Hilbert modular double octic Calabi-Yau 3-fold

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L–series of a Calabi–Yau manifold

Let X be a Calabi–Yau manifold defined over $\mathbb{Q}$ and let p be a prime of good reduction, i.e. the reduction X_p of X mod p is smooth. By the Weil Conjecture the Zeta function of X_p can be written as

$$\frac{P_{1,p}(t)P_{3,p}(t)\dots P_{2n-1,p}(t)}{P_{0,p}(t)P_{2,p}(t)\dots P_{2n-2,p}(t)P_{2n,p}(t)}$$

where $P_{i,p}$ is a polynomial of degree b_i. We define the i–th cohomological L–series of X as

$$L(H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l),s) = (*) \prod_{p \text{ good prime}} \frac{1}{P_{i,p}(p^{-s})}$$

where $(*)$ stands for the Euler factors corresponding to the primes of bad reduction. The most interesting is the middle L–series

$$L(X,s) = L(H^n_{\text{ét}}(\bar{X},\mathbb{Q}_l),s).$$

Frobenius morphism

The L–series has expansion $L(X,s)=\sum_{k=1}^{\infty}\frac{a_k(X)}{k^s}$.
By the proof of Weil Conjecture

$$P_{i,p}(t)=det(1-t\,\mathrm{Frob}_p^*\,|H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l))$$

and so

$$a_p(X)=\mathrm{tr}(\mathrm{Frob}_p^*\,|H^i_{\text{ét}})$$

for p prime, and $a_k(X)$ can be recovered from $a_p(X)$ for p prime factors of k.
Lefschetz fixed point formula in dimension 3

$$\#X_{p^r}=\sum_{i=0}^{6}(-1)^i\,\mathrm{tr}(\mathrm{Frob}_p^*\,|H^i_{\text{ét}})=1+p^3+\mathrm{tr}(\mathrm{Frob}_p^*\,|H^2_{\text{ét}})(1+p)-a_p$$

Every Calabi–Yau manifold is modular in the sense that its L–series is L–series of some automorphic form. In dimension one it is the Taniyama–Shimura–Weil Conjecture. More generally, from Serre's Conjecture follows the following Modularity Theorem

Theorem

Lest X be an n dimesnional Calabi-Yau manifold defined over $\mathbb{Q}$ and such that $b_n(X) = 2$. Then there exists a modular form f of weight $n+1$ for the congruence sungroup $\Gamma_0(N)$, where N is a natural number divisible only by primes of bad reduction of X, such that

$$L(X, s) = (*)L(f, s).$$

There exist Calabi-Yau threefolds with $h^{1,2} > 0$ such that $L(X, s) = L(f, s) \prod_{i=1}^{h^{1,2}} L(g_i, s-1)$.
B. van Geemen and J. Werner constructed a quintic $X \subset \mathbb{P}^4$ with 120 nodes and $h^{1,1}(X) = 21$, $h^{1,2}(X) = 1$, Consani and Scholten constructed a Hilbert modular form h, Dieulefait, Pacetti and Schütt proved that small resolution $\hat{X}$ of X is modular with modular form h.

Let X be the double octic Calabi-Yau threefolds constructed as a resolution of the double covering of $\mathbb{P}^3$ branched along the following 8 hyperplanes:

$$\{u^2 = x(x-z)(x-v)(x-z-v)y(y-z)(y-v)(y+v+2z) = 0\} \subset \mathbb{P}(1^4, 4).$$

Variety X has Hodge numbers

$$h^{11}(X) = 37, \quad h^{12}(X) = 1,$$

the only primes of bad reduction of X are 2 and 3 and it is birational to the Kummer fibration (induced by the map $(x, y, z, v, u) \mapsto (z, v)$) corresponding to the following fiber product

∞	0	1	-1	$-\frac{1}{2}$	$-\frac{1}{3}$
I_4	I_4	I_2	I_2		
I_2	I_2	I_2	I_2	I_2	I_2

One parameter family

X is isomorphic to element corresponding to $t = -1/2$ of the one parameter family defined by Arr. No 250 in [Meyer].
This family has a corresponding Picard-Fuchs operator, the order 4 ordinary differential operator satisfied by the period integral

$$f(t) = \int_{\gamma_t} \omega_t.$$

The Picard-Fuchs operator has the followng local exponents

$$\left\{\begin{array}{cccccc} -2 & -1 & -1/2 & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\ 2 & 1 & 4 & 1 & 2 & 3/2 \end{array}\right\}$$

The operator is symmetric with respect to the reflection $t \mapsto -1-t$ with fixed point $t = -1/2$. The family is not symmetric in an obvious way, but the elements X_t and X_{-1-t} are related.

The Kummer fibration for X_t is

∞	0	1	-1	t	$\frac{t}{2+t}$
I_4	I_4	I_2	I_2		
I_2	I_2	I_2	I_2	I_2	I_2

Pulling back by the map $t \mapsto \frac{t+1}{t-1}$ we get

∞	0	1	-1	$\frac{-1-t}{2+(-1-t)}$	$-1-t$
I_2	I_2	I_4	I_4		
I_2	I_2	I_2	I_2	I_2	I_2

Swapping I_2 and I_4 fibers of the first surface is given by an isogeny, swapping last two I_2 fibers on the second surface is a birational map.

