Lecture Notes in Mathematics 1478
Editors:
A. Dold, Heidelberg
B. Eckmann, Zürich
F. Takens, Groningen

M.-P. Malliavin (Ed.)

Topics in Invariant Theory

Séminaire d'Algèbre P. Dubreil
et M.-P. Malliavin
1989-1990 (40ème Année)

(Revised by the author in 1995)

ALGEBRO - GEOMETRIC APPLICATIONS OF SCHUR S- AND Q-POLYNOMIALS

Piotr Pragacz

To I.M. Gel'fand on his 75-th birthday

INTRODUCTION

The theory of symmetric functions was a significant part of classical algebraic knowledge, starting in the 18-th century with classical elimination theory. Nowadays, one observes a renewal of interest concerning symmetric functions, especially in connection with representation theory and cohomological computations in geometry and topology.

The aim of this paper is to investigate two families of symmetric po-

This paper is in final form and no version of it will be submitted for publication elsewhere.

lynomials, which are useful for algebro-geometric applications. The first family is formed by a variant of Schur S-polynomials symmetric in two sets of independent variables. The second family consists of the so called Schur Q-polynomials. Both families have a lot of parallel properties, which we discuss in the first three sections.

In Section 1 we give three definitions of those polynomials: determinantal, using symmetrizing operators and combinatorial (via tableaux). The first one seems to be the most "compact"; the remaining two are often more convenient to handle in computations.

In Section 2 we discuss factorizations formulas, Pieri's formulas for multiplication and, moreover we characterize S- and Q-polynomials via some cancellations properties (Theorem 2.11).

Not all symmetric polynomials are useful in geometry. Those polynomials which behave regularly with respect to Gysin push forwards are usually important. In Section 3 we show a regular behaviour of S- and Q- polynomials in Gysin push forwards for Grassmannian and flag bundles, adding a new formula in Proposition 3.3(ii). This Section is a supplement of Section 2 in $[P]_3$.

It seems that the source of a similar behaviour of S- and Q-polynomials lies in the fact that they are characters of representations of superalgebras of type gl(n/m) and Q(n) (see [B-R], [K] and $[Se]_1$).

In Section 4 we recall results of Stembridge concerning combinatorial rules for multiplication of two general Q-polynomials and for the decomposition of a Q-polynomial in terms of Schur S-polynomials.

The elimination theory of two polynomials in one variable is developed in Section 5. As a generalization of the resultant, we describe the ideal of polynomials in the coefficients of two generic polynomials, which vanish if after specialization to a field, the resulting polynomials have r+1 roots in common (Theorem 5.3). Some variants of the generic situation, as well as numerous properties of the ideals involved are also discussed. This Section should be considered as a continuation of [L] and $[P]_2$. Theorem 5.3 (i) has already appeared in $[P]_2$; here, by extending the results of the cited paper, we present an alternative method of proof of the main result of $[P]_2$.

In Section 6, by developing Remark 8.7 in $[P]_3$, we answer the following two questions raised in [H-B] p.63 about the Schubert Calculus for Grassmannians of isotropic subspaces $Sp(n)/U(n)$,$SO(2n+1)/U(n)$:

"Is there a Giambelli-type formula that expresses each Schubert class as a polynomial in the special Schubert classes?

Is there a Littlewood-Richardson rule for multiplying arbitrary Schubert classes?"

The answer is given with the help of Schur Q- polynomials (see Theorem 6.17). This section should be treated as a supplement of the paper of Hiller and Boe [H-B].

The symmetric Schur S-polynomials "in a difference of alphabets" that we use in this paper appeared in an implicit way in the works of the 19-th century elimination theory (see e.g. [Po],[T],[Mu]). Nowadays, they have been used in [Th], [K-V], [B-R], $[L\text{-}S]_1$, $[Se]_1$ and $[P]_{1\text{-}3}$. The Schur Q-polynomials were introduced by Schur in [Sch]. Their combinatorial properties were studied recently in [Sa], [W], and $[Ste]_2$.

This paper arose as a by-product of $[P]_{1\text{-}3}$ and it has both research and expository character. We hope that it will be useful for those mathematicians who use in their work symmetric polynomials. The author's contributions are : a characterization of Q-polynomials via the Q-cancellation property (Theorem 2.11(Q)) , an elementary proof of the symmetrization operator formula for the $s_I(A-B)$ (Lemma 2.15), push-forward formulas for S- and especially Q-polynomials in Grassmannian bundles (Theorem 3.3), a generalization of the resultant and some of its analogues (Theorem 5.3), and a Giambelli-type formula for Grassmannians of isotropic subspaces (Proposition 6.6, 6.14 and Theorem 6.17).

During the preparation of this material I benefited from correspondence with R.C.King, A.Lascoux, I.G.Macdonald, A.N.Sergeev, and R.P.Stanley. This work owes a lot to [So]; I am very grateful to I.M.Gel'fand not only for bringing [So] to my attention , but also for giving me some valuable mathematical hints during my visits in Moscow in 1986,87 and 89. I am grateful also to M.Nazarov, V.Serganova, A.N.Tiurin and A.Zelevinsky for making these visits valuable and enjoyable ones.

The results of this paper were reported at the conferences in Banach Center (Warsaw, June 1988) and Oberwolfach (July 1988) and in Séminaire d'Algèbre Dubreil-Malliavin (Paris, October 1988).

The author started to work on this material during his stay in Strasbourg in 1984. The final version of this work has been written during the author's visit in the Department of Mathematics of K.T.H. in Stockholm. My thanks are due to Dan Laksov for his warm hospitality.

Finally, I thank S.Balcerzyk for pointing me out several defects in

the previous version of this paper, and M.P.Malliavin for her patience in waiting for this manuscript.

Notations and conventions

Partitions

By a partition we mean a weakly decreasing sequence $I=(i_1,\dots,i_k)$ of integers where $i_1 \ge i_2 \ge \dots \ge i_k \ge 0$.

Instead of $(\underbrace{i,\dots,i}_{k})$ we will write $(i)^k$.

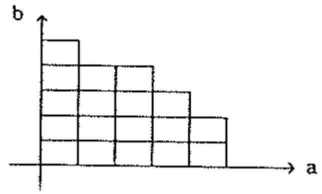
If for some h, $i_1 > i_2 > \dots > i_h > i_{h+1} = i_{h+2} = \dots = i_k = 0$, then I will be called strict.

For given partitions $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_k)$, $I \pm J$ will denote the sequence $(i_1 \pm j_1,\dots,i_k \pm j_k)$ and $I \subset J$ will mean that $i_h \le j_h$ for every h .

If $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_\ell)$ are two sequences of integers, then the juxtaposition sequence $(i_1,\dots,i_k,j_1,\dots,j_\ell)$ will be denoted by I,J .

To every partition $I=(i_1,\dots,i_k)$ one associates its Ferrers' diagram $D_I = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : 1 \le b \le k ,\ 1 \le a \le i_b\}$ .Pictorially, it is convenient to replace (a,b) by 1x1 square with the right upper vertex in (a,b).

For example, $D_{(55431)}$ is equal to



The partition $I\sim$ is called conjugate of I if: $(a,b) \in D_I \iff (b,a) \in D_{I\sim}$.

Moreover, for a given partition I its weight : $\sum_{p=1} i_p$ will be denoted by $|I|$ and its length : card { p , $i_p \ne 0$ } will be denoted by $\ell(I)$.

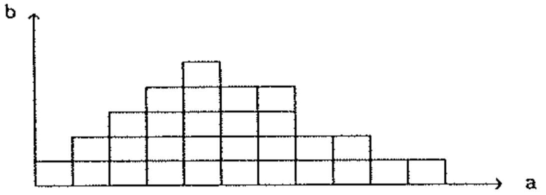
By ρ_k we denote the partition $(k,k-1,\dots,2,1)$.

With a strict partition $I=(i_1,\dots,i_k)$, $\ell(I)=k$, one associates the shifted Ferrers' diagram

$$D^s = \{ (a,b) \in \mathbb{Z}\times\mathbb{Z} : 1 \le b \le k ,\ b \le a \le i_b + b - 1 \}$$

For example,

$D^{s}_{(11,8,5,4,1)}$ =



<u>Symmetric polynomials</u>

If $A=(a_1,\dots,a_n)$ is a sequence of independent indeterminates, then by $\mathcal{S}ym(A)$ we denote a ring all symmetric polynomials in A with coefficients in $\mathbb{Z}$. If $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$, $C=(c_1,\dots,c_r)$,... are sequences of independent indeterminates then by AB, ABC ,... we will denote the corresponding juxtaposition sequences. Moreover, if AB, ABC are sequences of independent variables then we will write

$\mathcal{S}ym\ (A,B)\ :=\ \mathcal{S}ym(A) \otimes \mathcal{S}ym(B)$

$\mathcal{S}ym\ (A,B,C)\ :=\ \mathcal{S}ym(A) \otimes \mathcal{S}ym(B) \otimes \mathcal{S}ym(C)$

etc.

The k-th power sum polynomial will be denoted by $\psi_k(A) = \sum_{i=1}^{n} a_i^k$.

For a given sequence $A=(a_1,\dots,a_n)$ we will write (A) for the class of A under the equivalence relation ~ , where $(a_1,\dots,a_n) \sim (b_1,\dots,b_n)$ iff $a_1=b_{w(1)},\dots,a_n=b_{w(n)}$ for some permutation $w\in S_n$. In other words the sets $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_n\}$ are equal and the multiplicities of pairwaise equal elements in both sequences are the same. Moreover we write A^- for the sequence $(-a_1,\dots,-a_n)$.

1. THREE DEFINITIONS OF SCHUR S- AND Q-POLYNOMIALS.

Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_m)$ be two sequences of elements of a commutative ring R .

