RIBBON SCHUR FUNCTIONS

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We present a new determinantal expression of a Schur function. Previous expressions were due to Jacobi, Trudi, Giambelli and others (see [7]) and involved elementary symmetric functions or hook-functions. We give in Theorem 0.1 the decomposition of a Schur function into ribbon-functions (also called skew-hook-functions, new functions by MacMahon, and MacMahon functions by others). We provide two different proofs of this result in § 1 and § 2.

In §1, we use Bazin's formula for the minors of a general matrix, as we already did in [6] to decompose a skew Schur function into hooks.

In §2, we show how to pass from hooks to ribbons and conversely.

In §3, we generalize to skew Schur functions.

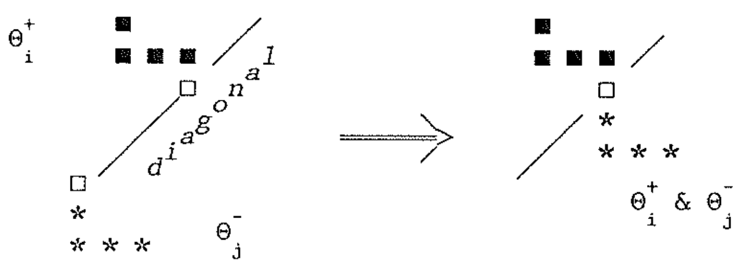
In §4, we give some applications, and show how such constructions, in the case of staircase partitions, generalize the classical continued fraction for the tangent function due to Euler

0. DECOMPOSITION INTO SUCCESIVE RIMS

Many properties of symmetric functions can be visualized graphically. For example, Schur Functions involve properties of Ferrers' diagrams : given a *partition* $J = (j_1, j_2, \dots, j_n)$, i.e. an increasing sequence of numbers : $0 \le j_1 \le \dots \le j_n$, one represents it by a diagram of boxes which is called its *(Ferrers') diagram* (see [7]) ; more generally, a *skew diagram* or *skew partition* is the complement of a diagram I into another bigger one J .

A skew diagram Θ which contains no 2x2 block of boxes is called a *ribbon* (*skew hook* for Anglo-saxons) ; the *rim* of a diagram (*outer strip* for [7, p.31]) is the maximal outer ribbon of the diagram. Given a partition, we can peel its diagram off into successive rims $\Theta_p, \dots, \Theta_1$ (see example 0.2) beginning from the outside. Such a ribbon Θ_i is cut by the diagonal into three disjoint parts : Θ_i^+ , $\Box_i$, Θ_i^- which are respectively the boxes of Θ_i strictly above the diagonal, the diagonal box, the boxes strictly under the diagonal.

Given two ribbons Θ_i , Θ_j , we denote by $\Theta_i^+ \& \Theta_j^-$ the ribbon obtained by replacing the lower part Θ_i^- of Θ_i by Θ_j^-. Pictorially, this can be represented by a displacement along the diagonal and superposition of the two diagonal boxes of Θ_i and Θ_j:

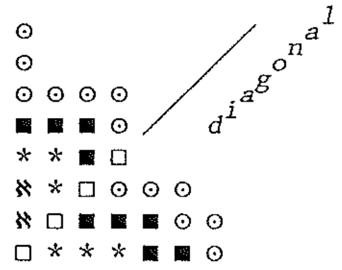


The following theorem, for which we give two different proofs in the next two paragraphs, shows that the decomposition of a partition into ribbons provides a determinantal expression of a Schur function.

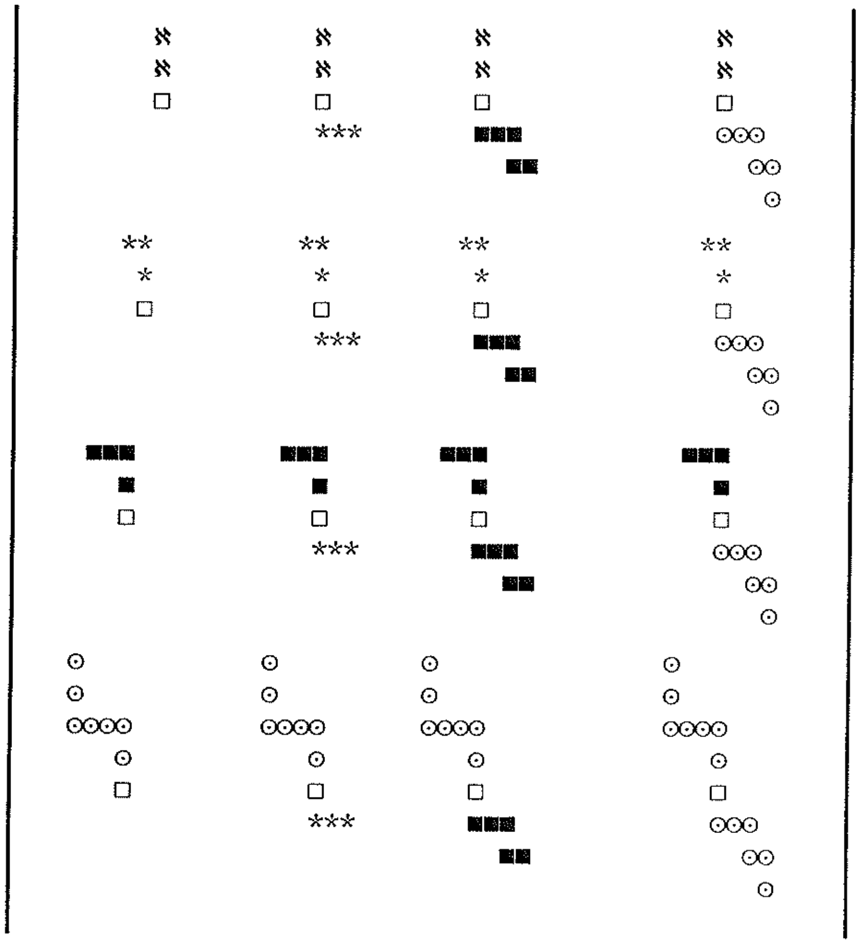
THEOREM 0.1. *Given a partition J , let (Θ_p, ..., Θ_1) be its decomposition into ribbons. Then*

