ or $2n+1$ equipped with a non-degenerate orthogonal form $Q\colon E \otimes E \to L$ (with values in a certain line bundle $L \to X$). We say that a subbundle S of E is isotropic if S is a subbundle of its orthogonal complement $S^\perp$, where

$$S^\perp \coloneqq \{w \in E \mid \forall v \in S\colon Q(w,v) = 0\}.$$

The maximal rank of an isotropic subbundle is n.

Let $1 \leq d_1 < \cdots < d_m \leq n$ be a sequence of integers. We denote by $\pi\colon \mathbf{F}^Q(d_1, \ldots, d_m)(E) \to X$ the bundle of flags of isotropic subspaces of dimensions $d_1, \ldots, d_m$ in E. On $\mathbf{F}^Q(d_1, \ldots, d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of π^*E, where $\operatorname{rank}(U_{d_k}) = d_k$.

3.2. *Step-by-step construction of isotropic full flag bundles*

Let us recall the construction of $\mathbf{F}^Q(1, 2, \ldots, d)(E)$ for $d = 1, 2, \ldots, n$ as chains of quadric bundles of isotropic lines [6, §6]. We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq E$$

such that for each i, the dimension of V_i is i and V_i is isotropic. In particular, if the dimension of E is even, $V_n = V_n^\perp$. For V_1, we can take any isotropic line in E. Thus, by definition $\mathbf{F}^Q(1)(E) \simeq \mathbf{Q}(E)$, the codimension one subbundle of $\mathbf{P}(E)$ cut out by Q. Next, we consider $\mathbf{F}^Q(1,2)(E) \to \mathbf{F}^Q(1)(E)$ above V_1. In order to get an isotropic subspace $V_1 \subsetneq V_2 \subsetneq V_1^\perp$, it suffices to pick one more line in $\mathbf{Q}(V_1^\perp/V_1)$. Iterating this construction,

$$\begin{array}{c} \mathbf{F}^Q(1, \ldots, i, i+1)(E) \\ \downarrow \\ \mathbf{F}^Q(1, \ldots, i)(E) \end{array} \qquad (\text{fiber} = \mathbf{Q}(V_i^\perp/V_i))$$

picking one isotropic line in $\mathbf{Q}(V_i^\perp/V_i)$ at each step, one ends up with $V_n^\perp/V_n$, which is either zero dimensional or one dimensional. In the first case, when the rank of E is even, it is usual to take only one of the two connected components, but we will not, as we do not want to treat this case separately.

Globalizing this construction over X, we obtain a chain of quadric bundles of isotropic lines

$$\mathbf{F}^Q(1, \ldots, n)(E) \to \mathbf{F}^Q(1, \ldots, n-1)(E) \to \cdots \to \mathbf{F}^Q(1,2)(E) \to \mathbf{F}^Q(1)(E) \to X,$$

which is the same as

$$\mathbf{Q}(U_{n-1}^{\perp}/U_{n-1}) \to \mathbf{Q}(U_{n-2}^{\perp}/U_{n-2}) \to \cdots \to \mathbf{Q}(U_1^{\perp}/U_1) \to \mathbf{Q}(E) \to X.$$

3.3. *A push-forward formula for the quadric bundle*

The quadric bundle $\iota\colon \mathbf{Q}(E) \hookrightarrow \mathbf{P}(E)$ of isotropic lines of E, cut out by the quadratic form $Q\colon E \otimes E \to L$, is the zero set of a section of the line bundle

$$\mathrm{Hom}(\mathcal{O}_{\mathbf{P}(E)}(-1) \otimes \mathcal{O}_{\mathbf{P}(E)}(-1), L) \simeq \mathcal{O}_{\mathbf{P}(E)}(2) \otimes L.$$