I. <u>Determinantal definition</u>

(S) Define $s_i(A-B) \in R$ by

$$\prod_{i=1}^{n}(1-ta_i)^{-1} \prod_{j=1}^{m}(1-tb_j) = \sum_{i=-\infty}^{\infty} s_i(A-B)\ t^i$$

In particular $s_i(A-B) = 0$ for $i < 0$, and for $i > 0$ $s_i(A-B)$ interpolate

between $s_i(A)$ -the i-th complete symmetric polynomial in A and $s_i(-B) = (-1)^i$ (i-th elementary symmetric polynomial in B).

Let $I = (i_1,\dots,i_k) \in \mathbb{Z}^k$. Define

$$s_I(A-B)\ :=\ \mathrm{Det}\left[\ s_{i_p-p+q}(A-B)\ \right] \qquad 1\le p,q \le k$$

If I is a partition (i.e. $i_1 \ge i_2 \ge \dots \ge i_k \ge 0$) and $B = (0,\dots,0)$, then $s_I(A-B) = s_I(A)$ - the classical Schur polynomial in $a_1,\dots,a_n$.

(Q) Define $q_i(A) \in R$ $(i\in\mathbb{Z})$ by

$$\prod_{i=1}^{n}(1+a_it)(1-a_it)^{-1} = \sum_{i=-\infty}^{\infty} q_i(A)\ t^i$$

In particular $q_i(A) = 0$ for $i < 0$ and for $i \ge 0$

$$q_i(A) = \sum_{p=0}^{i} s_p(A)\ s_{(1)^{i-p}}(A) = 2\sum_{I} s_I(A)$$

where the sum is over all hook partitions of length i .

Now, for $i,j \in \mathbb{N}$ ($\mathbb{N}$ denotes here and in the sequel the set of nonnegative integers) we put

$$Q_{i,j}(A) = q_i(A)q_j(A) + 2\sum_{p=1}^{j}(-1)^p q_{i+p}(A)q_{j-p}(A)\ .$$

It is easy to see that for $i>0$, $Q_{(i,0)}(A) = q_i(A)$, and for $i+j>0$, $Q_{i,j}(A) = -Q_{j,i}(A)$.

Finally, if $I = (i_1,\dots,i_k) \in \mathbb{N}^k$ and k is even, we define

$$Q_I(A)\ :=\ \mathrm{Pfaffian}\left[\ Q_{i_s,i_t}(A)\ \right] \qquad 1 \le s,t \le k\ ,$$

and for k-odd $Q_I(A) := Q_{(i_1,\dots,i_k,0)}(A)$.

By Laplace - type expansions for pfaffians we obtain the following equivalent definition through the recurrence formulas:

If k is odd,

$$Q_{i_1,\dots,i_k}(A) = \sum_{p=1}^{k}(-1)^{p-1}q_{i_p}(A)\ Q_{i_1,\dots,\hat{i}_p,\dots,i_k}(A)\ ,$$

and if k is even,

$$Q_{i_1,\dots,i_k}(A) = \sum_{p=1}^{k}(-1)^{p}Q_{i_1 i_p}(A)\ Q_{i_2,\dots,\hat{i}_p,\dots,i_k}(A)\ .$$

Using the recurrence relations it is possible to extend the definition of $Q_I(A)$ for $I\in\mathbb{Z}^k$. The following rule stems from [H-H], Section 9. Let $I=(i_1,\dots,i_k)\in\mathbb{Z}^k$. If, for some $r>0$, the subsequence of those i_p for which

$|i_p| = r$ does not have one of the forms

$$(r,-r,r,-r,\dots,r,-r,r), \qquad (-r,r,-r,r,\dots,-r,r),$$

then $Q_I(A)=0$. Otherwise there is a permutation w which rearranges I into a sequence of the form

$$J,(-r_1,r_1),\dots,(-r_s,r_s),0,\dots,0 ,$$

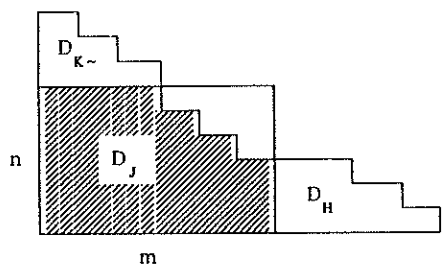
where J is a strict partition, $r_p>0$ $p=1,\dots,s$. A definition of w is as follows. Move all zeros to the right hand end. Move all left most positive parts to the left and rearrange into a decreasing order. Rearrange the middle section so that all $(-r,r)$ occur as adjacent pairs without changing the order of the subsequences $(-r,r,\dots,-r,r)$. Then,

$$Q_I(A) = (\mathrm{sgn}\ w)(-1)^{r_1+\dots+r_s}\ 2^s\ Q_J(A).$$

II. The symmetrization operator definition

Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_m)$ be two sequences of independent indeterminates.

(S) Let $I=(i_1,i_2,\dots)$ be a partition. Write D_I as



where $H = (h_1,\dots,h_n)$, $K = (k_1,\dots,k_m)$.
Then define

$$s_I^{(S)}(A-B) := (-1)^{|K|} \sum_{w\in S_n\times S_m} w \left[\frac{\prod_{(i,j)\in D_J}(a_i-b_j)\cdot a_1^{h_1+n-1}\cdot\ldots\cdot a_n^{h_n}\cdot b_1^{k_1+m-1}\cdot\ldots\cdot b_m^{k_m}}{\prod_{i<j}(a_i-a_j)\ \prod_{i<j}(b_i-b_j)} \right]$$

Here, $w\ f(a_1,\dots,a_n,b_1,\dots,b_m)$ means $f(a_{v(1)},\dots,a_{v(n)},b_{u(1)},\dots,b_{u(m)})$ for $w = vxu \in S_n \times S_m$. Of course, the above expression is a symmetric polynomial in A and B.

(Q) Let $I=(i_1,\dots,i_k)$,$k\le n$, be a strict partition of length k.
We define

$$Q_I^{(S)}(A) \ := \ 2^k \sum_{w\in S_n/(S_1)^k\times S_{n-k}} w \left[a_1^{i_1}a_2^{i_2}\dots a_k^{i_k} \prod_{\substack{1\le i<j\le n\\ i\le k}} \frac{a_i+a_j}{a_i-a_j} \right]$$

Here, $w\ f(a_1,\dots,a_n)$ means $f(a_{w(1)},\dots,a_{w(n)})$. Note that the sum does not depend on the choice of representatives in $S_n/(S_1)^k\times S_{n-k}$, because of the form of the summand. Sometimes it is more convenient to rewrite $Q_I^{(S)}(A)$ as (see [H-H] (9.11)):

$$Q_I^{(S)}(A)= 2^k \sum_{w\in S_n} \mathrm{sign}(w) w\Big[a_1^{i_1}a_2^{i_2}\dots a_k^{i_k} \prod_{\substack{1\le i<j\le n\\ i\le k}} (1+a_i^{-1}a_j)\Big]$$

Such summations have been extensively studied by Sylvester. A more satisfactory approach is to interpret them as some symmetrizing operators. For a theory of the symmetrizing operators see $[L\text{-}S]_2$. The importance of symmetrizing operators in geometry was illuminated in [B-G-G]. Here, we recall some elementary facts about the <u>total symmetrizer</u>, which will be useful in Section 2 (for more informations concerning this operator, and for examples of its applications, see $[L\text{-}P]_1$). If $A = (a_1,\dots,a_n)$ is a sequence of independent indeterminates, then one defines

$$\Pi_A : \mathbb{Z}[a_1,\dots,a_n] \longrightarrow \mathcal{S}ym(A) ,$$

where for $P \in \mathbb{Z}[a_1,\dots,a_n]$,

$$\Pi_A(P) := \sum_{w\in S_n} w \left[P\cdot a_1^{n-1}a_2^{n-2}\dots a_n^0 \prod_{i<j} (a_i-a_j)^{-1} \right] .$$

In particular, in the above notation ,

$$s_I^{(S)}(A-B) \ = \ \Pi_B\ \Pi_A \left[\prod_{(i,j)\in D_J} (a_i-b_j)\ a_1^{h_1}\dots a_n^{h_n}\ (-b_1)^{k_1}\dots(-b_m)^{k_m} \right] .$$

We have

<u>Lemma 1.1</u> For every sequence $I = (i_1,\dots,i_n) \in \mathbb{N}^n$,

$$\Pi_A(a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}) = s_I(A) .$$

Note that if I is a partition , then the Lemma is equivalent to the Jacobi definition of Schur S-polynomial (see [M] I.3). In fact, one sees easily that the proof of I.(3.6) in [M] allows us to obtain the above equality for $I \in \mathbb{N}^n$.

Observe, that with the help of the rule

$$s_{(\ldots,i,j,\ldots)}(A) = \pm\, s_{(\ldots,j-1,i+1,\ldots)}(A) ,$$

we can write

$$\Pi_A \,(\, a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n}) = \pm\, s_J(A) \quad \text{or} \quad 0 ,$$

where J is a partition. We see that if $i_k \neq 0$ for some k, then the corresponding partition J has length $\geq k$. This observation implies the following

Corollary 1.2 Let us fix $n' \geq n$ and $m' \geq m$. Moreover, let $A' = (a_1,\ldots,a_n,a_{n+1},\ldots,a_{n'})$, $B' = (b_1,\ldots,b_m,b_{m+1},\ldots,b_{m'})$ be two sequences of independent indeterminates. Then the polynomial $s_I^{(S)}(A-B)$ can be obtained as the specialization of $s_I^{(S)}(A'-B')$ with $a_{n+1} = \ldots = a_{n'} = 0$, $b_{m+1} = \ldots = b_{m'} = 0$.