$$S_J = \begin{vmatrix} S_{\Theta_1} & S_{\Theta_1^+ \& \Theta_2^-} & \cdots & S_{\Theta_1^+ \& \Theta_p^-} \\ S_{\Theta_2^+ \& \Theta_1^-} & S_{\Theta_2} & \cdots & S_{\Theta_2^+ \& \Theta_p^-} \\ \vdots & \vdots & & \vdots \\ S_{\Theta_p^+ \& \Theta_1^-} & S_{\Theta_p^+ \& \Theta_2^-} & \cdots & S_{\Theta_p} \end{vmatrix}$$

Example 0.2. Let J = 11 444 677 = (2347 & 0356).
The decomposition of J into the succesive rims ⊙...⊙ , ■...■, *...*, ℵ...ℵ , representing moreover the diagonal boxes by □ , is



and, according to theorem 0.1, the Schur function $S_{11444677}$ is equal to the following determinant of ribbon-functions (writing the ribbons instead of the corresponding functions) :



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1 RELATIONS BETWEEN MINORS

A *sequence* A is a finite sequence of positive integers. The concatenation product $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$ of two sequences A and B will be denoted A B. If C is a subsequence of A, then the complementary sequence of C in A will be denoted $A\setminus C$.

Given an ∞ x n matrix and a sequence A of cardinal n, we note [A] the maximal minor taken on rows $a_1, \ldots, a_n$. Then one has the following determinantal relation between the minors (due to Bazin and Sylvester independently) :

Lemma 1.1 (Bazin). *Given an* ∞xn *matrix and 3 sequences* A, B, C *such that* card(A) = card(C) = $p \le n$, card(A) + card(B) = n , *one has :*

$$\det \left| \; [\ ABc \setminus a\ ] \; \right|_{a\in A,\, c\in C} = [\ AB\ ]^{p-1} . [\ BC\ ]$$

(see [6] for more details).

In the ring of symmetric functions (in an infinite number of variables, see [7], we consider the *complete symmetric functions* S_i (i.e. the sum of all monomials of degree i ; S_i=0 if $i<0$) and the infinite matrix $\mathbb{S} = (S_{j-i})_{i,j\ge1}$. Given two partitions of cardinal r , the *skew-Schur Function* $S_{J/I}$ is the minor of $\mathbb{S}$ taken on rows i_1+1, i_2+2, ..., i_r+r and columns j_1+1, j_2+2, ..., j_r+r. The usual Schur function S_J is the special case where I = 0 . In other words,

$$(1.2) \qquad S_{J/I} = \det \left| \; S_{j_k - i_h + k - h} \; \right|_{1\le h,k\le r}$$

We shall write $S_J(\mathbb{A})$, $S_{J/I}(\mathbb{A})$ when we want to specify or specialize the variables to $\mathbb{A} = \{a_1, a_2, \dots\}$.

To prove Theorem 0.1 , we use the submatrix of $\mathbb{S}$ taken on columns j_1+1, j_2+1, ..., j_r+1 and rows 1, 2, ..., r, α_1+r+1, ..., α_p+r+1, where ($\beta_1, \dots, \beta_p$ & $\alpha_1, \dots, \alpha_p$) is the *Frobenius decomposition* of the partition J (see [7, p.3] ; here, $0\le \beta_1 < \dots < \beta_p$ and $0\le \alpha_1 < \dots < \alpha_p$). Now, in Bazin's theorem, we take $A = \{\alpha_1+r+1, \alpha_2+r+1, \dots, \alpha_p+r+1\}$

$$B = \{1, 2, \dots, r\} - \{r-\beta_p, \dots, r-\beta_1\}$$

$$C = \{r-\beta_p, r-\beta_{p-1}, \dots, r-\beta_1\}$$

and we find the wanted formula, since [BC] = S_J and [AB] = 1 .

Example 1.3. Take J = 11 444 677 as in Example 0.2. We have to use the matrix

$$\begin{vmatrix}
S_1 & S_2 & S_6 & S_7 & S_8 & S_{11} & S_{13} & S_{14} \\
S_0 & S_1 & S_5 & S_6 & S_7 & S_{10} & S_{12} & S_{13} \\
. & S_0 & S_4 & S_5 & S_6 & S_9 & S_{11} & S_{12} \\
. & . & S_3 & S_4 & S_5 & S_8 & S_{10} & S_{11} \\
. & . & S_2 & S_3 & S_4 & S_7 & S_9 & S_{10} \\
. & . & S_1 & S_2 & S_3 & S_6 & S_8 & S_9 \\
. & . & S_0 & S_1 & S_2 & S_5 & S_7 & S_8 \\
. & . & . & S_0 & S_1 & S_4 & S_6 & S_7 \\
. & . & . & . & S_0 & S_3 & S_5 & S_6 \\
. & . & . & . & . & S_0 & S_2 & S_3 \\
. & . & . & . & . & . & S_0 & S_1 \\
. & . & . & . & . & . & . & S_0
\end{vmatrix}$$

where we note . for S_i, $i<0$. Let $A = \{9, 12, 14, 15\}$, $B = \{2, 3, 7, 8\}$, $C = \{1, 4, 5, 6\}$. Then Bazin's formula expresses $S_{11\ 444\ 677}$ as the following determinant of minors :