From this description we get a two-to-one rational map $\Psi : X \longrightarrow X$ defined over $\mathbb{Q}[\sqrt{2}]$ by

$$\Psi : \begin{pmatrix} x \\ y \\ z \\ v \\ u \end{pmatrix} \mapsto \begin{pmatrix} x\,(x - v - z)\,(z - v)\,(3\,y + v)\,, \\ \frac{1}{2}\,(3\,z + v)\left(v^2 - 2\,xv + zv + 2\,x^2 - 2\,xz\right)(y - v)\,, \\ \frac{1}{2}\left(v^2 - 2\,xv + zv + 2\,x^2 - 2\,xz\right)(3\,y + v)\,(z + v)\,, \\ \frac{1}{2}\left(v^2 - 2\,xv + zv + 2\,x^2 - 2\,xz\right)(3\,y + v)\,(z - v)\,, \\ \frac{\sqrt{2}}{2}\,(v - z)\,(v + 3\,y)^2\,v^2\,(2\,x - v - z)\,(v + z) \times \\ \times (3\,z + v)\left(v^2 - 2\,xv + zv + 2\,x^2 - 2\,xz\right)^2 u \end{pmatrix}$$

We have

$$\Psi^*\omega_X = \sqrt{2}\omega_X \text{ and } \Psi^*\bar{\omega}_X = \sqrt{2}\bar{\omega}_X$$

so Ψ^* acts as multiplication by $\sqrt{2}$ on $H^{3,0}(X) \oplus H^{0,3}(X)$.
Similarly Ψ^* acts as multiplication by $-\sqrt{2}$ on $H^{1,2}(X) \oplus H^{2,1}(X)$.

The map Ψ decomposes the motive $H^3(X)$ into a direct sum of two two-dimensional submotives

$$H^3(X) = H^3_+ \oplus H^3_-$$

The Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ preserves H^3_+ and H^3_-, so it defines two Galois-conjugate Galois representations

$$\rho, \bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}]) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_2[\sqrt{2}]).$$

Using point count in $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ and Lefschetz fixed point formula we compute the traces

$$a_p = \mathrm{tr}(\mathrm{Frob}_p\,|H^3(X_p)) \quad \text{and} \quad a_{p^2} = \mathrm{tr}(\mathrm{Frob}_{p^2}\,|H^3(X_p)).$$

Frobenius polynomial equals

$$F_p = X^4 - a_p X^3 - \tfrac{1}{2}(a_p^2 + a_{p^2})X^2 - a_p p^3 X + p^6.$$

By "real multiplication" $a_p = 0$ when p is innert in $\mathbb{Q}[\sqrt{2}]$.

Frobenius polynomials

p	a_p	a_{p^2}	F_p
5	0	20	$X^4 - 10\,X^2 + 15625$
7	32	-796	$X^4 - 32\,X^3 + 910\,X^2 - 10976\,X + 117649$ $(X^2 + 4\sqrt{2}X - 16\,X + 343) \times (X^2 - 4\sqrt{2}X - 16\,X + 343)$
11	0	-1452	$X^4 + 726\,X^2 + 1771561$
13	0	5876	$X^4 - 2938\,X^2 + 4826809$
17	-124	-10940	$X^4 + 124\,X^3 + 13158\,X^2 + 609212\,X + 24137569$ $(X^2 + 16\sqrt{2}X + 62\,X + 4913) \times (X^2 - 16\sqrt{2}X + 62\,X + 4913)$
23	80	-45212	$X^4 - 80\,X^3 + 25806\,X^2 - 973360\,X + 148035889$ $(X^2 + 8\sqrt{2}X - 40\,X + 12167) \times (X^2 - 8\sqrt{2}X - 40\,X + 12167)$
31	272	-59068	$X^4 - 272\,X^3 + 66526\,X^2 - 8103152\,X + 887503681$ $(X^2 - 76\sqrt{2}X - 136\,X + 29791) \times (X^2 + 76\sqrt{2}X - 136\,X + 29791)$
41	84	-148252	$X^4 - 84\,X^3 + 77654\,X^2 - 5789364\,X + 4750104241$ $(X^2 - 176\sqrt{2}X - 42\,X + 68921) \times (X^2 + 176\sqrt{2}X - 42\,X + 68921)$
47	-64	-134460	$X^4 + 64\,X^3 + 69278\,X^2 + 6644672\,X + 10779215329$ $(X^2 + 264\sqrt{2}X + 32\,X + 103823) \times (X^2 - 264\sqrt{2}X + 32\,X + 103823)$
89	-2476	507556	$X^4 + 2476\,X^3 + 2811510\,X^2 + 1745503244\,X + 496981290961$ $(X^2 + 256\sqrt{2}X + 1238\,X + 704969) \times (X^2 - 256\sqrt{2}X + 1238\,X + 704969)$
97	1284	-2822268	$X^4 - 1284\,X^3 + 2235462\,X^2 - 1171872132\,X + 832972004929$ $(X^2 + 32\sqrt{2}X - 642\,X + 912673) \times (X^2 - 32\sqrt{2}X - 642\,X + 912673)$