III. Combinatorial definition – via tableaux.

(S) Let $I=(i_1,\ldots,i_k)$ be a partition. Consider two sequences of symbols $(1,2,\ldots,n)$ and $(1',2',\ldots,m')$ ordered in such a way that $1 < 2 < \ldots < n < 1' < 2' < \ldots < m'$.

A bitableau T of shape I is an assignment

$$T : D_I \longrightarrow \{1,2,\ldots,n,1',2',\ldots,m'\}$$

such that (1) $T(a,b) \leq T(a+1,b)$, $T(a,b) \leq T(a,b+1)$

(2) Each i (=1,2,…,n) appears at most once in each column.

(3) Each j' (=1',2',…,m') appears at most once in each row.

For example,

$$\begin{array}{lllll} 1' & 2' & & & \\ 3 & 1' & & & \\ 2 & 1' & 2' & & \\ 1 & 1 & 2 & 1' & 2' \end{array}$$

is a bitableau of shape (5322).

Define $\alpha_i := \text{card}\left\{ (a,b) \in D_I : T(a,b) = i \right\}$

$\beta_j := \text{card}\left\{ (a,b) \in D_I : T(a,b) = j' \right\}$

Then with each bitableau T we associate a monomial

$$m(T) = (-1)^{\beta_1+\ldots+\beta_m} \prod_{i=1}^{n} a_i^{\alpha_i} \prod_{j=1}^{m} b_j^{\beta_j}$$

Define

$$s_I^{(T)}(A-B) \;:=\; \sum_T m(T) ,$$

where the sum is over all bitableaux of shape I .

(Q) Let $I=(i_1,\ldots,i_k)$ be a strict partition of length k. One associates with I the so called shifted Ferrers' diagram :

$$D_I^s = \left\{ (a,b) \in \mathbb{Z}\times\mathbb{Z} : 1 \leq b \leq k ,\ b \leq a \leq i_b+b-1 \right\}$$

Let $(1,2,..,n)$ and $(1',2',\ldots,n')$ be two sequences of symbols ordered in such a way that $1'<1<2'<2<\ldots<n'<n$.

A shifted tableau T of shape I is an assignment

$$T : D_I^s \longrightarrow \{1,2,\ldots,n,1',2',\ldots,n'\}$$

such that (1) $T(a,b) \leq T(a+1,b)$, $T(a,b) \leq T(a,b+1)$

(2) Each i (=1,2,…,n) appears at most once in each column.

(3) Each j' (=1',2',…,n') appears at most once in each row.

For example,

$$\begin{array}{cccccc} & & 3' & & & \\ & 2' & 2 & 3' & 4 & \\ 1' & 1 & 1 & 3' & 3 & 4 \end{array}$$

is a shifted tableau of shape (641).

Let $\alpha_i = \text{card}\left\{ (a,b) \in D_I^s : T(a,b) = i \text{ or } i' \right\}$

Then with each shifted tableau T we associate a monomial

$$m(T) \;=\; \prod_{i=1}^{n} a_i^{\alpha_i}$$

Define

$$Q_I^{(T)}(A) := \sum_T m(T) ,$$

where the sum is over all shifted tableaux of shape I .

For example, let $I=(2,1)$, n=3. We have the following shifted tableaux :

 2'   2   3'   3   ;   2'   2   3'   3
1'1 , 1'1 , 1'1 , 1'1 ; 1'2' , 1'2' , 1'2', 1'2 ;

 3'   3   ;   3'   3   ;   2'   2   3'   3
1'2 , 1'2 ; 1'3', 1'3' ; 1 1 , 1 1 , 1 1 , 1 1 ;

 2'   2   3'   3   ;   3'   3   ;   3   ;   3'   3
1 2', 1 2', 1 2', 1 2' ; 1 2 , 1 2 ; 1 3' ; 2'2 , 2'2 ;

$$\begin{matrix} 3' \\ 2'3' \end{matrix}\ ,\ \begin{matrix} 3 \\ 2'3' \end{matrix}\ ;\ \begin{matrix} 3' \\ 2\,2 \end{matrix}\ ,\ \begin{matrix} 3 \\ 2\,2 \end{matrix}\ ;\ \begin{matrix} 3' \\ 2\,3' \end{matrix}\ ,\ \begin{matrix} 3 \\ 2\,3' \end{matrix}\ .$$

We obtain

$$Q_{21}^{(T)}(a_1,a_2,a_3) = 4 \sum_{1 \le i < j \le 3} a_i^2 a_j + 4 \sum_{1 \le i < j \le 3} a_i a_j^2 + 8 a_1 a_2 a_3 =$$

$$= 4 s_{21}(a_1,a_2,a_3) .$$

Theorem 1.3 (S) Let I be a partition. Then

$$s_I(A-B) = s_I^{(S)}(A-B) = s_I^{(T)}(A-B)$$

(Q) Let $I=(i_1,\dots,i_k)$ be a strict partition of length k .Then

$$Q_I(A) = Q_I^{(S)}(A) = Q_I^{(T)}(A) .$$

The first equality holds also for I being a sequence of positive integers.

Remark on a proof. (S) $s_I(A-B) = s_I^{(T)}(A-B)$ is proved in [B-R],[R]. The equality $s_I(A-B) = s_I^{(S)}(A-B)$ was communicated to me (without proof) by Sergeev in $[Se]_2$.As Serganova points out, this is an unpublished result of Sergeev. According to Serganova, the method used by Sergeev is based on some general facts from the theory of representations of superalgebras.

We will give at the end of Section 2 a new, independent and elementary proof of $s_I(A-B) = s_I^{(S)}(A-B)$, which uses nothing but simple algebraic properties of symmetric polynomials.

(Q) $Q_I(A) = Q_I^{(S)}(A)$ was proved essentially by Schur in [Sch] also in the case of I being a sequence of positive integers. The proof of $Q_I^{(S)}(A) = Q_I^{(T)}(A)$ can be found in $[Sa] \cup [Ste]_2 \cup [Wo]$.

2. ALGEBRAIC PROPERTIES AND CHARACTERIZATIONS OF S- AND Q-POLYNOMIALS.

Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_m)$ be two sequences of elements in a commutative ring R . Let us begin with the following observation. Comparing $Q_I^{(S)}(A)$ with Definition 2.2 of the Hall-Littlewood polynomials in [M] p.104 we see that $Q_I(A)$ is a specialization of the Hall-Littlewood polynomial $Q_I(A;t)$ for $t=-1$. Let I,J be partitions. Let $Q_{I/J}(A;t)$ be the skew Hall-Littlewood polynomial as defined in [M].III.5.

Define the skew Q-polynomial $Q_{I/J}(A)$ by

$$Q_{I/J}(A) = Q_{I/J}(A;t=-1)$$

A combinatorial version of this definition is as follows

A shifted skew diagram is a subset of $\mathbb{Z}\times\mathbb{Z}$ of the form $D_{I/J}^s = D_I^s \setminus D_J^s$, where D_I^s and D_J^s are any pair of shifted diagrams with $D_J^s \subset D_I^s$.
A shifted skew tableau T of skew shape I/J is an assignment

$$T : D_{I/J}^s \longrightarrow \{1,2,\dots,n,1',2',\dots,n'\}$$

satisfying the shifted standardness rules specified in (1.III.(Q)).

Let $\alpha_i = \operatorname{card}\left\{ (a,b) \in D_I^s : T(a,b) = i \text{ or } i' \right\}$

For example,

$$\begin{matrix} & & & 3 & & \\ & & 2 & 2 & & \\ & \cdot & \cdot & 1 & 1 & 2' \\ \cdot & \cdot & \cdot & \cdot & 1' & 2' \end{matrix}$$

is a shifted skew tableau of shape (6521)/(42).

Then with each shifted tableau T we associate a monomial

$$m(T) = \prod_{i=1}^{n} a_i^{\alpha_i}$$

It turns out (see [W]) that $Q_{I/J}(A) = \sum m(T)$, where the sum is over all shifted tableaux of shape I/J .

T.Józefiak and the author have recently found the following determinantal formula for $Q_{I/J}(A)$. Let $\ell(I)=k$, $\ell(J)=r$. If k+r is even, then $Q_{I/J}(A)$ is the Pfaffian of the k+r-order alternating matrix $M=(m_{s,t})$, where $m_{s,t} = Q_{i_s,i_t}(A)$, $1\le s,t \le k$; $m_{s,k+t} = q_{i_s - j_{r+1-t}}(A)$, $1 \le s \le k$, $1 \le t \le r$ and $m_{s,t}=0$, $s>k$, $t>k$. If k+r is odd, then $Q_{I/J}(A)=Q_{I/J'}(A)$, where we put $J'=(j_1,\dots,j_r,0)$. For details see a forthcoming paper [J-P].

Lemma 2.1 (Linearity Formula). (S) For every partition I

$$s_I(A-B) = \sum_J (-1)^{|I|-|J|} s_J(A)\, s_{I^\sim/J^\sim}(B) ,$$

where the sum is over all partitions J, and $s_{I^\sim/J^\sim}(B)$ is the corresponding skew Schur polynomial (see [M] I.5).

(S') For every partition I,

$$s_I(A) = \sum_J s_J(A-B)\, s_{I/J}(B) ,$$

where the sum is over all partitions J .

(Q) Let I be a strict partition. Then

$$Q_I(AB) = \sum_J Q_J(A)\, Q_{I/J}(B) ,$$

where the sum is over all (strict) partitions J .

<u>Remark on a proof.</u> (S) and (S') are consequences of λ-ring calculus (see [M],Remark I.2.15 and $[L\text{-}S]_1$). The assertion (Q) is a consequence of the formulas III.5.2 and III.5.5 in [M].