Its fundamental class in $\mathbf{P}(E)$ is thus $[\mathbf{Q}(E)] = 2\xi + c_1(L)$. Let $\pi\colon \mathbf{P}(E) \to X$ be the natural projection, and $\rho = \pi \circ \iota$. Let $\tilde{\xi} := \iota^*\xi$ denote the restriction of $\xi = c_1(\mathcal{O}_{P(E)}(1))$. One has

$$\begin{aligned}\rho_*(\tilde{\xi}^i \rho^*\alpha) &= \pi_*([\mathbf{Q}(E)] \cdot \xi^i \pi^*\alpha)\\ &= \pi_*((2\xi + c_1(L))\xi^i \pi^*\alpha) = (2s_{i-r+2}(E) + c_1(L)s_{i-r+1}(E))\alpha,\end{aligned}$$

where $r = \mathrm{rank}(E)$. Similarly as we get (2) from (1), we now infer

$$\rho_* f(\tilde{\xi}) = [t^{r-1}](f(t)(2t + c_1(L))s_{1/t}(E)). \tag{8}$$

Since we are working with towers of quadric bundles in the orthogonal case, this formula will play the analogous role as formula (2) in this section.

3.4. *Universal push-forward formula for orthogonal flag bundles*

Let $(E, Q) \to X$ be a vector bundle of rank $2n$ or $2n+1$, equipped with a everywhere non-degenerate quadratic form $Q\colon E \otimes E \to L$, for a certain line bundle $L \to X$.

Given a sequence of integers $0 = d_0 < d_1 < \cdots < d_m \leq n$ as in Sec. 3.1, we set $d := d_m$ and for $k = 1, \ldots, m$ we write $r_k := d_k - d_{k-1}$. We also set for $i = 1, \ldots, d$

$$\xi_i := -c_1(U_{d+1-i}/U_{d-i}).$$

The following Gysin formula holds for the isotropic partial flag bundle $\mathbf{F}^Q(d_1, \ldots, d_m)(E) \to X$.

Theorem 3.1. *For any rational equivalence class* $f(\xi_1, \ldots, \xi_d) \in A^\bullet(\mathbf{F}^Q(d_1, \ldots, d_m)(E))$, *one has*

$$\int_{\mathbf{F}^Q(d_1,\ldots,d_m)(E)}^X f(\xi_1, \ldots, \xi_d) = [{t_1}^{e_1} \cdots {t_d}^{e_d}] \Bigg(f(t_1, \ldots, t_d) \prod_{1 \leq i \leq d} (2t_i + c_1(L)) \times \prod_{1 \leq i < j \leq d} (t_i - t_j)(t_i + t_j + c_1(L)) \prod_{1 \leq i \leq d} s_{1/t_i}(E) \Bigg),$$

where for $j = d - d_k + i$ *with* $i = 1, \ldots, r_k$, *we denote* $e_j := \mathrm{rank}(E) - i$.

Note that, if the rank is $2n$ and $d = n$, we consider *both* of the two isomorphic connected components of the flag bundle. Thus, if one is interested in only one of the two components, the result should be divided by 2. When $c_1(L) = 0$, this makes appear the usual coefficient 2^{n-1}.

Proof. We first prove the formula for full flags. Since the quadratic form Q is everywhere non-degenerate, one can consider the isomorphism $\iota_Q \colon E \ni v \mapsto Q(v, \cdot) \in \mathrm{Hom}(E, L)$. As in the symplectic case, this isomorphism yields the relation in the Grothendieck group of $\mathbf{F}^Q(1, \ldots, j)(E)$

$$[U_j^{\perp}/U_j] = [E] - \sum_{i=1}^{j}([U_i/U_{i-1}] + [(U_i/U_{i-1})^{\vee} \otimes L]).$$

The proof goes in the very same way as for type C, replacing formula (2) for $\mathbf{P}(E)$ by formula (8) for $\mathbf{Q}(E)$. So we skip the details.