Traces of the Hilbert modular form h computed with MAGMA

2	0	3	9
5	10	$(7,\sqrt{2}+3)$	$4\sqrt{2}+16$
$(7,\sqrt{2}+4)$	$-4\sqrt{2}+16$	11	-726
13	2938	$(17,\sqrt{2}+11)$	$-16\sqrt{2}-62$
$(17,\sqrt{2}+6)$	$16\sqrt{2}-62$	19	6650
$(23,\sqrt{2}+18)$	$8\sqrt{2}+40$	$(23,\sqrt{2}+5)$	$-8\sqrt{2}+40$
29	23258	$(31,\sqrt{2}+23)$	$-76\sqrt{2}+136$
$(31,\sqrt{2}+8)$	$76\sqrt{2}+136$	37	4810
$(41,\sqrt{2}+17)$	$176\sqrt{2}+42$	$(41,\sqrt{2}+24)$	$-176\sqrt{2}+42$
43	-74390	$(47,\sqrt{2}+40)$	$264\sqrt{2}-32$
$(47,\sqrt{2}+7)$	$-264\sqrt{2}-32$	53	-60950
59	-143606	61	107482
67	122074	$(71,\sqrt{2}+12)$	$56\sqrt{2}-104$
$(71,\sqrt{2}+59)$	$-56\sqrt{2}-104$	$(73,\sqrt{2}+32)$	$544\sqrt{2}-326$
$(73,\sqrt{2}+41)$	$-544\sqrt{2}-326$	$(79,\sqrt{2}+70)$	$-812\sqrt{2}-40$
$(79,\sqrt{2}+9)$	$812\sqrt{2}-40$	83	-55942
$(89,\sqrt{2}+64)$	$-256\sqrt{2}-1238$	$(89,\sqrt{2}+25)$	$256\sqrt{2}-1238$
$(97,\sqrt{2}+14)$	$-32\sqrt{2}+642$	$(97,\sqrt{2}+83)$	$32\sqrt{2}+642$

For an inert prime p (in the previous slide) the trace of the modular form agrees with the trace $\mathrm{tr}(\rho(F_p))$ for the Frobenius element F_p of the field $\mathbb{Q}[\sqrt{2}]$. For a prime $\mathfrak{p}$ in $\mathbb{Q}[\sqrt{2}]$ over a split prime p however the trace of the modular form agrees with the trace $\mathrm{tr}(\rho(F_\mathfrak{p}))$ or $\mathrm{tr}(\bar{\rho}(F_\mathfrak{p}))$.
For instance let $p = 7$, then $(7) = (7, \sqrt{2}+4)(7, \sqrt{2}+3)$ and the traces of h equals:
$(7, \sqrt{2}+4)$: $-4\sqrt{2}+16$, $\quad (7, \sqrt{2}+3)$: $4\sqrt{2}+16$
while the Frobenius polynomial equals:
$(X^2 - (4\sqrt{2}+16)X + 343) \times (X^2 - (-4\sqrt{2}+16)X + 343)$.
We shall prove however

Theorem

The Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ *on the motive* H^3_+ *is isomorphic to the Galois representation of the Hilbert modular form* h *for* $K = \mathbb{Q}[\sqrt{2}]$ *of weight* $[4, 2]$ *and level* $6\sqrt{2}\mathcal{O}_K$.

Proposition

Let $K=\mathbb{Q}[\sqrt{2}]$ *and* $E=\mathbb{Q}_2[\sqrt{2}]$ *and let* $\mathcal{P}:=\sqrt{2}\,\mathbb{Z}_2$ *be the maximal ideal of the ring of integers of* E. *Let* $S:=\{\sqrt{2},3\}$ *and*

$$\begin{aligned} T &= \{5, 11, \sqrt{2}+3,\ \sqrt{2}-3,\ 3\sqrt{2}-1,\ \sqrt{2}+5,\ \sqrt{2}-5,\\ &\quad 4\sqrt{2}-1,\ 4\sqrt{2}+1,\ 5\sqrt{2}-3,\ \sqrt{2}-7,\ \sqrt{2}+7,\\ &\quad 4\sqrt{2}-11,\ 1-7\sqrt{2}\}\\ U &= \{5, 11, 13,\ \sqrt{2}-3,\ 3\sqrt{2}-1,\ \sqrt{2}-5,\ 4\sqrt{2}-1,\ 5\sqrt{2}-3\} \end{aligned}$$

be two sets of primes in $\mathcal{O}_K$. *Suppose that* $\rho_1,\rho_2:\mathrm{Gal}(\bar{K}/K)\longrightarrow \mathrm{GL}_2(E)$ *are continuous Galois representations unramified outside* S *and satisfying*

1. $\mathrm{Tr}(\rho_1(\mathrm{Frob}_\mathfrak{p}))\equiv \mathrm{Tr}(\rho_2(\mathrm{Frob}_\mathfrak{p}))\equiv 0 \pmod{\mathcal{P}}$ *for* $\mathfrak{p}\in U$,
2. $\det(\rho_1)\equiv\det(\rho_2) \pmod{\mathcal{P}}$,
3. $\mathrm{Tr}(\rho_1(\mathrm{Frob}_\mathfrak{p}))=\mathrm{Tr}(\rho_2(\mathrm{Frob}_\mathfrak{p}))$ *and* $\det(\rho_1(\mathrm{Frob}_\mathfrak{p}))=\det(\rho_2(\mathrm{Frob}_\mathfrak{p}))$ *for* $\mathfrak{p}\in T$.