For every k let ρ_k denote the partition (k,k-1,...,2,1). We now state the following important property:

<u>Proposition 2.2</u> (Factorization Formula) Let $I = (i_1,\dots,i_n)$, $J = (j_1,\dots,j_p)$ be two partitions, $j_1 \le m$. Then

$$\text{(S)}\qquad s_{(m)^n+I,J}(A-B) = s_I(A)\, s_{(m)^n}(A-B)\, s_J(-B)$$

$$= (-1)^{|J|}\, s_I(A)\, s_{(m)^n}(A-B)\, s_{J\sim}(B)$$

$$\text{(Q)}\qquad Q_{\rho_{n-1}+I}(A) = Q_{\rho_{n-1}}(A)\, s_I(A) .$$

In particular if n<n', m<m', then $s_{(m')^{n'}+I,J}(A-B) = 0$ and $Q_{\rho_{n'}+I}(A) = 0$. Moreover, $Q_{\rho_n+I}(A) = Q_{\rho_n}(A)\, s_I(A)$.

<u>Remark on a proof.</u> (S) is proved in [B-R] Theorem 6.20 and in $[L\text{-}S]_1$ 7.6. Remmel has a combinatorial proof (see [R] and references there). Note that this assertion follows immediately from the equality $s_I(A-B) = s_I^{(S)}(A-B)$. The equality (Q) stems from [St].

It follows e.g. from the form of $Q_I^{(S)}(A)$ that for every strict partition I there exists a polynomial $P_I(A)$ in $\mathbb{Z}[a_1,\dots,a_n]$ such that $Q_I(A) = 2^{l(I)} P_I(A)$. Let us recall some particular examples of Schur S- and Q-polynomials.

<u>Lemma 2.3</u> (i) $s_{(m)^n}(A-B) = \prod_{i,j}(a_i - b_j)$

(ii) $P_{\rho_{n-1}}(A) = \prod_{i<j}(a_i + a_j)$

(iii) $P_{\rho_n}(A) = \prod_i a_i \cdot \prod_{i<j}(a_i + a_j)$

(iv) For every k, $P_{\rho_k}(A) = s_{\rho_k}(A)$.

For proofs of (i) see $[L\text{-}S]_1$ 7.6 and of (ii) -(iv)- [St] ,[W].

Let us mention some other useful properties of Q-polynomials, which stem essentially from [Sch].

<u>Proposition 2.4</u> If $a_1,\dots,a_n$ are independent indeterminates, then the Q-polynomials $Q_I(A)$, $\ell(I) \le n$, are Z-linearly independent.

Let $\psi_k(A)$ be the k-th power sum polynomial $\psi_k(A) = \sum_{i=1}^n a_i^k$.

<u>Proposition 2.5</u> $\mathbb{Q}[q_1(A),q_2(A),\dots] = \mathbb{Q}[\psi_1(A),\psi_3(A),\dots]$.

<u>Proof.</u> Write $Q(t) = \sum_{k\ge 0} q_k(A)t^k$, $H(t) = \sum_{k\ge 0} s_k(A)t^k$, $E(t) = \sum_{k\ge 0} s_{(1)^k}(A)t^k$, $\Psi(t) = \sum_{k\ge 1} \psi_k(A)t^{k-1}$. Then $Q(t) = H(t)\,E(t)$, and by [M], $\Psi(t) = H'(t)/H(t)$, $\Psi(-t) = E'(t)/E(t)$. Therefore $Q'(t) = Q(t)(\Psi(t)+\Psi(-t))$, and we have

$$(2k+1)q_{2k+1}(A) = 2\psi_{2k+1}(A) + 2q_2(A)\psi_{2k-1}(A) + \dots + 2q_{2k}(A)\psi_1(A) \qquad (k\ge 0),$$

$$2k\, q_{2k}(A) = 2q_1(A)\psi_{2k-1}(A) + \dots + 2q_{2k-1}(A)\psi_1(A) \qquad (k\ge 1) .$$

The assertion follows. ∎

<u>Remark 2.6</u> We have

$$q_k(A) = \sum (2^r/i_1!i_3!\dots i_p!\dots)\ (\psi_1(A)/1)^{i_1}(\psi_3(A)/3)^{i_3} \dots (\psi_p(A)/p)^{i_p} \dots$$

where the sum is over all (finite) sequences $(i_1,i_3,\dots,i_p,\dots)$, p is odd, $k = i_1+3i_3+\dots+pi_p+\dots$, and $r = \mathrm{card}\{\, p : i_p \ne 0 \,\}$.

An explicit expression of a general $Q_I(A)$ in terms of power sum polynomials (and vice versa) requires the projective characters of the symmetric group (see [Sch] p.235). Note that if A is a countable set of independent indeterminates, then $\psi_1(A)$, $\psi_2(A)$, $\psi_3(A)$, ... are algebraically independent, and in this case the ring of Q-polynomials is the subring $\mathbb{Q}[\psi_1(A),\psi_3(A),\dots] \subset \mathcal{S}ym(A)_{\mathbb{Q}} = \mathbb{Q}[\psi_1(A),\psi_2(A),\dots]$.

One basic property of Schur Q-polynomials is the Pieri formula describing their multiplication.

Proposition 2.7 (Morris $[Mo]_2$) Let $I = (i_1,\ldots,i_k)$ be a strict partition of length k. Then

$$Q_I(A)\cdot q_r(A) = \sum e(I,(r);J)\ Q_J(A)$$

where the sum is over all (strict) partitions $J=(j_1,\ldots,j_{k+1})$, $i_{p-1} \geq j_p \geq i_p$ for $p = 1,\ldots,k+1$ $(i_0=\infty\ ,\ i_{k+1}=0)$, $|J|=r+|I|$ and $e(I,(r);J) = 2^{\bar{e}(I,r;J)}$, where $\bar{e}(I,(r);J) = \mathrm{card}\ \{1\leq p \leq k : j_{p+1} < i_p < j_p\}$.

Therefore, for P-polynomials we have

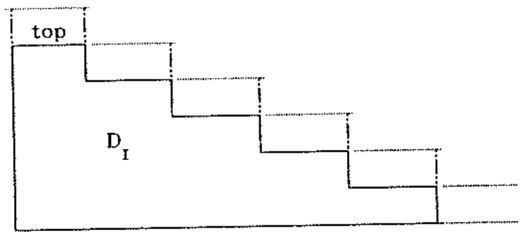
Corollary 2.8 With the above notation

$$P_I(A)\ P_r(A) = \sum f(I,(r);J)\ P_J(A)$$

where the summation is the same as above, $f(I,(r);J) = 2^{\bar{f}(I,(r);J)}$, and

$$\bar{f}(I,(r);J) = \begin{cases} \bar{e}(I,(r);J)-1 & \text{if } j_{k+1}= 0 \\ \bar{e}(I,(r);J) & \text{if } j_{k+1}\neq 0 \end{cases} .$$

Consider the following picture



Pictorially, we can interpret Proposition 2.7 as follows. If D_I is the shape determined by the solid lines, D_J can have any shape extending it contained in the dotted lines. The multiplicity of such a J contains a factor 2 for each row (of the original diagram) that is extended but for which the row above it fails to reach the maximum value. In the case of P-polynomials a factor of 2 is subtracted if no new top row is added.

Remark 2.9 The Pieri formula for the polynomials $s_I(A-B)$ is the same as for the classical Schur polynomials $s_I(A)$ (see [M]). For a proof, see for example [R], [K] and $[Se]_1$.

For a sake of completeness we describe the behaviour of Q-polynomials with respect to differentiation. Here, A is a countable sequence of independent indeterminates.

Lemma 2.10 Let $I=(i_1,\ldots,i_k)$ be a strict partition. Then the differentiation with respect to the p-th power sum polynomial satisfies:

$$p\ \frac{\partial}{\partial\psi_p(A)}\ Q_I(A) = 2\sum_{j=1}^{k} Q_{(i_1,i_2,\ldots,i_j-p,\ldots,i_k)}(A)$$

For a proof see $[Mo]_1$ p.451; recall that the description of $Q_I(A)$ for $I\in\mathbb{Z}^k$ was given in the end of 1.I(Q).

Characterization via cancellation properties

Let $A = (a_1,\ldots,a_n)$, $B = (b_1,\ldots,b_m)$ be two sequences of indeterminates such that AB is a sequence of independent variables.

We say that $f\in \mathcal{S}ym(A,B)$ has the S-cancellation property if the following holds: when the substitution $a_1= t,\ b_1= t$ is made in f, the resulting polynomial is independent of t.

We say that $f\in \mathcal{S}ym(A)$ has the Q-cancellation property if the following holds: when the substitution $a_1= t\ ,\ a_2=-t$ is made in f , the resulting polynomial is independent of t $(n\geq 2)$.

The polynomials with the S- (resp. Q-) cancellation property form a subring in $\mathcal{S}ym(A,B)$ (resp. in $\mathcal{S}ym(A)$).

Theorem 2.11 (S) Every polynomial f in $\mathcal{S}ym(A,B)$ with the S-cancellation property is a $\mathbb{Z}$-linear combination of the $s_I(A-B)$'s .
(Q) Every polynomial f in $\mathcal{S}ym(A)$ with the Q-cancellation property is a $\mathbb{Z}$-linear combination of the $P_I(A)$'s .