Then, to go from full flags to partial flags, the argument is the same as in the symplectic case. Clearly, any flag inside an isotropic subbundle is an isotropic flag. It yields the expected formulas with shifted exponents. □

Note that in the basic case where the quadratic form Q takes values in the trivial line bundle $L = \mathcal{O}_X$ the theorem has a simpler form, and reads as follows.

Corollary 3.2. *When $L = \mathcal{O}_X$, for any rational equivalence class $f(\xi_1, \ldots, \xi_d) \in A^\bullet(\mathbf{F}^Q(d_1, \ldots, d_m)(E))$, one has*

$$\int_{\mathbf{F}^Q(d_1,\ldots,d_m)(E)}^{X} f(\xi_1, \ldots, \xi_d)$$
$$= 2^d [{t_1}^{e_1} \cdots {t_d}^{e_d}] \left(f(t_1, \ldots, t_d) \prod_{1 \le i < j \le d} (t_i^2 - t_j^2) \prod_{1 \le i \le d} s_{1/t_i}(E) \right),$$

where for $j = d - d_k + i$ with $i = 1, \ldots, r_k$, we denote $e_j := \mathrm{rank}(E) - 1 - i$.

4. Applications: New Determinantal Formulas

To finish, we would like to compute new determinantal formulas, for types A, B, C, D, in order to illustrate the usefulness and the efficiency of our approach.

We shall use the following linearity result, whose proof is left to the reader.

Lemma 4.1. *For any $f_{ij} \in A^\bullet X[t_j, {t_j}^{-1}]$ where $1 \le i, j \le d$, and for any exponents $e_1, \ldots, e_d \in \mathbb{Z}$*

$$[{t_1}^{e_1} \cdots {t_d}^{e_d}](\det(f_{ij})) = \det([t_j^{e_j}](f_{ij})).$$

4.1. *Schur functions*

For $r \ge 0$ and for any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, recall ([7, 17]) that the Schur polynomial $s_\lambda \in \mathbb{Z}[t_1, \ldots, t_r]$ can be defined by the formula:

$$s_\lambda(t_1, \ldots, t_r) := \det({t_j}^{\lambda_i + r - i})_{1 \le i,j \le r} \Big/ \prod\nolimits_{1 \le i < j \le r}(t_i - t_j). \tag{9}$$

Note that in particular for $\lambda = (i)$, one has $s_i = h_i$ the complete symmetric function of degree i.

For a vector bundle $E \to X$ of rank r over a variety, let $\xi_1, \ldots, \xi_r$ denote the Chern roots of $E^\vee$, the dual of E, it is well known that (under the monomorphism

$A^\bullet(X) \hookrightarrow A^\bullet(\mathbf{F}(E))$):

$$s_i(E) = s_i(\xi_1, \ldots, \xi_r).$$

Then the Jacobi–Trudi identity states

$$s_\lambda(t_1, \ldots, t_r) = \det(s_{\lambda_i - i + j}(t_1, \ldots, t_r))_{1 \leq i,j \leq r},$$

where $s_i = 0$ for $i < 0$. This identity allows one to generalize the Segre classes of E for any sequence of integers $\lambda \in \mathbb{Z}^r$ in a natural way, by setting

$$s_\lambda(E) \coloneqq \det(s_{\lambda_i + (j-i)}(E))_{1 \leq i,j \leq r}. \tag{10}$$

Note that if the partition λ has (at most) d parts, for some $d \leq r$:

$$s_\lambda(E) \coloneqq \det(s_{\lambda_i + (j-i)}(E))_{1 \leq i,j \leq r} = \det(s_{\lambda_i + (j-i)}(E))_{1 \leq i,j \leq d}.$$

In what follows, the difference of two sequences in $\mathbb{Z}^d$ is defined componentwise.

4.2. *Formulas using monomials*

These Schur functions naturally appear for type A when one uses the additive basis of monomials, as illustrated below.

Let $E \to X$ be a rank n vector bundle. Recall that on the partial flag bundle $\mathbf{F}(d_1, \ldots, d_m)(E)$, we denote $d \coloneqq d_m$ and $\xi_1, \ldots, \xi_d$ the Chern roots of $(U_d)^\vee$.