Then ρ_1 *and* ρ_2 *have isomorphic semisimplifications.*

Following the arguments of [Livné] we first verify that assumption 1. implies that $\mathrm{Tr}(\rho_1) \equiv \mathrm{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$. If $\mathrm{Tr}(\rho_i) \not\equiv 0 \pmod{\mathcal{P}}$ denote by L/K the Galois extension cut out by the kernel $\mathrm{Ker}\,\bar{\rho}_i$ of the reduction $\bar{\rho}_i$ of ρ_i modulo $\mathcal{P}$.
The Galois group of the extension L/K is isomorphic to S_3 or C_3, so it is the Galois closure of a degree 3 extension M/K. Then M is a degree 6 extension of $\mathbb{Q}$ unramified outside $\{2, 3\}$.
Jones and Roberts made a list 398 such fields presented as a splitting field of a monic degree 6 polynomial with rational coefficients. The assumption that M contains the subfield $K = \mathbb{Q}[\sqrt{2}]$ implies that the minimal polynomial of any primitive element of the extension $M/\mathbb{Q}$ factors over $\mathbb{Q}[\sqrt{2}]$.
Exactly 25 of these 398 polynomials from satisfy this condition. For each of them we determine a prime integer p such that the reduction of the degree 3 polynomial over K modulo a prime $\mathfrak{p}$ in $\mathbb{O}_K$ over p stays irreducible over $\mathbb{O}_K/\mathfrak{p} \cong \mathbb{F}_{\mathfrak{p}}$.

We list these data below.

$$x^6-2x^3-1=\left(x^3+\sqrt{2}-1\right)\times\left(x^3-\sqrt{2}-1\right),\ p=5$$
$$x^6-12x^4+36x^2-8=\left(x^3-6x-2\sqrt{2}\right)\times\left(x^3-6x+2\sqrt{2}\right),\ p=5$$
$$x^6-2=\left(x^3+\sqrt{2}\right)\times\left(x^3-\sqrt{2}\right),\ p=7$$
$$x^6-4x^3+2=\left(x^3+\sqrt{2}-2\right)\times\left(x^3-\sqrt{2}-2\right),\ p=5$$
$$x^6+6x^4+9x^2-8=\left(x^3+3x+2\sqrt{2}\right)\times\left(x^3+3x-2\sqrt{2}\right),\ p=11$$
$$x^6+6x^4+9x^2-2=\left(x^3+3x-\sqrt{2}\right)\times\left(x^3+3x+\sqrt{2}\right),\ p=5$$
$$x^6-6x^4-6x^3+12x^2-36x+1=\left(x^3+3\sqrt{2}x^2+6x+2\sqrt{2}-3\right)\times\left(x^3-3\sqrt{2}x^2+6x-2\sqrt{2}-3\right),\ p=7$$
$$x^6-18=\left(x^3-3\sqrt{2}\right)\times\left(x^3+3\sqrt{2}\right),\ p=7$$
$$x^6-6x^4-12x^3+12x^2-72x+28=\left(x^3-3\sqrt{2}x^2+6x-2\sqrt{2}-6\right)\times\left(x^3+3\sqrt{2}x^2+6x+2\sqrt{2}-6\right),\ p=13$$

Faltings–Serre–Livné method – proof III

$$x^6 - 6\,x^3 - 9 = \left(x^3 - 3\sqrt{2} - 3\right) \times \left(x^3 + 3\sqrt{2} - 3\right),\ p = 5$$

$$x^6 - 6\,x^4 - 4\,x^3 + 9\,x^2 + 12\,x - 14 = \left(x^3 - 3\,x - 3\sqrt{2} - 2\right) \times \left(x^3 - 3\,x + 3\sqrt{2} - 2\right),\ p = 5$$

$$x^6 - 18\,x^4 - 12\,x^3 + 81\,x^2 + 108\,x + 18 = \left(x^3 - 9\,x + 3\sqrt{2} - 6\right) \times \left(x^3 - 9\,x - 3\sqrt{2} - 6\right),\ p = 5$$

$$x^6 + 6\,x^4 - 4\,x^3 - 9\,x^2 + 12\,x - 4 = \left(x^3 + 3\sqrt{2}x + 3\,x - 2\sqrt{2} - 2\right) \times \left(x^3 - 3\sqrt{2}x + 3\,x + 2\sqrt{2} - 2\right),\ p = 31$$