$$\begin{vmatrix}
[23678,12,14,15] & [236789,14,15] & [236789,12,15] & [236789,12,14] \\
[23578,12,14,15] & [235789,14,15] & [235789,12,15] & [235789,12,14] \\
[23478,12,14,15] & [234789,14,15] & [234789,12,15] & [234789,12,14] \\
[12378,12,14,15] & [123789,14,15] & [123789,12,15] & [123789,12,14]
\end{vmatrix}$$

which in turn is identified to

$$\begin{vmatrix}
/11\ 333\ 677 & /11\ 333\ 377 & /11\ 333\ 357 & /11\ 333\ 356 \\
/11\ 233\ 677 & /11\ 233\ 377 & /11\ 233\ 377 & /11\ 233\ 356 \\
/11\ 133\ 677 & /11\ 133\ 377 & /11\ 133\ 357 & /11\ 133\ 356 \\
/00\ 033\ 677 & /00\ 033\ 377 & /00\ 033\ 357 & /00\ 033\ 356
\end{vmatrix}$$

where we write $/I$ for the Schur function $S_{11\ 444\ 677\ /\ I}$ .Now, these Schur functions are exactly the ribbon-functions that we announced in Example 0.2.

2. DEFORMATION OF DIAGRAMS

Given a (skew) diagram, its *right corner* is the box at the extreme right of the bottom row, and its *left corner* is the upper box of the left column. Given two skew diagrams H, K, let ℵ be the right corner of H and ■ the left corner of K . We define $H \rhd K$ to be the skew diagram obtained by glueing the two diagrams by their corners, ℵ , ■ being on the same horizontal, and H ‡ K to be the skew diagram obtained by glueing the two corners on a vertical :

```
                ⊙⊙                                ⊙⊙
                ⊙⊙⊙⊙                              ⊙⊙⊙⊙
                  ⊙⊙⊙⊙                              ⊙⊙⊙⊙
                    ⊙⊙※■*****                         ⊙⊙※
H ▷ K =                 ****        H ‡ K =              ■*****
                          **                               ****
                                                             **
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Lemma 2.1. *Let* H *and* K *be two skew diagrams. Then*

$$S_H \cdot S_K = S_{H \triangleright K} + S_{H \ddagger K}$$

Proof. The Schur function S_H can be looked as the sum $\sum t$ of all the *tableaux* of diagram H ; similarly S_K is the sum $\sum t'$ of all the tableaux of diagram K . For any pair of tableaux t, t', let y be the letter in the box ※ of t and z the letter in the box ■ of t' ; according as $y \le z$ or $y > z$, the product of tableaux t.t' is a tableau of diagram $H \triangleright K$ or $H \ddagger K$ and conversely, cutting into two pieces all the tableaux of diagrams $H \triangleright K$ or $H \ddagger K$, one obtains all the pairs of tableaux of respective diagrams H, K { all this is a trivial consequence of the *Jeu de Taquin* which allows to move parts of tableaux, see [10] ; this lemma is also given by Zelevinsky [11, p.69] } . □

Proposition 2.2. *Let* p *be a positive integer,* $q = \binom{p-1}{2}$, I, J, ..., H, K *be* p *skew partitions and* $\alpha, \beta, \ldots, \gamma, \delta$ *be* p-1 *other skew partitions. Then*

(i) *the determinant*

$$\begin{vmatrix} S_I & S_{I \triangleright_1 \alpha} & S_{I \triangleright_2 \alpha \triangleright_3 \beta} & \cdots & S_{I \triangleright_q \alpha \triangleright_{q+1} \beta \triangleright_{q+2} \cdots \triangleright_{q+p-2} \gamma} \\ S_J & S_{J \triangleright_1 \alpha} & S_{J \triangleright_2 \alpha \triangleright_3 \beta} & \cdots & S_{J \triangleright_q \alpha \triangleright_{q+1} \beta \triangleright_{q+2} \cdots \triangleright_{q+p-2} \gamma} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ S_K & S_{K \triangleright_1 \alpha} & S_{K \triangleright_2 \alpha \triangleright_3 \beta} & \cdots & S_{K \triangleright_q \alpha \triangleright_{q+1} \beta \triangleright_{q+2} \cdots \triangleright_{q+p-2} \gamma} \end{vmatrix}$$

where we have suffixed the symbols $\triangleright$ *in the order they were appearing in each row, is equal, up to a sign, to any determinant obtained from it by changing any subset* $\{ \triangleright_i , \ldots, \triangleright_j \}$ *of* $\triangleright$ *symbols into* ‡ *'s* .

(ii) *the determinant*

$$\begin{vmatrix} S_I & S_{\alpha\triangleright_1 I} & S_{\beta\triangleright_2\alpha\triangleright_3 I} & \cdots & S_{\gamma\triangleright_q\cdots\triangleright_{q+p-4}\beta\triangleright_{q+p-3}\alpha\triangleright_{q+p-2}I} \\ \vdots & \vdots & \vdots & & \vdots \\ S_K & S_{\alpha\triangleright_1 K} & S_{\beta\triangleright_2\alpha\triangleright_3 K} & \cdots & S_{\gamma\triangleright_q\cdots\triangleright_{q+p-4}\beta\triangleright_{q+p-3}\alpha\triangleright_{q+p-2}K} \end{vmatrix}$$