Proposition 4.2. *A rational equivalence class on the partial flag bundle* $\mathbf{F}(d_1, \ldots, d_m)(E) \to X$

$$f(\xi_1, \ldots, \xi_d) \coloneqq \sum_{\lambda \in (\mathbb{Z}_{\geq 0})^d} \alpha_\lambda {\xi_1}^{\lambda_1} \cdots {\xi_d}^{\lambda_d}$$

goes via the Gysin map to

$$\int_{\mathbf{F}(d_1,\ldots,d_m)(E)}^X f(\xi_1, \ldots, \xi_d) = \sum_{\lambda \in (\mathbb{Z}_{\geq 0})^d} \alpha_\lambda s_{(\lambda - \nu)^\leftarrow}(E),$$

where for $K = (k_1, \ldots, k_d)$, *one denotes* $K^\leftarrow \coloneqq (k_d, \ldots, k_1)$ *and where* ν *is an increasing sequence of integers determined by the flag data, viz.*

$$\nu_i \coloneqq n - d_k \quad \textit{for } d - d_k < i \leq d - d_{k-1}.$$

Proof. The push-forward formula for type A yields

$$\int_{\mathbf{F}(d_1,\ldots,d_m)(E)}^X f(\xi_1, \ldots, \xi_d)$$
$$= [{t_1}^{e_1} \cdots {t_d}^{e_d}] \left(\sum \alpha_\lambda t_1^{\lambda_1} \cdots t_d^{\lambda_d} \prod_{1 \leq i < j \leq d} (t_i - t_j) \prod_{1 \leq j \leq d} s_{1/t_j}(E) \right),$$

where $e_j = n + d - d_k - j$, for $d - d_k < j \leq d - d_{k-1}$.

Now, it is well known (and due to Vandermonde) that $\prod_{1\leq i<j\leq d}(t_i - t_j) = \det(t_j^{d-i})_{1\leq i,j\leq d}$, and then by Lemma 4.1

$$\int_{\mathbf{F}(d_1,\ldots,d_m)(E)}^{X} f(\xi_1,\ldots,\xi_d) = \sum \alpha_\lambda \det([t_j^{e_j}](t_j^{d-i+\lambda_j} s_{1/t_j}(E)))_{1\leq i,j\leq d}$$

$$= \sum \alpha_\lambda \det(s_{\lambda_j+d-i-e_j}(E))_{1\leq i,j\leq d}.$$

For $d - d_k < j \leq d - d_{k-1}$, one has

$$\lambda_j + d - i - e_j = \lambda_j - (n - d_k) + j - i = \lambda_j - \nu_j + j - i.$$

In order to get the result, first transpose the determinant ($i \leftrightarrow j$), and then permute respectively the rows and the columns using $i \leftarrow (d+1-i)$ and $j \leftarrow (d+1-j)$. □

In the symplectic case and in the orthogonal case, we will now assume that $L = \mathcal{O}_X$ is the trivial bundle. In this case

$$\prod_{1\leq i<j\leq d}(t_i - t_j)(t_i + t_j + c_1(L)) = \prod_{1\leq i<j\leq d}(t_i^2 - t_j^2) = \det(t_j^{2(d-i)})_{1\leq i,j\leq d}.$$

Hence, the proof of Proposition 4.2 is easily adapted in order to obtain determinantal formulas for types B, C, D. We first introduce the classes

$$s_\lambda^{(2)}(E) := \det(s_{\lambda_i+2(j-i)}(E))_{1\leq i,j\leq d}, \tag{11}$$

for all sequences of integers $\lambda \in \mathbb{Z}^d$. It seems adequate to call them *quadratic Schur functions* (compare with (10)). They are closely related to the classes $s_\lambda^{[2]}(E)$ defined in [21], as it will soon appear (see Sec. 4.3).