$$x^6 + 6\,x^4 - 4\,x^3 + 9\,x^2 - 12\,x - 4 = \left(x^3 + 3\,x + 2\sqrt{2} - 2\right) \times \left(x^3 + 3\,x - 2\sqrt{2} - 2\right),\ p = 23$$

$$x^6 - 6\,x^4 - 4\,x^3 + 9\,x^2 + 12\,x - 4 = \left(x^3 - 3\,x - 2\sqrt{2} - 2\right) \times \left(x^3 - 3\,x + 2\sqrt{2} - 2\right),\ p = 5$$

$$x^6 - 12\,x^3 + 18 = \left(x^3 + 3\sqrt{2} - 6\right) \times \left(x^3 - 3\sqrt{2} - 6\right),\ p = 5$$

$$x^6-12\,x^3-36=\left(x^3+6\,\sqrt{2}-6\right)\times\left(x^3-6\,\sqrt{2}-6\right),\ p=5$$

$$x^6-6\,x^4-4\,x^3-9\,x^2-12\,x-4=\left(x^3-3\,\sqrt{2}x-3\,x-2\,\sqrt{2}-2\right)\times\left(x^3+3\,\sqrt{2}x-3\,x+2\,\sqrt{2}-2\right),\ p=41$$

$$x^6-8\,x^3-18\,x^2-48\,x-16=\left(x^3-3\,\sqrt{2}x-4\,\sqrt{2}-4\right)\times\left(x^3+3\,\sqrt{2}x+4\,\sqrt{2}-4\right),\ p=7$$

$$x^6+6\,x^4-12\,x^3+9\,x^2-36\,x+28=\left(x^3+3\,x-2\,\sqrt{2}-6\right)\times\left(x^3+3\,x+2\,\sqrt{2}-6\right),\ p=17$$

$$x^6-8\,x^3-18\,x^2+24\,x+8=\left(x^3-3\,\sqrt{2}x+2\,\sqrt{2}-4\right)\times\left(x^3+3\,\sqrt{2}x-2\,\sqrt{2}-4\right),\ p=13$$

$$x^6-16\,x^3-18\,x^2+48\,x+32=\left(x^3+3\,\sqrt{2}x-4\,\sqrt{2}-8\right)\times\left(x^3-3\,\sqrt{2}x+4\,\sqrt{2}-8\right),\ p=11$$

$$x^6 - 18\,x^4 - 36\,x^3 - 81\,x^2 - 108\,x + 36 = \left(x^3 - 9\sqrt{2}x - 9\,x - 12\sqrt{2} - 18\right) \times \left(x^3 + 9\sqrt{2}x - 9\,x + 12\sqrt{2} - 18\right), \quad p = 11$$

$$x^6 - 18\,x^4 - 12\,x^3 + 81\,x^2 + 108\,x - 36 = \left(x^3 - 9\,x - 6\sqrt{2} - 6\right) \times \left(x^3 - 9\,x + 6\sqrt{2} - 6\right), \quad p = 11$$

$$x^6 - 18\,x^4 - 36\,x^3 + 81\,x^2 + 324\,x + 252 = \left(x^3 - 9\,x - 6\sqrt{2} - 18\right) \times \left(x^3 - 9\,x + 6\sqrt{2} - 18\right), \quad p = 11$$

Given M and $\mathfrak{p}$ as above, it follows that any element in the conjugacy class of $\mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{Gal}(L/K)$ has order 3; consequently $\mathrm{Tr}(\rho_i(\mathrm{Frob}_{\mathfrak{p}})) \equiv 1 \pmod{\mathcal{P}}$, contradicting our assumptions. Thus we see that the set U was indeed chosen in such a way that condition 1. implies that

$$\mathrm{Tr}(\rho_1) \equiv \mathrm{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}.$$

Faltings–Serre–Livné method – proof VI

Let K_S be the compositum of all quadratic extensions of K unramified outside S. Since the ring $\mathcal{O}_K$ is a unique factorization domain, generators of K_S/K can be taken as

$$\sqrt{-1}, \sqrt[4]{2}, \sqrt{\sqrt{2}-1}, \sqrt{3}.$$

The table of quadratic characters $\mathrm{Gal}(K_S/K) \longrightarrow (\mathbb{Z}/2)^4$ at the primes from T as follows:

$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{2}$	3	-1	$\sqrt{2}-1$	$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{2}$	3	-1	$\sqrt{2}-1$
5	25	1	0	0	0	$4\sqrt{2}-1$	31	0	1	1	0
11	121	0	0	0	1	$4\sqrt{2}+1$	31	1	1	1	1
$\sqrt{2}+3$	7	0	1	1	1	$5\sqrt{2}-3$	41	1	1	0	0
$\sqrt{2}-3$	7	1	1	1	0	$\sqrt{2}-7$	47	0	0	1	0
$3\sqrt{2}-1$	17	1	1	0	1	$\sqrt{2}+7$	47	1	0	1	1
$\sqrt{2}+5$	23	0	0	1	1	$4\sqrt{2}-11$	89	0	1	0	1
$\sqrt{2}-5$	23	1	0	1	0	$1-7\sqrt{2}$	97	1	0	0	1

The image of the Frobenius elements $\mathrm{Frob}_t, t \in T$, contains 14 different non-zero elements, hence it is non–cubic. By [Livné, Thm. 4.3] the Galois representations ρ_1, ρ_2 have isomorphic semisimplifications.

