Tangency and regular separation

with W. Domitrz and P. Mormul

Two plane curves both nonsingular at x^0 have a contact of order $\geq k$ if in properly chosen regular parametrizations they have identical Taylor polynomials of degree k about x^0.

Let $M, \tilde{M} \subset \mathbb{R}^m$ be C^r mfds, $\dim M = \dim \tilde{M} = p$, $x^0 \in M \cap \tilde{M}$.

$k \leq r$ $\quad M, \tilde{M}$ have at x^0 the order of tangency $\geq k$ if $\exists$ nghb $\mathcal{U} \ni u^0$ in $\mathbb{R}^p$ and parametrizations

$$q : (\mathcal{U}, u^0) \to (M, x^0) \qquad \tilde{q} : (\mathcal{U}, u^0) \to (\tilde{M}, x^0)$$

s.t. $(\tilde{q} - q)(u) = o(|u-u^0|^k)$ when $\mathcal{U} \ni u \to u^0$

$\uparrow$

Prop. is equivalent to $T^k_{u^0}(q) = T^k_{u^0}(\tilde{q})$ (Taylor polys).

Pf $\Rightarrow$

$$o(|u-u^0|^k) = \tilde{q}(u) - q(u) = \left(\tilde{q}(u) - T^k_{u^0}(\tilde{q})(u-u^0)\right) + \underbrace{\left(T^k_{u^0}(\tilde{q})(u-u^0) - T^k_{u_0}(q)(u-u^0)\right)}_{= o(|u-u^0|^k)} + \underbrace{\left(T^k_{u_0}(q)(u-u^0) - q(u)\right)}_{o(|u-u^0|^k)}$$

$\Uparrow \; = 0$

Lemma Let $w \in \mathbb{R}[u_1, \dots, u_p]$, $\deg w \leq k$, $w(u) = o|u|^k$ when $u \to 0$ in $\mathbb{R}^p$. Then $w \equiv 0$. $\square$

$\Leftarrow$ Exc.

If the order of tangency is $\geq k$ but not $\geq k+1$, we say that is k. Suppose that is k. Assume that $k < r$.

Mini-max procedure $\qquad T_{x^0} M = T_{x^0} \tilde{M} = T_{x^0}$

Thm If $k < r$

$$\min_v \left(\max_{\gamma, \tilde{\gamma}} \left(\max \{ l \in \{0\} \cup \mathbb{N} : |\gamma(t) - \tilde{\gamma}(t)| = o(|t|^l) \text{ when } t \to 0 \} \right) \right) = k$$

$0 \neq v \in T_{x^0}$; $\quad \gamma \subset M$, $\tilde{\gamma} \subset \tilde{M}$ $\quad \gamma(0) = x^0 = \tilde{\gamma}(0)$ $\quad \dot{\gamma}(0), \dot{\tilde{\gamma}}(0)$ parallel to v

min max t.o.

$L \geq 0$ in T_{x_0} $\gamma \subset M, \tilde{\gamma} \subset \tilde{M}$: span $T_\gamma, T_{\tilde{\gamma}} = L$

Grassmann towers attached to q

C^1 immersion $H : N^n \to N'^{n'} \rightsquigarrow \mathcal{G}H : N \to G_n(N')$ $\quad G_n(T_{N'}) \to N'$

$z \in N$ $\quad \mathcal{G}H(z) = dH(z)(T_z N)$.

Have

$$\mathcal{G}_q : U \to G_p(\mathbb{R}^m) \qquad \mathcal{G}_{\tilde{q}} : U \to G_p(\mathbb{R}^m)$$

$$M^{(0)} = \mathbb{R}^m, \ \mathcal{G}^{(1)} = \mathcal{G} \qquad l \geq 1$$

$$\mathcal{G}^{(l)}q : U \to G_p(M^{(l-1)}) \qquad \mathcal{G}^{(l+1)}q = \mathcal{G}(\mathcal{G}^{(l)}q)$$

$$\mathcal{G}^{(l)}\tilde{q} : U \to G_p(M^{(l-1)}) \qquad \mathcal{G}^{(l+1)}\tilde{q} = \mathcal{G}(\mathcal{G}^{(l)}\tilde{q})$$

Here $M^{(l)} = G_p(M^{(l-1)})$.