Let $E \to X$ be a symplectic vector bundle of rank $2n$, equipped with the symplectic form $\omega\colon E \otimes E \to \mathcal{O}_X$.

Proposition 4.3. *A rational equivalence class on the isotropic flag bundle* $\mathbf{F}^\omega(d_1, \ldots, d_m)(E) \to X$

$$f(\xi_1,\ldots,\xi_d) := \sum_{\lambda\in(\mathbb{Z}_{\geq 0})^d} \alpha_\lambda \xi_1^{\lambda_1}\cdots\xi_d^{\lambda_d}$$

goes via the Gysin map to

$$\int_{\mathbf{F}^\omega(d_1,\ldots,d_m)(E)}^{X} f(\xi_1,\ldots,\xi_d) = \sum_{\lambda\in(\mathbb{Z}_{\geq 0})^d} \alpha_\lambda s^{(2)}_{(\lambda-\nu)^{\leftarrow}}(E),$$

where ν *is an increasing sequence of integers determined by the flag data, viz.*

$$\nu_i = 2n - d - d_k + i \quad \textit{for } d - d_k < i \leq d - d_{k-1}.$$

Let $E \to X$ be an orthogonal vector bundle, equipped with the quadratic form $Q\colon E \otimes E \to \mathcal{O}_X$.

Proposition 4.4. *A rational equivalence class on the isotropic flag bundle* $\mathbf{F}^Q(d_1, \ldots, d_m)(E) \to X$

$$f(\xi_1, \ldots, \xi_d) \coloneqq \sum_{\lambda \in (\mathbb{Z}_{\geq 0})^d} \alpha_\lambda \xi_1^{\lambda_1} \cdots \xi_d^{\lambda_d}$$

goes via the Gysin map to

$$\int_{\mathbf{F}^Q(d_1,\ldots,d_m)(E)}^X f(\xi_1, \ldots, \xi_d) = 2^d \sum_{\lambda \in (\mathbb{Z}_{\geq 0})^d} \alpha_\lambda s^{(2)}_{(\lambda-\nu)^{\leftarrow}}(E),$$

where ν *is an increasing sequence of integers determined by the flag data, viz.*

$$\nu_i = (\mathrm{rank}(E) - 1) - d - d_k + i \quad \text{for } d - d_k < i \leq d - d_{k-1}.$$

4.3. *Formulas using Schur functions*

We now focus on the Grassmann bundles and use the additive basis of Schur functions. We start with type A, and show an alternative deduction of [13, Corollary 1].

Proposition 4.5. *Let* $E \to X$ *be a rank* n *vector bundle. For* $d = 1, \ldots, n-1$, *for any* $\lambda \in \mathbb{Z}^d$, *one has the Gysin formula*

$$\int_{\mathbf{F}(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d) = s_{\lambda-(n-d)^d}(E).$$

Proof. The push-forward formula for the polynomial $s_\lambda(\xi_1, \ldots, \xi_d)$ is

$$\int_{\mathbf{F}(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d)$$

$$= [{t_1}^{e_1} \cdots {t_d}^{e_d}] \left(s_\lambda(t_1, \ldots, t_d) \prod_{1 \leq i < j \leq d} (t_i - t_j) \prod_{1 \leq j \leq d} s_{1/t_j}(E) \right),$$

where for $j = 1, \ldots, d$, the exponents are $e_j = n - j$.