is equal, up to a sign, to any determinant obtained from it by the exchange of some $\triangleright_i$ *'s into* $\ddagger$ *'s .*

Proof: This is a direct consequence of Lemma 2.1, since in a determinant, adding a linear combination of columns to a given column does not change the value of the determinant. For example, the fifth column of the first determinant, $S_{I\triangleright\alpha\triangleright\beta\triangleright\gamma\triangleright\delta}$ can be transformed into $S_{I\triangleright\alpha\ddagger\beta\triangleright\gamma\ddagger\delta}$ because of the identities $S_{I\triangleright\alpha\triangleright\beta\triangleright\gamma\ddagger\delta} = S_{I\triangleright\alpha\triangleright\beta\triangleright\gamma} \cdot S_\delta - S_{I\triangleright\alpha\triangleright\beta\triangleright\gamma\triangleright\delta}$; $S_{I\triangleright\alpha\ddagger\beta\triangleright\gamma\ddagger\delta} = S_{I\triangleright\alpha} \cdot S_{\beta\triangleright\gamma\ddagger\delta} - S_{I\alpha\triangleright\beta\triangleright\gamma\ddagger\delta}$.

Proof of Theorem 0.1. Given a partition J whose Frobenius decomposition is $(\beta_1,\ldots,\beta_p \;\&\; \alpha_1, \ldots, \alpha_p)$, we have the following equality due to Giambelli (see [7 p.30]) :

$$(2.3) \qquad S_J = \begin{vmatrix} S_{1^{\beta_1}\&\alpha_1} & S_{1^{\beta_1}\&\alpha_2} & \cdots & S_{1^{\beta_1}\&\alpha_p} \\ \vdots & \vdots & & \vdots \\ S_{1^{\beta_p}\&\alpha_1} & S_{1^{\beta_p}\&\alpha_2} & \cdots & S_{1^{\beta_p}\&\alpha_p} \end{vmatrix}$$

where $1^\beta \& \alpha$ denotes the partition $(\overbrace{1, \ldots, 1}^{\beta}, \alpha+1)$.

Now, using Proposition 2.2 (i) with $(I, J, \ldots, K) = (1^{\beta_1} \& \alpha_1 , \ldots, 1^{\beta_p} \& \alpha_1)$, $\alpha, \beta, \ldots, \gamma$ being the one-part-partitions $\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots, \alpha_p - \alpha_{p-1}$, we get that

$$S_J = (-1)^{\binom{p}{2}} \begin{vmatrix} S_{1^{\beta_1}\&\Theta_1^-} & S_{1^{\beta_1}\&\Theta_2^-} & \cdots & S_{1^{\beta_1}\&\Theta_p^-} \\ \vdots & \vdots & & \vdots \\ S_{1^{\beta_p}\&\Theta_1^-} & S_{1^{\beta_p}\&\Theta_2^-} & \cdots & S_{1^{\beta_p}\&\Theta_p^-} \end{vmatrix}$$

since $\Theta_1^- = \alpha_1$, $\Theta_2^- = \alpha_1 \ddagger (\alpha_2 - \alpha_1)$, ..., $\Theta_p^- = \alpha_1 \ddagger (\alpha_2 - \alpha_1) \ddagger \ldots \ddagger (\alpha_p - \alpha_{p-1})$

Using the second part of Proposition 2.2, we can further transform the determinant and get the desired result.

Example 2.4. Let I = 1 3 5 5 7. The decompositions of the diagram of I into respectively hooks and ribbons are

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⊙                          ⊙
⊙ * ■                      ⊙ ⊙ ⊙
⊙ * □ ■ ■          and     * * □ ⊙ ⊙
⊙ □ * * *                  ■ □ * * ⊙
□ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙              □ ■ ■ * ⊙ ⊙ ⊙
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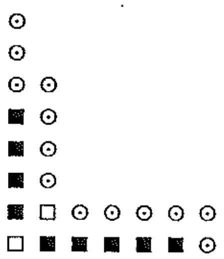
$$S_{13557} = \begin{vmatrix} S_{13} & S_{14} & S_{17} \\ S_{113} & S_{114} & S_{117} \\ S_{11113} & S_{11114} & S_{11117} \end{vmatrix} =$$

$$\begin{vmatrix} S_{13} & S_{13\ddagger 1} & S_{13\ddagger 1\ddagger 3} \\ S_{113} & S_{113\ddagger 1} & S_{13\ddagger 1\ddagger 3} \\ S_{11113} & S_{11113\ddagger 1} & S_{11113\ddagger 1\ddagger 3} \end{vmatrix} = \begin{vmatrix} S_{13} & S_{13\ddagger 1} & S_{13\ddagger 1\ddagger 3} \\ S_{1\triangleright 13} & S_{1\triangleright 13\ddagger 1} & S_{1\triangleright 13\ddagger 1\ddagger 3} \\ S_{11\triangleright 1\triangleright 13} & S_{11\triangleright 1\triangleright 13\ddagger 1} & S_{11\triangleright 1\triangleright 13\ddagger 1\ddagger 3} \end{vmatrix}$$

Proposition 2.2. not only allows to pass from Giambelli's determinant 2.3 to the determinant of rims 0.1, but moreover produces $2^{(p-1)(p-2)}$ determinants equal up to a sign, by exchanging symbols $\triangleright$ and $\ddagger$. Certain choices of $\triangleright$, $\ddagger$ correspond to a *block decomposition* of a partition. For example, let J = 11 444 677 as in 0.2 , and Θ = 111. Then S_J =