Trace of Ψ^* on $H^i(X)$

To prove theorem we have to compute $\mathrm{tr}(\mathrm{Frob}_\mathfrak{p}\,|H^3_+)$ for $\mathfrak{p} \in T \cup U$, we need to care only for $\mathfrak{p}$ over a split prime $p \in \mathbb{Z}$. Let start with traces of Ψ^* on $H^i(X)$. The following are obvious

$$\mathrm{tr}(\Psi^*|H^0) = 1, \;\; \mathrm{tr}(\Psi^*|H^1) = \mathrm{tr}(\Psi^*|H^5) = 0, \;\; \mathrm{tr}(\Psi^*|H^6) = 2.$$

As Ψ^* the eigenvalues of Ψ^* on $H^3(X)$ equal $\sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}$

$$\mathrm{tr}(\Psi^*|H^3) = 0.$$

Ψ preserves the Kummer fibration and maps the fiber at (z, v) into the fiber at $(z + v, z - v)$. To compute Lefschetz number of Ψ we can restrict ourselves to the fibers at $(1 \pm \sqrt{2}, 1)$ where the fiber is the Kummer surface of the product of the elliptic curves

$$u^2 = x^3 - 30x + 56 \quad \text{and} \quad u^2 = y^3 - y,$$

and the map Ψ is induced by the complex multiplications given by

$$x \longmapsto -\frac{x^2 - 4x + 18}{2(x - 4)} \quad \text{and} \quad y \longmapsto -\frac{y + 1}{y - 1}.$$

Trace of Ψ^* on $H^i(X)$

Using MAGMA we computed $\mathcal{L} = 12$ and so

$$\operatorname{tr}(\Psi^*|H^2) + \operatorname{tr}(\Psi^*|H^4) = 9.$$

In a similar manner we computed the Lefschetz numbers of $\operatorname{Frob}_{\mathfrak{p}} \circ \Psi$ on $H^3(X)$

$\mathfrak{p}$	$3+\sqrt{2}$	$3-\sqrt{2}$	$3\sqrt{2}-1$	$5+\sqrt{2}$
$N(\mathfrak{p})$	7	7	17	23
$\mathcal{L}$	944	976	11404	27104
$\mathfrak{p}$	$5-\sqrt{2}$	$4\sqrt{2}+1$	$4\sqrt{2}-1$	$5\sqrt{2}-3$
$N(\mathfrak{p})$	23	31	31	41
$\mathcal{L}$	27040	64208	64816	147116
$\mathfrak{p}$	$\sqrt{2}+7$	$\sqrt{2}-7$	$4\sqrt{2}-11$	$1-7\sqrt{2}$
$N(\mathfrak{p})$	47	47	89	97
$\mathcal{L}$	219936	217824	1450924	1872652

Proof of thm.

We have for any split prime $p \in \mathbb{Z}$ and any prime $\mathfrak{p} \in \mathbb{Z}[\sqrt{2}]$ over p

$$\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}}^* \circ \Psi) = 1 + p\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}\left(\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_+) - \mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_-)\right) + p^2\,\mathrm{tr}(\Psi^*|H^4) + 2p^3.$$

In the case $p = 7$, $\mathfrak{p} = 3 + \sqrt{2}$ we get two possibilities

$$976 = 1 + 7\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}((16 - 4\sqrt{2}) - (16 + 4\sqrt{2})) + 49\,\mathrm{tr}(\Psi^*|H^4) + 686$$

or

$$976 = 1 + 7\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}((16 + 4\sqrt{2}) - (16 - 4\sqrt{2})) + 49\,\mathrm{tr}(\Psi^*|H^4) + 686,$$

Equivalently,

$$273 = 7(\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4)) \text{ or } 305 = 7(\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4))$$

As $7 \nmid 305$, the second option is impossible and consequently

$$\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4) = 39.$$

Proof of thm.

Together with $\operatorname{tr}(\Psi^*|H^2) + \operatorname{tr}(\Psi^*|H^4) = 9$, this yields

$$\operatorname{tr}(\Psi^*|H^2) = 4, \qquad \operatorname{tr}(\Psi^*|H^4) = 5.$$

Now, we get

$$\sqrt{2}(\operatorname{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_+) - \operatorname{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_-)) = -\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}}^* \circ \Psi) + 1 + 4p + 5p^2 + 2p^3.$$

Since

$$\operatorname{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_+) + \operatorname{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_-) = \operatorname{tr}(\mathrm{Frob}_{p}^*|H^3(X))$$

and we can compute $\operatorname{tr}(\mathrm{Frob}_{\mathfrak{p}}^*|H^3_+)$ and conclude the proof.