If for some $i = 1, \ldots, d$, one has $\lambda_i < i - d$, then for $j = 1, \ldots, d$ one has $s_{\lambda_i + (j-i)} = 0$ and accordingly $s_\lambda = 0$. One checks that the stated formula becomes $\int_{\mathbf{F}(d)(E)}^X 0 = 0$. So we can assume that definition (9) holds:

$$s_\lambda(t_1, \ldots, t_d) \prod_{1 \leq i < j \leq d} (t_i - t_j) = \det({t_j}^{\lambda_i + d - i})_{1 \leq i,j \leq d},$$

and then by linearity of the determinant with respect to the columns and by Lemma 4.1

$$\int_{\mathbf{F}(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d) = \det([{t_j}^{e_j}]({t_j}^{\lambda_i + d - i} s_{1/t_j}(E)))_{1 \leq i,j \leq d}$$

$$= \det(s_{\lambda_i + d - i - e_j}(E))_{1 \leq i,j \leq d},$$

which is the announced Schur function since

$$\lambda_i + d - i - e_j = \lambda_i - (n-d) + (j-i). \qquad \square$$

We now treat types B, C, D. Here, the quadratic Schur functions $s_\lambda^{(2)}$ will appear again. The comparison with the results of [21] (for the case of maximal rank $d = n$; see also [24]), reveals a deep connection between the Schur type functions $s_\lambda^{(2)}(E)$ of this work and the Schur type functions $s_\lambda^{[2]}(E)$ of [21], that should probably be investigated.

But first, we need to prove the following combinatorial result.

Lemma 4.6. *For $e \in \mathbb{Z}$ and $\Lambda(t_1, \ldots, t_d)$ antisymmetric, one has*

$$\left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(\Lambda(t_1,\ldots,t_d)\prod_{1\le i<j\le d}(t_i+t_j)\right) = \left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(\Lambda(t_1,\ldots,t_d)\prod_{j=1}^{d} t_j^{j-1}\right).$$

Proof. We prove this formula by induction on d. Denote by $(i,j) \prec (i_0, j_0)$ the set of pairs of integers $i < j$ such that (i,j) is smaller than (i_0, j_0) in inverse lexicographic order. Notice that

$$\prod_{1\le i<j\le d}(t_i+t_j) = \sum_{p=1}^{d-1} t_d^{p-1} t_{d-p} \prod_{(i,j)\prec(d-p,d)}(t_i+t_j) + t_d^{d-1}\prod_{1\le i<j\le d-1}(t_i+t_j).$$

Hence:

$$\begin{aligned}&\left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(\Lambda(t_1,\ldots,t_d)\prod_{1\le i<j\le d}(t_i+t_j)\right)\\ &\quad= \sum_{p=1}^{d-1}\left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(t_d^{p-1}t_{d-p}\Lambda(t_1,\ldots,t_d)\prod_{(i,j)\prec(d-p,d)}(t_i+t_j)\right)\\ &\qquad+\left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(t_d^{d-1}\Lambda(t_1,\ldots,t_d)\prod_{1\le i<j\le d-1}(t_i+t_j)\right).\end{aligned}$$

All the $d-1$ first terms in the sum are $= 0$, since after shifting by t_{d-p}, one takes the coefficient of a monomial symmetric in t_{d-p} and t_{d-p+1} in an antisymmetric function in t_{d-p} and t_{d-p+1}. Thus

$$\begin{aligned}&\left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(\Lambda(t_1,\ldots,t_d)\prod_{1\le i<j\le d}(t_i+t_j)\right)\\ &\quad= \left[\prod_{j=1}^{d} t_j^{e-j}\right]\left(t_d^{d-1}\Lambda(t_1,\ldots,t_d)\prod_{1\le i<j\le d-1}(t_i+t_j)\right).\end{aligned}$$

One concludes by induction on d, by considering $[t_d^{e-d}](t_d^{d-1}\Lambda(t_1,\ldots,t_d))$ as an antisymmetric function in $t_1, \ldots, t_{d-1}$. $\square$

We can now proceed.

Proposition 4.7. *Let $E \to X$ be a symplectic vector bundle of rank $2n$, equipped with the symplectic form $\omega\colon E \otimes E \to \mathcal{O}_X$. For $d = 1, \ldots, n$, for any $\lambda \in \mathbb{Z}^d$, one has the Gysin formula*

$$\int_{\mathbf{F}^\omega(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d) = s^{(2)}_{\lambda-\mu}(E),$$

where $\mu_i = 2n - d + 1 - i$ for $i = 1, \ldots, d$.