Proof. (S) We use induction on m . If m=0, the assertion is true. Consider the polynomial $f(a_n=b_m=0)$ (this means that the substitution $a_n=b_m=0$ is made in f). This polynomial has the S-cancellation property. By induction ,$f(a_n=b_m=0)$ is a $\mathbb{Z}$-linear combination of the $s_I(A'-B')$'s , where $A'=(a_1,\ldots,a_{n-1})$, $B'=(b_1,\ldots,b_{m-1})$. Let

$$f(a_n=b_m=0) = \sum_I \alpha_I\ s_I(A'-B')\ ,\quad \alpha_I \in \mathbb{Z}\ .$$

Consider the polynomial

$$\bar{f} = \sum_I \alpha_I \, s_I(A-B).$$

Since f and $\bar{f}$ have the S-cancellation property, we have

$$(f-\bar{f})(a_n=b_m=t) = (f-\bar{f})(a_n=b_m=0) = 0 .$$

This implies that a_n-b_m divides $f-\bar{f}$. But $f-\bar{f} \in \mathcal{S}ym(A,B)$, so by Lemma 2.3(i), $s_{(m)^n}(A-B)$ divides $f-\bar{f}$. Thus

$$f = \bar{f} + s_{(m)^n}(A-B)\cdot g \quad ,$$

where $g \in \mathcal{S}ym(A,B)$. Write

$$g = \sum_{I,J} \alpha_{I,J} \, s_I(A) \, s_J(-B) \quad , \quad \alpha_{I,J} \in \mathbb{Z} .$$

By the factorization formula,

$$f = \bar{f} + \sum_{I,J} \alpha_{I,J} \, s_{(m)^n+I,J}(A-B)$$

i.e. f is a $\mathbb{Z}$-linear combination of the $s_K(A-B)$'s .

(Q) We use induction on n. Let n=2 and let $f=f(a_1,a_2)$ has the Q-cancellation property. Then a_1+a_2 divides f . Thus f can be written

$$f = (a_1+a_2) \sum \alpha_{ij} \, s_{ij}(A) \quad , \quad \alpha_{ij} \in \mathbb{Z} .$$

It suffices to show that $(a_1+a_2) \, s_{ij}(A)$ is a $\mathbb{Z}$-linear combination of the $P_I(A)$'s . One checks readily that

$$(a_1+a_2) \, s_{ij}(A) = P_{i+1,j}(A), \text{ if } i\geq j.$$

If n>2 we proceed as follows. Consider the polynomial $f(a_{n-1}=a_n=0)$. This polynomial has the Q-cancellation property. By induction, $f(a_{n-1}=a_n=0)$ is a $\mathbb{Z}$-linear combination of the $P_I(A')$'s, where $A'=(a_1,\ldots,a_{n-2})$. Let

$$f(a_{n-1}=a_n=0) = \sum_I \alpha_I \, P_I(A') \quad , \quad \alpha_I \in \mathbb{Z} .$$

Consider the polynomial

$$\bar{f} = \sum_I \alpha_I \, P_I(A) .$$

Arguing as above we see that $P_{\rho_{n-1}}(A)$ divides $f-\bar{f}$. Hence

$$f = \bar{f} + P_{\rho_{n-1}}(A) \sum_I \alpha'_I \, s_I(A) \quad , \quad \alpha'_I \in \mathbb{Z}$$

$$= \bar{f} + \sum_I \alpha'_I \, P_{\rho_{n-1}+I}(A) \quad \text{(by the factorization formula)}$$

is a $\mathbb{Z}$-linear combination of the $P_I(A)$'s . ■

Corollary 2.12 (S) Every polynomial $f\in \mathcal{S}ym(A,B)$ with the S-cancellation property is a polynomial over $\mathbb{Q}$ in

$$\psi_k(A) - \psi_k(B) \quad , \quad k=1,2,\ldots .$$

(Q) Every polynomial $f\in \mathcal{S}ym(A)$ with the Q-cancellation property is a polynomial over $\mathbb{Q}$ in $\psi_k(A)$, $k=1,3,5,\ldots$.

Proof. (S) Recall that $\mathcal{S}ym(A,B)$ has a canonical λ-ring structure (see [M] I.2.15; more precisely, it is a quotient of the free λ-ring in two variables α and ψ modulo the ideal generated by $\lambda^i\alpha$, if $i > n$, and $\lambda^j\psi$, if $j > m$). Each Schur operator in a λ-ring is a polynomial over $\mathbb{Q}$ in the Adams operations $\psi_k(-)$. In particular $s_I(A-B)$ is a polynomial over $\mathbb{Q}$ in $\psi_k(A-B) = \psi_k(A)-\psi_k(B)$, and the assertion follows. ($\psi_k(A-B) = \psi_k(A)+\psi_k((-1)B) = \psi_k(A)+\psi_k(-1)\psi_k(B) = \psi_k(A)-\psi_k(B)$ in the λ-ring $\mathcal{S}ym(A,B)$.)

(Q) This follows from Proposition 2.6 and Theorem 2.11.■

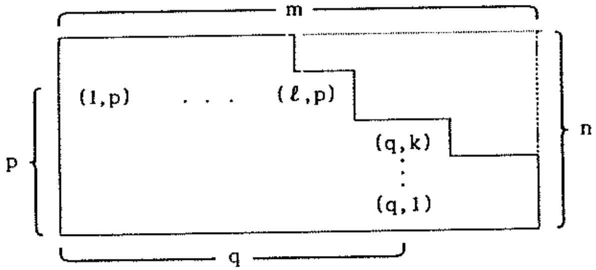
Remark 2.13 Theorem 2.11(S) was proved originally by J.Stembridge in $[\text{Ste}]_1$. The proof given here is different, seems to be more direct and extends uniformly to the Q-polynomials case. One can easily show that the above results remain valid when we replace A and B by countable sequences of independent variables (see $[\text{Ste}]_1$ Section 3).

Remark 2.14 In the next Section we will deal with Q-polynomials associated with vector bundles. In this language Theorem 2.11(Q) can be reformulated as follows. Every polynomial in the Chern classes of a bundle E which does not change its value if we replace E by $E\oplus F\oplus F^{\vee}$ (F–another vector bundle on the same base space) is a $\mathbb{Z}$-combination of the $P_I(E)$'s. Therefore the Schur P-polynomials form a family of polynomials wanted in [H-T].

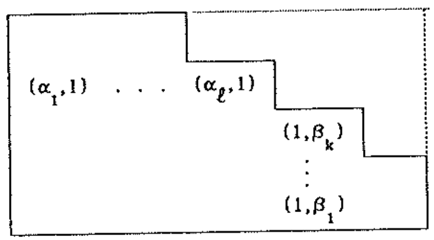
We give now a proof of the formula $s_I(A-B) = s_I^{(S)}(A-B)$ as promised in Section 1.

Lemma 2.15 The polynomial $s_I^{(S)}(A-B)$, as defined in Section 1.II(S), has the S-cancellation property.

Proof. Let $a_1=b_1=t$. All the summands in $s_I^{(S)}(A-B)$ which have (a_1-b_1) as a factor, vanish. We look at the summands which do not have this factor. More precisely, we compute the degree of the numerator and denominator of such a summand with respect to t. The degree of the denominator is $m+n-2$. Suppose that the summand in question stems from $w=vxu \in S_n x S_m$. Let $p=v^{-1}(1)$, $q=u^{-1}(1)$. In the notation of 1.II(S), let $\ell=j_p$ and $k=(\tilde{j})_q$, where $\tilde{J} = ((\tilde{j})_1,(\tilde{j})_2,\ldots)$ is the partition conjugate of J . The role of the numbers p,q,k,ℓ in D_J can be illustrated as follows:



Note that since (a_1-b_1) is not a factor in the summand under consideration, $(q,p) \notin D_J$. Thus $k < p$ and $\ell < q$. The set $\{ (u(x),v(y)) \,,\, (x,y) \in D_J \}$ looks like



for some $\alpha_1,\ldots,\alpha_\ell$ and $\beta_1,\ldots,\beta_k$. Then the contribution of t in the numerator stems from the product:

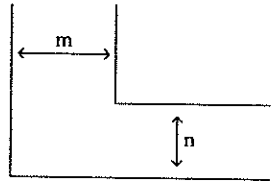
$$(a_1-b_{\alpha_1}) \ldots (a_1-b_{\alpha_\ell})(a_{\beta_1}-b_1) \ldots (a_{\beta_k}-b_1) \times a_1^{n-p} \times b_1^{m-q} \ldots\ldots$$

Since $k \le p-1$, $\ell \le q-1$, we see that the degree of the numerator with respect to t is

$$\ell+k+n-p+m-q \le m+n-2 \,.$$

Summing up, after the substitution $a_1=b_1=t$ in $s_I^{(S)}(A-B)$, we obtain a sum of the expressions of the form $f(t)/g(t)$ where $f(t),g(t) \in \mathbb{Z}[a_2,\ldots,a_n,b_2,\ldots,b_m][t]$ and $\deg_t f(t) \le \deg_t g(t)$. But we know that the expression under consideration is a polynomial in t . Therefore $s_I^{(S)}(A-B)$ with $a_1=b_1=t$, must be of degree zero with respect to t and the Lemma follows. ∎

It follows from Corollary 1.2 that we can assume $I \subset (m)^n$, where card A = n , card B = m . Moreover, we obtain from Lemma 2.15 that $s_I^{(S)}(A-B) = \sum_J \alpha_J s_J(A-B)$ for some $\alpha_J \in \mathbb{Z}$. Note that $mn \ge \deg s_I^{(S)}(A-B) = |I| = |J| = \deg s_J(A-B)$ for every J such that $\alpha_J \ne 0$ .Thus each D_J, where $\alpha_J \ne 0$, is contained in the (m,n)-hook:



Let $\mathcal{J} = \{ J : \alpha_J \ne 0 \}$. Our aim is to show that $\mathcal{J} = \{I\}$. Let $m' \ge m$, $n' \ge n$ be such that for all $J \in \mathcal{J}$, $J \subset (m')^{n'}$. Let $A' = (a_1,\ldots,a_n,a_{n+1},\ldots,a_{n'})$, $B' = (b_1,\ldots,b_m,b_{m+1},\ldots,b_{m'})$ be two sequences of independent variables. Then $s_I^{(S)}(A'-B') = \sum \beta_J \, s_J(A'-B')$ by Lemma 2.15. We claim that $\beta_J=\alpha_J$ for every partition J . By the same reason as above (i.e. by comparing the weights of partitions involved) if $\beta_J \ne 0$, then D_J is contained in the (m,n)-hook. Specialize $a_{n+1}=\ldots = a_{n'}=0$, $b_{m+1}=\ldots=b_{m'}=0$. Then, by Corollary 1.2, the left hand side gives $s_I^{(S)}(A-B)$. The right hand side gives $\sum_J \beta_J s_J(A-B)$. Thus $\sum_J \alpha_J s_J(A-B) = \sum_J \beta_J s_J(A-B)$. This implies $\alpha_J = \beta_J$ for every J , by [B-R] Lemma 6.4 (which asserts the $\mathbb{Z}$-linear independence of the $s_J(A-B)$'s, where $D_J \subset$ (m,n)-hook). Thus $s_I^{(S)}(A'-B') = \sum \alpha_J s_J(A'-B')$. We have

$$s_I^{(S)}(A'-B') = \Pi_{B'}\Pi_{A'}\Big[\prod_{(i,j)\in D_I} (a_i-b_j) \Big] = \Pi_{B'}\Pi_{A'}\Big[a^I + \sum \alpha_{K,L} a^K b^L \Big] \,.$$

where for given sequences $K = (k_1,\ldots,k_{n'})$, $L = (\ell_1,\ldots,\ell_{m'})$, $a^K := a_1^{k_1} a_2^{k_2} \ldots$, $b^L := b_1^{\ell_1} b_2^{\ell_2} \ldots$, and if $\alpha_{K,L} \ne 0$ then $\ell_1+\ell_2+\ldots > 0$. By Lemma 1.1 we obtain

$$s_I^{(S)}(A'-B') = s_I(A') + \sum \alpha_{K,L} s_K(A') s_L(B')$$

where $s_L(B') = 0$ or $s_L(B') = \pm\, s_{L'}(B')$, where L' is a partition and $|L'| \ge 1$. Consider the specialization $B' = (0,\ldots,0)$. We see that $s_I^{(S)}(A'-B')$ specializes to $s_I(A')$. On the other hand $\sum_J \alpha_J s_J(A'-B')$ specializes to $\sum_J \alpha_J s_J(A')$, the sum over the same set of J 's as above , because of the linearity formula (Lemma 2.1) and the assumption $\ell(J) \le n'$ if $\alpha_J \neq 0$.

We conclude that $\alpha_J=0$ if $J\neq I$, $\alpha_I=1$, because of $\mathbb{Z}$-linear independence of the $s_I(A')$'s (see [M] I.(3.2)). Therefore $\mathcal{J} = \{I\}$ and the proof of $s_I(A-B) = s_I^{(S)}(A-B)$ is finished.

Remark 2.16 In the above proof a characterization of the polynomials $s_I(A-B)$ via the S-cancellation property is used. Our derivation of this characterization uses the factorization formula. Therefore, in this approach the latter formula cannot be treated as a corollary of $s_I(A-B) = s_I^{(S)}(A-B)$. However, if one uses Stembridge's proof of the cited characterization, then the factorization formula is a corollary of the identity proved above.

3. SYMMETRIZING OPERATORS AND FORMULAS FOR GYSIN PUSH FORWARD.

Let $A = (a_1,\dots,a_n)$ be a sequence of independent indeterminates. We consider three symmetrizing operators defined as follows. At first, we have (see Section 1)

$$\Pi_A : \mathbb{Z}[a_1,\dots,a_n] \longrightarrow \mathcal{S}ym(A) \quad \text{, where}$$

$$P(a_1,\dots a_n) \longmapsto \sum_{w\in S_n} w\left[P(a_1,\dots,a_n)\, a_1^{n-1}a_2^{n-2}\dots a_n^{0} \prod_{i<j} (a_i-a_j)^{-1} \right].$$

Secondly, for $0<k<n$ define

$$\partial_A^k : \mathbb{Z}[a_1,\dots,a_k] \otimes \mathcal{S}ym(a_{k+1},\dots,a_n) \longrightarrow \mathcal{S}ym(A)$$

by
$$P(a_1,\dots,a_n) \longmapsto \sum_{w\in S_n/(S_1)^k\times S_{n-k}} w\left[P(a_1,\dots,a_n) \prod_{\substack{1\le i<j\le n\\ i\le k}} (a_i-a_j)^{-1} \right].$$

Similarly, for $0<q<n$ define

$$\delta_A^q : \mathcal{S}ym(a_1,\dots,a_q) \otimes \mathcal{S}ym(a_{q+1},\dots,a_n) \longrightarrow \mathcal{S}ym(A)$$

by
$$P(a_1,\dots,a_n) \longmapsto \sum_{w\in S_n/S_q\times S_{n-q}} w\left[P(a_1,\dots,a_n) \prod_{\substack{1\le i\le q\\ q+1\le j\le n}} (a_i-a_j)^{-1} \right].$$

In this Section we discuss some properties of S-and Q-polynomials as well as Schur functors applied to vector bundles (we take this opportunity to make smooth some points of exposition in $[P]_3$, Section 2). We use the

notations and conventions of [F]. If X is an algebraic scheme over a field, then A.(X) denotes the Chow group of cycles modulo rational equivalence. If E is a vector bundle over X, then $c_i(E)$–the i-th Chern class of E $i=1,\dots,\text{rank } E$, as well as polynomials in the Chern classes of E are treated as operators acting on A.(X) . More precisely, for $0<k\le n$, let $\tau_E^k : Fl^k(E) \longrightarrow X$ be the flag bundle parametrizing the flags of consecutive quotients of E of ranks $k,k-1,\dots,2,1$. Let

$$E \twoheadrightarrow Q^k \twoheadrightarrow Q^{k-1} \twoheadrightarrow \dots \twoheadrightarrow Q^2 \twoheadrightarrow Q^1$$

be the tautological sequence on $Fl^k(E)$. Define the line bundles $L_1^E,\dots,L_k^E$ on $Fl^k(E)$ by $L_i^E = \mathrm{Ker}(Q^i \to Q^{i-1})$. Let $a_i=c_1(L_i^E)$, $i=1,\dots,k$. Assume for a moment that $k=n$. Then $A=(a_1,\dots,a_n)$ is a sequence of the Chern roots of E operating on $A.(Fl^n(E))$. We define $s_I(E) := s_I(A)$. Since $s_I(A)$ is symmetric in $a_1,\dots,a_n$, $s_I(A)$ is a polynomial in the Chern classes of E , and thus, according to [F], $s_I(E)$ is an operator on A.(X).

For a vector bundle E, let $\pi_E : G_r(E) \longrightarrow X$ denote the Grassmannian bundle of r-subbundles of E. On $G_r(E)$ there exists a tautological sequence of vector bundles

$$0 \longrightarrow R_E \longrightarrow E_{G_r(E)} \longrightarrow Q_E \longrightarrow 0$$

of ranks r,n and q=n-r . Sometimes we will denote this Grassmannian by $G^q(E)$ or G , R_E by R and Q_E by Q .

The consecutive projections

$$Fl(E)=Fl^n(E) \longrightarrow Fl^k(E) \longrightarrow G^k(E) \longrightarrow X$$

induce the following chain of injections of the corresponding Chow groups

$$A(X) \longrightarrow A(G^k(E)) \longrightarrow A(Fl^k(E)) \longrightarrow A(Fl(E)).$$

Let $A=(a_1,\dots,a_n)$, $A_k=(a_1,\dots,a_k)$, $A^{n-k}=(a_{k+1},\dots,a_n)$. By considering polynomials in A as operators on A.(Fl(E)), the above sequence allows us to treat polynomials symmetric in A as operators on A(X), polynomials symmetric in A_k and A^{n-k} as operators on $A.(G^k(E))$, and finally polynomials symmetric in A_k as operators on $A.(Fl^k(E))$. Moreover, every element in $A.(Fl^k(E))$ can be written uniquely as $\sum a_1^{i_1}\dots a_k^{i_k} \cap \tau_E^{k*}\alpha_{i_1,\dots,i_k}$, where $i_p \le n-p$, $p=1,\dots,k$, and $\alpha_{i_1,\dots,i_k} \in A.(X)$. Similarly every element in $A.(G_r(E))$ can be written uniquely as $\sum s_I(R)\cap\pi_E^*\alpha_I$ (resp. $\sum s_I(Q)\cap\pi_E^*\alpha_I$) where $I \subset (n-r)^r$ (resp. $I \subset (r)^{n-r}$), $\alpha_I \in A.(X)$ (see [F] chap. 14).

If X is an algebraic scheme , let K(X) denote the Grothendieck group of vector bundles on X. If E is a vector bundle on X and I is a partition, let $S_I(E)$ denote the polynomial Schur functor associated to I,and applied to E (see [M] Appendix).Recall also that every element in K(Fl(E)) can be written uniquely as $L_1^{i_1} \otimes \dots \otimes L_n^{i_n} \otimes (\tau_E^n)^* \alpha_{i_1,\dots,i_n}$, where $L_i = L_i^E$, $\alpha_{i_1,\dots,i_n} \in K(X)$ and $i_p \le n-p$, $p=1,\dots,n$.