$$\begin{vmatrix} S_{\Theta} & S_{\Theta\triangleright 3} & S_{\Theta\triangleright 3\triangleright 2} & S_{\Theta\triangleright 3\triangleright 2\ddagger 1} \\ S_{1\ddagger\Theta} & S_{1\ddagger\Theta\triangleright 3} & S_{1\ddagger\Theta\triangleright 3\triangleright 2} & S_{1\ddagger\Theta\triangleright 3\triangleright 2\ddagger 1} \\ S_{1\ddagger 1\ddagger\Theta} & S_{1\ddagger 1\ddagger\Theta\triangleright 3} & S_{1\ddagger 1\ddagger\Theta\triangleright 3\triangleright 2} & S_{1\ddagger 1\ddagger\Theta\triangleright 3\triangleright 2\ddagger 1} \\ S_{111\triangleright 1\ddagger 1\ddagger\Theta} & S_{111\triangleright 1\ddagger 1\ddagger\Theta\triangleright 3} & S_{111\triangleright 1\ddagger 1\ddagger\Theta\triangleright 3\triangleright 2} & S_{111\triangleright 1\ddagger 1\ddagger\Theta\triangleright 3\triangleright 2\ddagger 1} \end{vmatrix}$$

and this determinantal expression corresponds to the following block-decomposition of J into two blocks :



and

Starting from the determinantal expression (1.2) of a Schur function, transforming symbols $\triangleright$ will produce other determinantal expressions of this Schur function : this is a special case of Proposition 2.2., $I\ J, \ldots, \alpha, \beta, \ldots$ being then one-part partitions. Thus, writing Θ for S_Θ , one has for example that

S_{357} which by definition is equal to $\begin{vmatrix} 3 & 6 & 8 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{vmatrix}$, i.e. to

$\begin{vmatrix} 3 & 3\triangleright 3 & 3\triangleright 3\triangleright 2 \\ 2 & 2\triangleright 3 & 2\triangleright 3\triangleright 2 \\ 1 & 1\triangleright 3 & 1\triangleright 3\triangleright 2 \end{vmatrix}$ is also equal, up to a sign, to one of the following determinants :

$$\begin{vmatrix} 3 & 3\ddagger 3 & 3\triangleright 3\triangleright 2 \\ 2 & 2\ddagger 3 & 2\triangleright 3\triangleright 2 \\ 1 & 1\ddagger 3 & 1\triangleright 3\triangleright 2 \end{vmatrix}, \begin{vmatrix} 3 & 3\triangleright 3 & 3\ddagger 3\triangleright 2 \\ 2 & 2\triangleright 3 & 2\ddagger 3\triangleright 2 \\ 1 & 1\triangleright 3 & 1\ddagger 3\triangleright 2 \end{vmatrix}, \begin{vmatrix} 3 & 3\triangleright 3 & 3\triangleright 3\ddagger 2 \\ 2 & 2\triangleright 3 & 2\triangleright 3\ddagger 2 \\ 1 & 1\triangleright 3 & 1\triangleright 3\ddagger 2 \end{vmatrix}, \begin{vmatrix} 3 & 3\triangleright 3 & 3\ddagger 3\ddagger 2 \\ 2 & 2\triangleright 3 & 2\ddagger 3\ddagger 2 \\ 1 & 1\triangleright 3 & 1\ddagger 3\ddagger 2 \end{vmatrix},$$

$$\begin{vmatrix} 3 & 3\ddagger 3 & 3\triangleright 3\ddagger 2 \\ 2 & 2\ddagger 3 & 2\triangleright 3\ddagger 2 \\ 1 & 1\ddagger 3 & 1\triangleright 3\ddagger 2 \end{vmatrix}, \begin{vmatrix} 3 & 3\triangleright 3 & 3\ddagger 3\ddagger 2 \\ 2 & 2\triangleright 3 & 2\ddagger 3\ddagger 2 \\ 1 & 1\triangleright 3 & 1\ddagger 3\ddagger 2 \end{vmatrix}, \begin{vmatrix} 3 & 3\ddagger 3 & 3\ddagger 3\ddagger 2 \\ 2 & 2\ddagger 3 & 2\ddagger 3\ddagger 2 \\ 1 & 1\ddagger 3 & 1\ddagger 3\ddagger 2 \end{vmatrix}.$$

3 GENERALISATION

In the previous paragraph, we have only used the fact that, for any skew partition J and any strictly positive integer i ,

(3.1) $$S_J \cdot S_i = S_{J\triangleright i} + S_{J\ddagger i}$$

(3.2) $$S_{1^i} \cdot S_J = S_{1^i\triangleright J} + S_{1^i\ddagger J}$$

As Schur functions can be expressed as determinants of functions S_{1^j} and S_j with j belonging to $\mathbb{Z}$ in general ($S_j = 0 = S_{1^j}$ if $j<0$) , it is desirable to extend (3.1) and (3.2) to the case $J = j$ or 1^j , $j<0$. This is done by putting

(3.3) $$j<0 \Longrightarrow \begin{cases} S_{j\triangleright i} = S_{j+i} = - S_{j\ddagger i} \\ \\ S_{1^i\ddagger 1^j} = S_{1^{i+j}} = - S_{1^i\triangleright 1^j} \end{cases}$$

and

(3.4) $\quad S_{0\rhd i} = S_i \;;\; S_{0\ddagger i} = 0 \;;\; S_{1^i\ddagger 1^0} = S_{1^i} \;;\; S_{1^i\rhd 1^0} = 0$

We shall never need $i = 0 = 1^0$, thus escaping the non-consistency of (3.4) in that case.

With these rules, we are able to transform more general determinants than (2.3), for example, the one expressing a skew Schur function in terms of hook functions.

Let J be a partition, $\Theta_1, \ldots, \Theta_p$ its decomposition into rims, $(\beta_1,\ldots, \beta_p \,\&\, \alpha_1,\ldots, \alpha_p)$ its Frobenius decomposition. Let I be a second partition and $(\delta_1, \ldots, \delta_r \,\&\, \gamma_1, \ldots, \gamma_r)$ its Frobenius decomposition. Then according to [6], $(-1)^{|I|} S_{J/I}$ is equal to the determinant