Modular forms h and $\bar{h}$

The Galois conjugate modular form has weight $[2, 4]$, there is an important difference between h and $\bar{h}$, for any split prime p and any prime $\mathfrak{p}$ over p we have $\mathfrak{p} \mid a_{\mathfrak{p}}$, where $a_{\mathfrak{p}}$ is the coefficient of $\bar{h}$. There is no such divisibility for h.

2	0	3	9
5	10	$(7, \sqrt{2}+3)$	$-4\sqrt{2}+16$
$(7, \sqrt{2}+4)$	$4\sqrt{2}+16$	11	-726
13	2938	$(17, \sqrt{2}+11)$	$16\sqrt{2}-62$
$(17, \sqrt{2}+6)$	$-16\sqrt{2}-62$	19	6650
$(23, \sqrt{2}+18)$	$-8\sqrt{2}+40$	$(23, \sqrt{2}+5)$	$8\sqrt{2}+40$
29	23258	$(31, \sqrt{2}+23)$	$76\sqrt{2}+136$
$(31, \sqrt{2}+8)$	$-76\sqrt{2}+136$	37	4810
$(41, \sqrt{2}+17)$	$-176\sqrt{2}+42$	$(41, \sqrt{2}+24)$	$176\sqrt{2}+42$
43	-74390	$(47, \sqrt{2}+40)$	$-264\sqrt{2}-32$
$(47, \sqrt{2}+7)$	$264\sqrt{2}-32$	53	-60950
59	-143606	61	107482
67	122074	$(71, \sqrt{2}+12)$	$-56\sqrt{2}-104$
$(71, \sqrt{2}+59)$	$56\sqrt{2}-104$	$(73, \sqrt{2}+32)$	$-544\sqrt{2}-326$
$(73, \sqrt{2}+41)$	$544\sqrt{2}-326$	$(79, \sqrt{2}+70)$	$812\sqrt{2}-40$
$(79, \sqrt{2}+9)$	$-812\sqrt{2}-40$	83	-55942
$(89, \sqrt{2}+64)$	$256\sqrt{2}-1238$	$(89, \sqrt{2}+25)$	$-256\sqrt{2}-1238$
$(97, \sqrt{2}+14)$	$32\sqrt{2}+642$	$(97, \sqrt{2}+83)$	$-32\sqrt{2}+642$

Modular forms h and $\bar{h}$

This property has geometric motivation: h is modular form for a Galois representation on $H^{3,0} \oplus H^{0,3}$, while h is a modular form for Galois representation on $H^{2,1} \oplus H^{1,2}$.
In the case of Consani-Scholten quintic we have the same divisibility for Hilbert modular form (in fact it is always the case for Hilbert modular form of weight [2,4]), so we can expect that the decomposition of the Galois action has a geometric origin.
There is at least one more Hilbert modular double octic with $h^{1,2} = 1$ (Arr. no 10 with $B/A = 1$ or $B/A = -1/2$), this time there is no geometric explanation - modularity follows by point count from Grenié (it is enough to find traces for $p = 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23, 31,\ 73,\ 137,\ 257,\ 337$ or characteristic polynomials for $p = 5, 7, 11, 17, 23, 31$.

There exist a rigid Calabi-Yau double octic defined over $\mathbb{Z}[\sqrt{5}]$, it is a double cover of $\mathbb{P}^3$ branched along an arrangement of eight planes (with $\varphi = \frac{1}{2}(-1+\sqrt{5})$)

$$xyzt\,(x+y+z)\,(\varphi y - z + t)\,(x+y+\varphi t)\,((1-\varphi)x + y - \varphi z + \varphi t)\,.$$

In this case couting points over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ is enough to get the traces of $\mathrm{Frob}_{\mathfrak{p}}$ and $\mathrm{Frob}_{\mathfrak{p}^2}$

p	$\mathfrak{p}$	φ	$n_{\mathfrak{p}}$	$n_{\mathfrak{p}^2}$	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}^2})$
11	$\sqrt{5}+4$	3	1459	1784297	60	938
	$\sqrt{5}-4$	7	1461	1786601	36	-1366
29	$2\sqrt{5}+7$	5	25217	595525129	-218	-1254
	$2\sqrt{5}-7$	23	25089	595564553	-90	-40678
31	$\sqrt{5}+6$	12	30685	888442233	192	-22718
	$\sqrt{5}-6$	18	31003	888475001	-64	-55486
61	$2\sqrt{5}-9$	17	230471	51534519081	354	-328646
	$2\sqrt{5}+9$	43	230215	51534272297	610	-81862

Hilbert modular form

Using MAGMA we found a Hecke eigenform of weight [4,4] and level 16 on $\mathbb{Q}[\sqrt{5}]$ with the same traces