Proof. The push-forward formula for the polynomial $s_\lambda(\xi_1, \ldots, \xi_d)$ is

$$\int_{\mathbf{F}^\omega(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d)$$

$$= [t_1{}^{e_1} \cdots t_d{}^{e_d}] \left(s_\lambda(t_1, \ldots, t_d) \prod_{1\le i<j\le d} (t_i - t_j) \prod_{1\le i<j\le d} (t_i + t_j) \prod_{1\le j\le d} s_{1/t_j}(E) \right),$$

where for $j = 1, \ldots, d$, the exponents are $e_j \coloneqq 2n - j$.

If for some $i = 1, \ldots, d$, one has $\lambda_i < i - d$, then one checks that the stated formula becomes $\int_{\mathbf{F}^\omega(d)(E)}^X 0 = 0$. So we can assume that definition (9) holds:

$$s_\lambda(t_1, \ldots, t_d) \prod_{1\le i<j\le d} (t_i - t_j) = \det(t_j{}^{\lambda_i+d-i})_{1\le i,j\le d},$$

and then by linearity of the determinant with respect to the columns

$$\int_{\mathbf{F}^\omega(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d)$$

$$= [t_1{}^{e_1} \cdots t_d{}^{e_d}] \left(\det(t_j{}^{\lambda_i+d-i} s_{1/t_j}(E))_{1\le i,j\le d} \prod_{1\le i<j\le d} (t_i + t_j) \right),$$

which according to Lemma 4.6 is the same as

$$\int_{\mathbf{F}^\omega(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d) = [t_1{}^{e_1} \cdots t_d{}^{e_d}] \left(\det(t_j{}^{\lambda_i+d-i} s_{1/t_j}(E))_{1\le i,j\le d} \prod_{1\le j\le d} t_j^{j-1} \right).$$

Thus, using again the linearity with respect to the columns and Lemma 4.1

$$\int_{\mathbf{F}^\omega(d)(E)}^X s_\lambda(\xi_1, \ldots, \xi_d) = \det([t_j{}^{e_j}](t_j{}^{\lambda_i+d-i+j-1} s_{1/t_j}(E)))_{1\le i,j\le d}$$

$$= \det(s_{\lambda_i+d-i+j-1-e_j}(E))_{1\le i,j\le d},$$

which is the announced quadratic Schur function since

$$\lambda_i + d - i + j - 1 - e_j = \lambda_i - (2n - d + 1 - i) + 2(j - i) = \lambda_i - \mu_i + 2(j - i).$$

□

The partition μ is the maximal strict partition in $(2n - d)^d$ and in the case where $d = n$, it is $\rho = (d, d - 1, \ldots, 1)$. As already noticed in the case $d = n$ in [21, Theorem 5.13], if one of the $\lambda_i - \mu_i$ is odd, one obtains 0. Indeed in one row of the determinantal form (11) of quadratic Schur functions the degree jumps by 2 between two columns, so all the degrees in the ith row would be odd; but all odd Segre classes $s_{2p+1}(E)$ are $= 0$ when $L = \mathcal{O}_X$.

The proof of Proposition 4.7 is easily adapted in the orthogonal case, in order to get the following statement.

Proposition 4.8. *Let $E \to X$ be an orthogonal vector bundle, equipped with the quadratic form $Q\colon E \otimes E \to \mathcal{O}_X$. For $d = 1, \ldots, \lfloor \mathrm{rank}(E)/2 \rfloor$, for $\lambda \in \mathbb{Z}^d$, one has the following Gysin formula*

$$\int_{\mathbf{F}^Q(d)(E)}^{X} s_\lambda(\xi_1, \ldots, \xi_d) = 2^d s^{(2)}_{\lambda-\mu}(E),$$

where $\mu_i = \mathrm{rank}(E) - d - i$ for $i = 1, \ldots, d$.

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