$$(3.5)\quad \begin{vmatrix} S_{1^{\beta_1}\&\alpha_1} & \cdots & S_{1^{\beta_1}\&\alpha_p} & S_{1^{\beta_1-\delta_1}} & \cdots & S_{1^{\beta_1-\delta_r}} \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{1^{\beta_p}\&\alpha_1} & \cdots & S_{1^{\beta_p}\&\alpha_p} & S_{1^{\beta_p-\delta_1}} & \cdots & S_{1^{\beta_p-\delta_r}} \\ S_{\alpha_1-\gamma_1} & \cdots & S_{\alpha_p-\gamma_1} & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{\alpha_1-\gamma_r} & \cdots & S_{\alpha_p-\gamma_r} & 0 & & 0 \end{vmatrix}$$

generalizing Giambelli's formula (2.3).

Performing the same linear combination of rows and columns as in § 2, one obtains the following theorem.

THEOREM 3.6. *The skew Schur function* $S_{J/I}$ *is equal to the determinant obtained from* (3.5) *by changing, for all* i, j, h, k : $1\le i, j\le p$, $1\le h, k\le r$

(1) the (i,j)*-th element (formerly* $= S_{1^{\beta_i}\&\alpha_j}$*) into* $S_{\Theta_i^+\&\Theta_j^-} =$

$$S_{1^{\beta_i-\beta_{i-1}}\rhd\ldots\rhd 1^{\beta_2-\beta_1}\rhd\Theta_1\ddagger\alpha_2-\alpha_1\ddagger\ldots\ddagger\alpha_j-\alpha_{j-1}}$$

(2) the (p+h,j)*-th element (formerly* $= S_{\alpha_j-\gamma_h}$ *) into*

$$S_{\alpha_1-\gamma_h\ddagger\alpha_2-\alpha_1\ddagger\ldots\ddagger\alpha_j-\alpha_{j-1}}$$

(3) the (i,p+k)*-th element (formerly* $= S_{1^{\beta_i-\delta_k}}$ *) into*

$$S_{1^{\beta_i-\beta_{i-1}}} \triangleright \ldots \triangleright 1^{\beta_2-\beta_1} \,\bar{\triangleright}\, 1^{\beta_1-\delta_k}$$

One notices that if in determinant (3.6), one of the elements of a row among the last r ones is the function S_0 (= 1), then all the other elements of this row are null thanks to (3.4) . Similarly, a function S_{1^0} in any of the last r columns imply that all the other elements of this column are null.

Corollary 3.7. *Let* J *be a partition,* $(\beta_1, \ldots, \beta_p \;\&\; \alpha_1, \ldots, \alpha_p)$ *its Frobenius decomposition and* $\Theta_1, \ldots, \Theta_p$ *its rim decomposition. Let further* r *be an integer* $\le p$, $\mathcal{H} = \{ h_1, \ldots, h_r \}$, $\mathcal{K} = \{ k_1, \ldots, k_r \}$ *two subsets of the set* $\{ 1, \ldots, p \}$. *Let at last* I *be the partition of Frobenius decomposition* $(\beta_{h_1}, \ldots, \beta_{h_r} \;\&\; \alpha_{k_1}, \ldots, \alpha_{k_r})$. *Then*

$$S_{J/I} = \left| \; S_{\Theta_i^+ \;\&\; \Theta_j^-} \; \right|_{i \notin \mathcal{H},\; j \notin \mathcal{K}}$$

Proof. with the hypothesis on I , each of the last r rows and the last r columns of the determinant (3.6) admits one and only one element which is different from 0, according to the preceeding remark. This reduces determinant (3.6) to the claimed one.

Example 3.8. Let J = 1 3 4 5 5 6 7 = (1346 & 1248), I = 1 2 3 5 = (13 & 14). Then J decomposes into the rims $\Theta = \Theta_1 = 1\ 2$, $\Theta_2 = 11 \triangleright \Theta \ddagger 1$, $\Theta_3 = 1 \triangleright 11 \triangleright \Theta \ddagger 1 \ddagger 2$, $\Theta_4 = 11 \triangleright 1 \triangleright 11 \triangleright \Theta \ddagger 1 \ddagger 2 \ddagger 4$ and

$$S_{J/I} = \begin{vmatrix} S_{\Theta_3^+ \;\&\; \Theta_2^-} & S_{\Theta_3^+ \;\&\; \Theta_4^-} \\ S_{\Theta_4^+ \;\&\; \Theta_2^-} & S_{\Theta_4^+ \;\&\; \Theta_4^-} \end{vmatrix}$$

since we have to throw away Θ_1^+ and Θ_2^- because 1, 3 are the first two elements of { 1,3,4,6 } , and to throw away Θ_1^- and Θ_3^- because 1, 4 are the first and the third element of {1, 2, 4, 8}.

4 EULERIAN DETERMINANTS

The tangent function admits a q-analog ; even better, since sinus and cosinus can be symmetrized, there exists a symmetrical analog of the tangent ; to compute its coefficients, one needs to know how to express the quotient of two formal series. This was done by Faure (1855, [9, II p.212]) ; Anglo-Saxons usually prefers Hammond (1875, [9, III p.232]), but could as well take Spottiswoode (1853, [9, II p.211]) who gives the coefficients of the succesive derivatives of f/g .

Proposition 4.1. *Let* $f = \alpha_0 + \alpha_1 z^1 + \alpha_2 z^2 + \ldots$, $g = 1 - \beta_1 z^1 + \beta_2 z^2 - \beta_3 z^3 + \ldots$ *be two formal series in* z . *Then*

$$f/g = \sum z^n S_n$$

with