Thm 2 C^r mfds M and $\tilde{M}$ have at x^0 o.t. $\geq k$ $(1 \leq k \leq r)$ if $\exists$ C^r $q, \tilde{q}$ parametrizations of nghbs of x_0 in $M, \tilde{M}$ s.t.

$$\mathcal{G}^{(k)}q(u^0) = \mathcal{G}^{(k)}\tilde{q}(u^0).$$

Suppose $M, \tilde{M}$ are the graphs of 2 mappings $f, g : \mathbb{R}^p \to \mathbb{R}^{m-p}$. Then M and $\tilde{M}$ have o.t. $\geq k$ if partials of f and g of order $\leq k$ coincide.

Assume $f(0) = g(0) = 0$.

We are interested in partial derivatives.

Lemma For $1 \leq l \leq k$ there exists a chart on $G_p(M^{(l-1)})$ in which $\mathcal{G}^{(l)}q(u)$ is $(u, f(u); \binom{l}{1} f_{[1]}(u), \binom{l}{2} f_{[2]}(u), \dots, \binom{l}{l} f_{[l]}(u))$

$f_{[i]}(u)$ – sum of all partials of i-th order of all components of f.

(in this Lemma we distinguish mixed derivatives taken in different orders)

Get 2 similar visualizations of $\mathcal{G}^{(k)}q(u^0)$, $\mathcal{G}^{(k)}\tilde{q}(u^0)$ using partials $\Rightarrow$ Prop. (Taylor)

Regular separation of pairs of sets (Łojasiewicz) (3

Simplified version : x^0 is an isolated point of $M \cap \tilde{M}$.

We shall call a positive number p a regular separation exponent if for some $c>0$ $\rho(q(u), \tilde{M}) \geq c|u-u^0|^p$. $(*)$

ρ - standard norm on $\mathbb{R}^m$, u belongs to a suitable nghb of u^0 ($q: (\mathcal{U}, u^0) \to (M, x^0)$ diffeo).

Let P = set of reg. sep. exps. $p_0 = \inf P$ - the minimal r.s.e. (Łojasiewicz's s.e.)

k order of tangency of M and $\tilde{M}$ at x_0. | In all examples o.t. $\leq$ Ł.e. in DMP

Lemma We have $k \leq p_0$.

Pf $p_0 = \inf P \Rightarrow$ it suffices to show (with k instead of p) an opposite inequality to $(*)$.

Or Def. $\Rightarrow$ $|q(u) - \tilde{q}(u)| < |u-u^0|^k$ when $u \to u^0$ in $\mathcal{U}$

$\rho(q(u), \tilde{M}) \leq |q(u) - \tilde{q}(u)| \Rightarrow \rho(q(u), \tilde{M}) < |u-u^0|^k$ -"-

opposite inequality to $(*)$ □

(Lemma also observed by Krasiński)

Ex. $C = \{(x,y) : (y-x^2)^2 = x^5\} \subset \mathbb{R}^2(x,y)$. Two semialgebraic branches

$C_- = \{y = x^2 - x^{5/2}, x \geq 0\}$ $C_+ = \{y = x^2 + x^{5/2}, x \geq 0\}$

extended to the graphs of $y_- = x^2 - |x|^{5/2}$ and $y_+ = x^2 + |x|^{5/2}$

The Taylor polynomials of deg 2 about $x=0$ of y_- and y_+ coincide. So these graphs have t.o. $= 2$.

The min. r. s. e. $= 5/2$.

Ex (Tworzewski) $N = \{y=0\}$, $Z = \{y^d + yx^{d-1} + x^s = 0\}$ $s > d$

At $(0,0)$ t.o. $s-d$

min. r. s. e $= s-d+1$.