2	0	3	14	$\sqrt{5}$	10
7	-74	$(11, \sqrt{5}+4)$	60	$(11, \sqrt{5}+7)$	36
13	-3942	17	2146	$(19, \sqrt{5}+9)$	-68
$(19, \sqrt{5}+10)$	100	23	-7210	$(29, \sqrt{5}+18)$	-218
$(29, \sqrt{5}+11)$	-90	$(31, \sqrt{5}+25)$	-64	$(31, \sqrt{5}+6)$	192
37	-31190	$(41, \sqrt{5}+28)$	-434	$(41, \sqrt{5}+13)$	334
43	-139522	47	182310	53	40330
$(59, \sqrt{5}+8)$	-820	$(59, \sqrt{5}+51)$	148	$(61, \sqrt{5}+35)$	610
$(61, \sqrt{5}+26)$	354	67	321614	$(71, \sqrt{5}+54)$	-24
$(71, \sqrt{5}+16)$	472	73	380146	$(79, \sqrt{5}+20)$	1040
$(79, \sqrt{5}+59)$	496	83	-47186	$(89, \sqrt{5}+19)$	-1302
$(89, \sqrt{5}+70)$	1002	97	977730		

Theorem

The Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{5}])$ on $H^3_{et}(Y, \mathbb{Q}_l)$ is Hilbert modular with the above Hilbert modular.

Let Z be the double octic defined as a resolution of singularities of the hypersurface

$$u^2 = xyzv\,(x+y)\,(x+y+z-v)\,(\zeta\,x-y+\zeta\,z)\,(y-\zeta\,z-v) \subset \mathbb{P}(1,1,1,1,4),$$

where $\zeta = \frac{1}{2}(-1+i\sqrt{3})$.Then Z is a rigid Calabi-Yau threefold defined over $\mathbb{Q}[\sqrt{-3}]$ with $h^{11} = 46$ (isomorphic to a member of the one-dimensional family of double octics given by arrangement No. 262 in [Meyer]).The only prime of bad reduction of Z is 2.

Proposition

Z is birational to a Calabi-Yau threefold defined over $\mathbb{Q}[i]$.

We can count points over $\mathbb{F}_p$ only if $p \equiv 1 \pmod 6$, i.e. p is a split prime in K. Above a given split prime p there are two prime ideals $\mathfrak{p}$ in the ring of integers of $\mathbb{Q}[\sqrt{-3}]$; this corresponds to two choices for $\zeta \in \mathbb{F}_{\mathfrak{p}}$ and two possibilities for the trace of Frobenius on $H^3(\bar{Z}_{\mathfrak{p}})$ which we list in the next slide.

p	ζ	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$	ζ	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$
7	4	-12	2	12
13	3	-58	9	-58
19	11	-136	7	136
31	25	20	5	-20
37	26	-18	10	-18
43	6	-200	36	200
61	47	-458	13	-458
67	29	-496	37	496
73	64	602	8	602
79	55	1108	23	-1108
97	61	-206	35	-206

The two traces of $\mathrm{Frob}_{\mathfrak{p}}$ coincide if $p \equiv 1 \pmod 4$ and are opposite if $p \equiv 3 \pmod 4$.

The computed traces agree up to sign with the Fourier coefficients of a modular form f of weight 4 for $\Gamma_0(72)$ (72/1 in Meyer's notation):

p	7	13	19	31	37	43	61	67	73	79	97
a_p	-12	58	-136	20	-18	-200	-458	-496	-602	1108	206

We observe that proper choices of signs are governed by the character corresponding to the extension $\mathbb{Q}[\sqrt[4]{-3}]/\mathbb{Q}[\sqrt{-3}]$.

Theorem

Consider the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{-3}])$ *on* $H^3_{et}(\bar{Z}, \mathbb{Q}_l)$ *and the one associated to the modular form* f *restricted to* $\mathbb{Q}[\sqrt{-3}]$ *and then twisted by the quadratic character associated to the extension* $\mathbb{Q}[\sqrt[4]{-3}]/\mathbb{Q}[\sqrt{-3}]$. *Then the Galois representations have isomorphic semi-simplifications.*

Let F be a totally real field of degree m over $\mathbb{Q}$ and let $\sigma_1, \dots, \sigma_m$ be the real embedings of F. We get a map $\mathrm{GL}_2(F) \longrightarrow \mathrm{GL}_2(\mathbb{R})^m$. The group $\mathrm{GL}_2^+(\mathcal{O}_F)$ acts on $\mathcal{H}^m$ as

$$\gamma \cdot z = (\sigma_1)(\gamma) \cdot z, \dots, \sigma_m)(\gamma) \cdot z).$$

A Hilbert modular form for the full modular group of weights $(k_1, \dots, k_m)$ is an analytic function $f : \mathcal{H}^m \longrightarrow \mathcal{C}$ such that for any $\gamma \in \mathrm{GL}_2^+(\mathcal{O}_F)$

$$f(\gamma z) = \prod_{j=1}^{m} \left(\det \sigma_i(\gamma)^{-k_i/2}\right) (cz_i + d)^{k_i} f(z).$$

If F is a quadratic field and $\mathfrak{a}$ is a fractional ideal of F, then we define

$$\Gamma(\mathcal{O}_F \oplus \mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \in \mathcal{O}_F, b \in \mathfrak{a}^{-1}, c \in \mathfrak{a} \right\}$$

is the Hilbert modular group corresponding to $\mathfrak{a}$.