$$S_n = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \beta_0 & \beta_1 & \cdots & \beta_n \\ \vdots & \vdots & & \vdots \\ \beta_{-n+1} & \beta_{-n+2} & \cdots & \beta_1 \end{vmatrix}$$

putting $\beta_0 = 1$, $\beta_i = 0 \ \forall\, i < 0$.

Let $\mathbb{A}$ be a set of indeterminates, $f = \sum z^{2i+1} S_{2i+1}(\mathbb{A})$, $g = \sum (-1)^j z^{2j} S_{2j}(\mathbb{A})$; define $tg(\mathbb{A}, z) = f / g$.

Lemma 4.2. $tg(\mathbb{A}, z) = \sum_0^\infty z^{2n+1} T_{2n+1}$ *with*

$$T_{2n+1} = S_{1\,2\,\ldots\,n+1\,/\,0\,0\,1\,2\,\ldots\,n-1}(\mathbb{A})$$

Proof : Taking $\alpha_i = S_{2i+1}(\mathbb{A})$, $\beta_j = S_{2j}(\mathbb{A})$ in Faure's determinant (4.1) , we recognize it to be the minor of $\mathbb{S}$ taken on columns 2, 4, 6 ... and rows 1, 2, 3, 5, 7, ... , i.e. to be $S_{123\ldots n+1/00\ldots n-1}(\mathbb{A})$. □

The q-analog of the tangent is obtained for $\mathbb{A} = \{1, q, q^2, \ldots\}$; $tg(\{1, q, \ldots\}, z(1-q))$ specializes to the usual tangent function for $q = 1$ (cf [3]).

The continued fraction-decomposition of any formal series $\sum z^j S_j$ has been obtained by many authors ; it is closely related to the computation of the succesive remainders in the division of a polynomial by another one (see e.g. Heilermann, 1845 [7,II p.36]).

Proposition 4.3. *For a series and its inverse, one has the following developments*

$$\sum z^j S_j = \cfrac{S_0}{1- \cfrac{zS_1/S_0}{1+ \cfrac{zS_{11}/S_0.S_1}{1- \cfrac{zS_0.S_{22}/S_1.S_{11}}{\ddots \cfrac{}{1- \cfrac{(-1)^n z\, \Delta_n.\Delta_{n+3} \,/\, \Delta_{n+1}.\Delta_{n+2}}{\ddots}}}}}}$$

with $\Delta_{2n} = S_{n^{n+1}}$ *and* $\Delta_{2n-1} = S_{n^n}$.

$$[\sum (-z)^j S_j]^{-1} = \cfrac{S_0}{1- \cfrac{zS_1/S_0}{1+ \cfrac{zS_2/S_0.S_1}{1- \cfrac{zS_0.S_{22}/S_1.S_2}{\ddots \; 1 - \cfrac{(-1)^n z\, \Delta_n \Delta_{n+3}/\Delta_{n+1} \Delta_{n+2}}{\ddots}}}}}$$

with $\Delta_{2n} = S_{(n+1)^n}$ *and* $\Delta_{2n-1} = S_{n^n}$.

To get the continued fraction for the tangent, we thus need to evaluate the determinants

$$\Delta'_{2n} = \begin{vmatrix} T_n & \cdots & T_{2n} \\ \vdots & & \vdots \\ T_0 & \cdots & T_n \end{vmatrix} \quad \text{and} \quad \Delta'_{2n-1} = \begin{vmatrix} T_n & \cdots & T_{2n-1} \\ \vdots & & \vdots \\ T_1 & \cdots & T_n \end{vmatrix}$$

with $T_n = S_{12\ldots n/0012\ldots n-2}$. Since the skew partitions $12..n$ / $0012...n-2$ are exactly the rims of the *staircase* partition 123..., we can apply Theorem 0.1 :

Corollary 4.4. $tg(\mathbb{A}, z)$ *is equal to the continued fraction*

$$\cfrac{z\,S_1}{1-\cfrac{z^2\,S_{12}/S_1}{1-\cfrac{z^2\,S_{123}/S_1.S_{12}}{\ddots\;1-\cfrac{z^2\,S_{12\ldots n}.S_{12\ldots n+3}\;/\;S_{12\ldots n+1}.S_{12\ldots n+2}}{\ddots}}}}$$

Similarly, the secant function is $\sec(z) = 1 / \cos(z)$; it admits the following symmetrical generalization :

$$\sec(\mathbb{A}, z) = \left[\sum_0^\infty (-1)^j\, S_{2j}(\mathbb{A}) \right]^{-1}$$

The specialisation $S_n \longrightarrow S_{2n}(\mathbb{A})$ in Faure's determinant (4.1) and in the expansion (4.3) leads to :

Lemma 4.5. $\sec(\mathbb{A},\sqrt{z}\,) = 1 + z\,S_2(\mathbb{A}) + z^2 S_{23/1}(\mathbb{A}) + \ldots + {}$

$$+ z^2\, S_{2\ldots n+1/01\ldots n-1}(\mathbb{A}) + \ldots$$

$$= \cfrac{1}{1-\cfrac{zS_2}{1+\cfrac{zS_4/S_2}{1-\cfrac{zS_{45/1}\;/S_2.S_4}{\ddots\;1+\cfrac{(-1)^n z\,\Delta_n.\Delta_{n+3}/\Delta_{n+1}.\Delta_{n+2}}{\ddots}}}}}$$

with $\Delta_0 = S_0,\ \Delta_2 = S_{4/0}\ ,\ \ldots,\ \Delta_{2n} = S_{2n+2\ldots 3n+1/01\ldots n-1}$ *and*

$\Delta_1 = S_2\ ,\ \Delta_3 = S_{45/01}\ ,\ \ldots,\ \Delta_{2n-1} = S_{2n\ldots 3n-1/01\ldots n-1}$.

If one prefers, one can use dimensions of $\mathbb{C}$-representations of the symmetric group. Recall (cf.[7,p.63]) :

Lemma 4.6. *Let* J/I *be a skew partition,* $\mathbb{V}_{J/I}$ *the corresponding representation of the symmetric group* $\mathfrak{S}_n$ *, n being the weight of* J/I *. Then, with* $\mathbb{A} = \{ 1, q, q^2, \ldots \}$,

$$\dim \mathbb{V}_{J/I} = \lim_{q\to 1} (1-q)(1-q^2)\ldots(1-q^n)\; S_{J/I}(\mathbb{A}) \quad .$$

Thus, any of the determinants (0.1), (1.2), (2.3), (3.5), (3.6) provides an expression of the dimension of $\mathbb{V}_{J/I}$. When I=0 one has also a "hook formula" (cf.[7, p.28]) .

The *Euler numbers* are defined by

$$\mathrm{tg}(y) + \sec(y) = \sum_0^\infty E_n\, y^n / n! \quad .$$