Lemma 3.1 Let E be a vector bundle on X. Let A be the set of the Chern roots of E. Then:

(i) the induced Gysin morphism $(\tau_E^k)_*: A.(Fl^k(E)) \longrightarrow A.(X)$ satisfies: for every $\alpha \in A(X)$, $P = P(a_1,\dots,a_n) \in \mathbb{Z}[A_k] \otimes \mathcal{S}ym(A^{n-k})$,

$$(\tau_E^k)_*\Big[P \cap (\tau_E^k)^*\alpha \Big] = \partial_A^k(P) \cap \alpha$$

(ii) the induced Gysin morphism $(\pi_E)_*: A.(G^q(E)) \longrightarrow A.(X)$ satisfies: for every $\alpha \in A(X)$, $P = P(a_1,\dots,a_n) \in \mathcal{S}ym(A_q) \otimes \mathcal{S}ym(A^{n-q})$,

$$(\pi_E)_*\Big[P \cap \pi_E^*\alpha \Big] = \delta_A^q(P) \cap \alpha .$$

Let now A denotes the sequence of classes of $L_1,\dots,L_n$ in $K(F\ell(E))$. Then
(iii) the induced Gysin morphism $(\tau_E^n)_!: K(F\ell(E)) \longrightarrow K(X)$ satisfies

$$(\tau_E^n)_!\Big[P(L_1,\dots,L_n) \Big] = \Pi_A\Big[P(L_1,\dots,L_n) \Big] .$$

Let F be a vector bundle of rank m on X. One can consider analogous objects $\tau_F^\ell: Fl^\ell(F) \longrightarrow X$, $0 < \ell \le m$. Write $b_j = c_1(L_j^F)$, $j=1,\dots,\ell$ and define $s_I(E-F) := s_I(A-B)$. Finally, we put $Q_I(E) := Q_I(A)$ and $P_I(E) := P_I(A)$.

The following fact follows easily as a consequence of Lemma 1.1, the Weyl-type formulas for $s_I(A-B)$, $Q_I(A)$ (see Section 1) and Lemma 3.1.

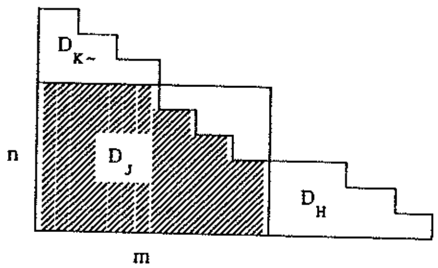
Proposition 3.2 Let $\alpha \in A.(X)$. (i) Let $I = (i_1,\dots,i_n)$ be a sequence of nonnegative integers. Write $a_i = a_i^E$, $i=1,\dots,n$. Then

$$s_I(E) \cap \alpha = (\tau_E^n)_*\Big[a_1^{i_1+n-1} a_2^{i_2+n-2} \dots a_{n-1}^{i_{n-1}+1} a_n^{i_n} \cap (\tau_E^n)^*\alpha \Big]$$

(ii) Let $I=(i_1,\dots,i_k)$ be a sequence of positive integers. Then

$$P_I(E) \cap \alpha = (\tau_E^k)_*\Big[a_1^{i_1} a_2^{i_2} \dots a_k^{i_k} \prod_{\substack{1\le i<j\le n \\ i \le k}} (a_i+a_j) \cap (\tau_E^k)^*\alpha \Big]$$

(iii) Let $I=(i_1,i_2,\dots)$ be a partition with the following diagram D_I:



where $H = (h_1,\dots,h_n)$, $K = (k_1,\dots,k_m)$. Then, in A.(X) ,

$$s_I(E-F) \cap \alpha = (-1)^{|K|}\tau_*\left\{\Big[\prod_{(i,j)\in D_J}(a_i-b_j)\, a_1^{h_1+n-1} \dots a_n^{h_n}\, b_1^{k_1+m-1} \dots b_m^{k_m}\Big] \cap \tau^*\alpha\right\}$$

($\tau = \tau_E^n \times_X \tau_F^m$ denotes here the (cartesian) projection

$$Fl^n(E) \times_X Fl^m(F) \longrightarrow X .)$$

(iv) With the notation from (iii) , the following formula holds in K(X) :

$$S_I(E-F) = \tau_!\left\{\Big[\prod_{(i,j)\in D_J}(a_i-b_j)\, a_1^{h_1} \dots a_n^{h_n} (-b_1)^{k_1} \dots (-b_m)^{k_m}\Big]\right\}$$

where $a_i = L_i^E$, $i=1,\dots,n$; $b_j = L_j^F$, $j=1,\dots,m$, and $\tau_!$ denotes the Gysin morphism of the corresponding Grothendieck groups.

The next theorem gives some useful formulas for calculations with Gysin push-forwards in Grassmannian bundles.The formula (ii) generalizes in an essential way the main computation of [J-L-P].

Theorem 3.3 Let $\alpha \in A.(X)$.
(i) With the above notation, for any vector bundle H on X ,and any sequences of integers $I=(i_1,\dots,i_q)$,$J=(j_1,\dots,j_r)$,

$$(\pi_E)_*\Big[s_J(R_E-H_G)\, s_I(Q_E-H_G) \cap \pi_E^*\alpha \Big] = s_{I-(r)^q,J}(E-H) \cap \alpha .$$

(ii) For any sequences $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_h)$ of positive integers, $k \le q$, $h \le r$,

$$(\pi_E)_*\left[\, c_{top}(R_E \otimes Q_E)\, P_J(R_E)\, P_I(Q_E) \cap \pi_E^*\alpha \,\right] \;=\; P_{I,J}(E) \cap \alpha$$

Proof. For a proof of (i) see $[P]_3$, Proposition 2.2 and [J-L-P], Proposition 1. To prove the statement (ii) consider first the projective bundle case. Let $\eta : G^1(E) \longrightarrow X$ be the Grassmannian (projective) bundle parametrizing 1-quotients of E. Let

$$0 \longrightarrow R \longrightarrow E_{G^1(E)} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

be the tautological exact sequence and write $\xi = c_1(\mathcal{O}(1))$.

Lemma 3.4 Let $I=(i_1,\dots,i_{k-1})$ be a sequence of positive integers, let i be an positive integer. Then for $\alpha \in A_*(X)$,

$$\eta_*\left[\, c_{top}(R\otimes\mathcal{O}(1))\cdot\xi^{i}\cdot P_I(R) \cap \eta^*\alpha \,\right] \;=\; P_{(i,i_1,\dots,i_{k-1})}(E) \cap \alpha\,.$$

Proof. Consider the following factorization

$$\tau^k = \tau^k_E : Fl^k(E) \xrightarrow{\;\omega\;} G^1(E) \xrightarrow{\;\eta\;} X$$

Write $a_i = c_1(L_i^E)$, $i=1,\dots,k$. Note that $\omega^*\mathcal{O}(1) = L_1^E$. By Proposition 3.2(ii)

$$P_{i,I}(E) \cap \alpha = \tau^k_*\left[\, a_1^{i} a_2^{i_1} a_3^{i_2}\cdots a_k^{i_{k-1}} \prod_{\substack{1\le p<q\le n\\ p\le k}} (a_p + a_q) \cap (\tau^k)^*\alpha \,\right]$$

$$= \eta_*\omega_*\left[\, a_2^{i_1}\cdots a_k^{i_{k-1}} \prod_{\substack{2\le p<q\le n\\ p\le k}} (a_p+a_q)\, c_1(\omega^*\mathcal{O}(1))^{i}\, c_{top}(\omega^*(R\otimes\mathcal{O}(1))) \cap (\tau^k)^*\alpha \,\right]$$

$$= \eta_*\left[\, c_{top}(R\otimes\mathcal{O}(1))\cdot P_I(R)\cdot\xi^{i} \cap \eta^*\alpha \,\right]\,.$$

We use here the projection formula for ω, Proposition 3.2(ii) for $P_I(R)$, and the splitting principle for $R\otimes\mathcal{O}(1)$. This proves the Lemma. ∎

Proof of Theorem 3.3 (ii)

For a given Grassmannian bundle $\pi : G^q(E) \longrightarrow X$, let $e(\pi) = c_{top}(R\otimes Q)$ where $0 \longrightarrow R \longrightarrow E_G \longrightarrow Q \longrightarrow 0$ is the tautological sequence on $G = G^q(E)$.

Consider the commutative diagram of Grassmannian extensions (with the corresponding tautological sequences) :

$$\begin{array}{ccccc}
 & & G^{q-1}(R') \;\simeq\; Fl^{q,1}(E) \;\simeq\; G^1(Q) & & \\
0\to R\to R'\to S\to 0 & \swarrow^{\pi_1} & & {}^{\pi_3}\searrow & 0\to S\to Q\to\mathcal{O}(1)\to 0 \\
 G^1(E) & & & & G^q(E) \\
0\to R'\to E\to\mathcal{O}(1)\to 0 & \searrow_{\pi_2} & & {}_{\pi}\swarrow & 0\to R\to E\to Q\to 0 \\
 & & X & &
\end{array}$$

where $Fl^{q,1}(E)$ parametrizes the flags of quotients of ranks q and 1 of E

Let $e_i = e(\pi_i)$ $i=1,2,3$. We want to show that

$$\pi_*\left[\, c_{top}(R\otimes Q)\cdot P_J(R)\; P_I(Q) \cap \pi^*\alpha \,\right] = P_{I,J}(E) \cap \alpha \quad , \text{ where } \; \alpha \in A_*(X)$$

For $q=1$ this follows from Lemma 3.4. Now let $q > 1$. Let $I'=(i_2,\dots,i_q)$ It follows from Lemma 3.4 that with $\xi = c_1(\mathcal{O}(1))$,

$$\pi_{3*}\left[\, e_3\cdot P_J(R)\; P_{I'}(S)\; \xi^{i_1} \cap (\pi\,\pi_3)^*\alpha \,\right] = P_J(R)\; P_I(Q) \cap \pi^*\alpha$$

Thus

$$\pi_*\left[\, e\cdot P_J(R)\; P_I(Q) \cap \pi^*\alpha \,\right] = \pi_*\pi_{3*}\left[\, e\cdot e_3\cdot P_J(R)\; P_{I'}(S)\cdot\xi^{i_1} \cap (\pi\,\pi_3)^*\alpha \,\right]$$

$$= \pi_{2*}\pi_{1*}\left[\, e_1\cdot e_2\cdot P_J(R)\; P_{I'}(S)\cdot\xi^{i_1} \cap (\pi_2\pi_1)^*\alpha \,\right]$$

By inductive hypothesis we have

$$\pi_{1*}\left[\, e_1\cdot P_J(R)\; P_{I'}(S) \cap \pi_1^*\pi_2^*\alpha \,\right] = P_{I',J}(R') \cap \pi_2^*\alpha$$

Thus finally

$$\pi_*\left[\, e\cdot P_J(R)\; P_I(Q) \cap \pi^*\alpha \,\right] = \pi_{2*}\left[\, e_2\cdot P_{I',J}(R')\cdot\xi^{i_1} \cap \pi_2^*\alpha \,\right] = P_{I,J}(E) \cap \alpha$$

using Lemma 3.4 again. ∎

Remark 3.5 For $J=(0)$, $k = q-1, q$, Theorem 3.3 (ii) can be proved in a shorter way - see $[P]_3$, where this formula was also used.