They are obtained through the specialisation $\mathbb{A} = \{1, q, q^2, \ldots\}$, $z = (1-q)y$, $q = 1$ of $\mathrm{tg}(\mathbb{A},z) + \sec(\mathbb{A},z) = \sum_0^\infty T_n\, z^n$. Thus, from (4.2) and (4.5), one gets the following interpretation of the Euler numbers due to Desiré André [1, 2], once knowing, thanks to Young and his school, that standard tableaux of any shape give bases of representations :

Lemma 4.7. *E_{2n-1} is the dimension of $\mathbb{V}_{12\ldots n/0012\ldots n-2}$ and of $\mathbb{V}_{2304\ldots nn/012\ldots n-1}$; E_{2n} is the dimension of $\mathbb{V}_{123..nn/001..n-1}$ and of $\mathbb{V}_{23\ldots n+1/01\ldots n-1}$.*

Much more properties of the characters of the above representations have been obtained by Foulkes [4, 5] .

The preceedings paragraphs allow to evaluate determinants in the $E_n/\ n!$ and to interpret them in terms of dimension of representations. Specially noteworthy are the following determinants, denoting by [n] the integral part of the real n .

Proposition 4.8. *Let the $T'_n = E_n/n!$ be the coefficients of the function* $\mathrm{tg}(y) + \sec(y)$ *. Then*

(1) For the following determinant of order [(n+1)/2] *, we have*

$$\begin{vmatrix} T'_{2n-1} & T'_{2n-3} & T'_{2n-5} & \cdots \\ T'_{2n-3} & T'_{2n-5} & T'_{2n-7} & \cdots \\ T'_{2n-5} & T'_{2n-7} & T'_{2n-9} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = \left(1^n\ 3^{n-1} 5^{n-2} \ldots\ (2n-1)\ \right)^{-1}$$

(2) For the following determinant of order [(n+2)/2], *we have*

$$\begin{vmatrix} T'_{2n} & T'_{2n-1} & T'_{2n-3} & \cdots \\ T'_{2n-2} & T'_{2n-3} & T'_{2n-5} & \cdots \\ T'_{2n-4} & T'_{2n-5} & T'_{2n-7} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = \left(1^n . 3^{n-1} . 5^{n-2} \ldots\ n!\ 2^n \right)^{-1}$$

(3) For the following determinant of order [(n+3)/2], *we have*

$$\begin{vmatrix} T'_{2n+1} & T'_{2n} & T'_{2n-2} & \cdots \\ T'_{2n} & T'_{2n-1} & T'_{2n-3} & \cdots \\ T'_{2n-2} & T'_{2n-3} & T'_{2n-5} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = (1^n.3^{n-1}\ldots(n!)^2\ 2^{2n}(2n+1))^{-1}$$

Proof: up to a symmetry with respect to the anti-diagonal, the three previous determinants are the specialisation of the rim-decomposition of the respective Schur-functions $S_{12\ldots n}$, $S_{23\ldots n+1}$, $S_{23\ldots n+1\ n+1}$. Lemma 4.6 together with the already mentionned hook-formula, gives the desired explicit values.

The same determinants, once truncated, provide the dimensions of the following representations.

Proposition 4.9. *Let* m, n, p *be positive integers such that* n-m = 2p. *Then for the minors taken on the first* p *rows and columns, one has*

(1) *the minor of* det 4.8*(1) is equal to*

$$\dim \mathbb{V}_{12\ldots n/12\ldots m} \Big/ \left[\binom{n+1}{2} - \binom{m+1}{2}\right]!$$

(2) *the minor of* det 4.8*(2) is equal to*

$$\dim \mathbb{V}_{2\ldots n+1/12\ldots m+1} \Big/ \left[\binom{n+1}{2} - \binom{m+1}{2} - 1\right]!$$

(3) *the minor of* det 4.8*(3) is equal to*

$$\dim \mathbb{V}_{2\ldots n+1\ n+1\ /\ 12\ldots m+2} \Big/ \left[\binom{n+1}{2} + n - \binom{m+2}{2}\right]!$$

Specializing det 4.4 in $z = \sqrt{y}\,(1-q)$, $\mathbb{A} = (1, q, \ldots)$, $q = 1$, one finds back the following expression of the tangent function due to Euler .

Corollary 4.10.

$$\mathrm{tg}\sqrt{y} \,/\, \sqrt{y} = \cfrac{1}{1 - \cfrac{y/1.3}{1 - \cfrac{y/3.5}{1 - \cfrac{y/5.7}{\ddots}}}}$$

However, a similar simplification does not arise for the cosecant function.

Finally, let us illustrate the rôle of rims in representation theory through the following fact (see also [12, p.69]).

Lemma 4.11. *Let* n *be a positive integer. Then*

$$(S_1)^n = \sum S_\Theta$$

sum on all different rims Θ *of weight* n .

Proof: by induction on n , with the help of Lemma 2.1. □
Combinatorially, the proof is even simpler : every word or every permutation can be written into a ribbon-diagram and only one, i.e. the "up-down sequence" of a permutation is well defined (see [7]).

In other words, the regular representation of the symmetric group $\mathfrak{S}_n$ decomposes into the multiplicity-free sum of all the different ribbon representations. There is a more general decomposition for a finite Coxeter group, see [11] .

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