Finally, we note the following useful expression for Schur S-polynomials, which can be obtained from Theorem 3.3 (i) with the help of the S-factorization formula (for a proof see $[P]_3$, Lemma 3.1).

<u>Proposition 3.6</u> Let $\pi_F : G^r(F) \longrightarrow X$ be the Grassmannian bundle parametrizing r-quotients of F . Let

$$0 \longrightarrow R_E^{(r)} \longrightarrow E_{G_r(E)} \longrightarrow Q_E^{(n-r)} \longrightarrow 0$$

$$0 \longrightarrow R_F^{(m-r)} \longrightarrow F_{G^r(F)} \longrightarrow Q_F^{(r)} \longrightarrow 0$$

be the tautological sequences on $G_r(E)$ and $G^r(F)$. Let $\pi = \pi_E \times_X \pi_F : G = G_r(E) \times_X G^r(F) \longrightarrow X$ be the (cartesian) projection. Write $R_E, Q_F, \ldots$ instead of $(R_E)_G, (Q_F)_G, \ldots$. Then for any partitions I,J such that $\ell(I) \le n-r$, $\ell(J\sim) \le m-r$,

$$\pi_*\left[s_I(Q_E)s_J(-R_F)\, c_{top}\left(\frac{\mathrm{Hom}(F,E)_G}{\mathrm{Hom}(Q_F,R_E)}\right) \cap \pi^*\alpha\right] = s_{(m-r)^{n-r}+I,J}(E-F) \cap \alpha \, .$$

4. COMBINATORIAL RULES FOR MULTIPLICATION AND DECOMPOSITION OF Q-POLYNOMIALS.

In this Section, we describe, following $[Ste]_2$, combinatorial rules for computing the coefficients f(I,J;L), g(I;J) appearing in the formulas

$$P_I(A)\cdot P_J(A) = \sum_L f(I,J;L)\, P_L(A)$$

and

$$P_I(A) = \sum_J g(I;J)\, s_J(A) \, .$$

Here, $A=(a_1,a_2,\ldots)$ is a countable sequence of independent indeterminates (i.e. we put $n=\infty$ in Definition 1.I.(Q) and $n=n'=\infty$ in Definition 1.III.(Q)).

To start with, consider a (possibly skew) shifted tableau T . Define the word $w(T) = w_1, w_2 \ldots$ of T to be the sequence obtained by reading the rows of T from right to left starting from the first (i.e. bottom row).

For example if T is the shifted skew tableau

$$\begin{array}{cccccc} & & & 3 & & \\ & & 2 & 2 & & \\ & \cdot & \cdot & 1 & 1 & 2' \\ \cdot & \cdot & \cdot & \cdot & 1' & 1 \end{array}$$

of skew shape (6521)/(42) , then $w(T) = 11'2'11223$.

Given any word $w=w_1w_2\ldots w_k$ over the alphabet $(1',1,2',2,\ldots)$ define for $i\ge1$

$$\alpha_i(j) := \mathrm{card}\left\{ p : w_p = i \, , \, 1\le p\le j \right\} \quad (1\le j\le k)$$

$$\alpha_i(k+j) := \alpha_i(k) + \mathrm{card}\left\{ p : w_p = i' \, , \, k-j+1\le p\le k \right\} \quad (1\le j\le k)$$

Moreover one puts $\alpha_i(0) = 0$. Observe that $(\alpha_1(2k), \alpha_2(2k), \ldots)$ is the sequence $(\alpha_1, \alpha_2, \ldots)$ defined in (1.III(Q)). For example for the shifted skew tableau above we get

$\alpha_1(1)=\alpha_1(2)=\alpha_1(3)=1$, $\alpha_1(4)=2$, $\alpha_1(5)=\alpha_1(6)=\ldots=\alpha_1(14)=3$,
$\alpha_1(15)=\alpha_1(16)=4$, $\alpha_2(1)=\ldots=\alpha_2(5)=0$, $\alpha_2(6)=1$,
$\alpha_2(7)=\alpha_2(8)=\ldots=\alpha_2(13)=2$, $\alpha_2(14)=\ldots=\alpha_2(16)=3$,
$\alpha_3(1)=\alpha_3(2)=\ldots=\alpha_3(7)=0$, $\alpha_3(8)=\alpha_3(9)=\ldots=\alpha_3(16)=1$.

We say that $w = w_1\ldots w_k$ satisfies the <u>lattice property</u> iff the following implications hold

$$\begin{cases} \alpha_i(j)=\alpha_{i-1}(j) \Rightarrow w_{j+1} \ne i, i' & \text{for } 0\le j<k \\ \alpha_i(j)=\alpha_{i-1}(j) \Rightarrow w_{2k-j} \ne i-1, i' & \text{for } k\le j<2k \end{cases}$$

Equivalently one can rewrite this rule as follows. Given a word w , we define a second word v by reading w from right to left, changing i to i' and i' to i for every i, and finally by changing each i' to (i+1)'.

For example, for the above $w = 1\,1'2'1\,1\,2\,2\,3$, we have $v = 4'3'3'2'2'2\,1\,2'$.

Then the lattice property reads in a more homogenous way : for the word u=wv , whenever $\alpha_i(j) = \alpha_{i-1}(j)$ $(0\le j\le 2k)$, then $u_{j+1} \ne i, i'$.

For example 1 1'2'1 1 2 2 3 has the lattice property.

Observe that the lattice property implies that $\alpha_1(j) \ge \alpha_2(j) \ge \ldots$ for $j=0,1,\ldots,2k$.

<u>Theorem 4.1</u> (Stembridge $[\mathrm{Ste}]_2$)

(i) The coefficient $f(I,J;L)$ is equal to the number of shifted skew tableaux T of skew shape L/I filled up with j_1 symbols 1 or 1', j_2 symbols 2 or 2' etc. satisfying the following conditions:

(1) $w = w(T)$ has the lattice property.

(2) For every i, $1 \le i \le \ell(J)$ the first symbol i or i', which appears in reading w from right to left is the symbol i.

(ii) The coefficient $g(I;J)$ is the number of "shifted tableaux" T of unshifted shape J (i.e. the assignments $T: D_J \longrightarrow \{1',1,2',2,\dots\}$ satisfying the "shifted standardness" conditions (1,III(Q)) filled up with i_1 symbols 1 or 1', i_2 symbols 2 or 2' etc. such that

(1) $w = w(T)$ satisfies the lattice property.

(2) For every i, $1 \le i \le \ell(I)$, the first symbol i or i' which appears in reading w from right to left is the symbol i.

<u>Examples 4.2</u>

(1) $$P_{321}(A)\cdot P_{42}(A) = P_{741}(A) + P_{732}(A) + P_{651}(A) + 2P_{642}(A) + P_{543}(A) + P_{5421}(A) + P_{6321}(A)$$

The summand $2P_{642}(A)$ stems from

```
    . 2                    . 2
  . . 1 2'               . . 1 2
. . . 1' 1 1           . . . 1' 1 1      .
```

(2) $f((42),(431);(6521)) = 4$. It stems from :

```
        3                      3
      2 2                    1 2
    . . 1 1 2'             . . 1 2 2
  . . . . 1' 1  ,        . . . . 1 1  ,

      3                      3
    1 2                    1 2
  . . 1 2' 2             . . 1' 2' 2
. . . . 1 1  ,         . . . . 1 1       .
```


(3) $P_{\rho_k}(A)\cdot P_{\rho_\ell}(A) = P_{\rho_k+\rho_\ell}(A)$. For example for $k=4$, $\ell=3$ the only shifted skew tableau satisfying the requirements of Theorem 4.1 is :

```
        .
      . . 3
    . . . 2 2
  . . . . 1 1 1      .
```

(4) (Pieri's formula). Let us look at $f(I,(r);J)$. Let $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_{k+1})$ and let $1 \le p \le k$. If the $p+1$ -th row in $D^s_{J/I}$ has cardinality $\ge (i_p - i_{p+1})$, then the p -th row in $D^s_{J/I}$ is of the form

$$\boxed{1}\ \cdots\ \boxed{1} \qquad (j_p - i_p \text{ - times})$$

If the $p+1$ -th row in $D^s_{J/I}$ has cardinality less than $i_p - i_{p+1}$, then there are two possibilities for the content of the p-th row of $D^s_{J/I}$:

$$\left\{\begin{array}{l} \boxed{1}\boxed{1}\ \cdots\ \boxed{1} \\ \boxed{1'}\boxed{1}\ \cdots\ \boxed{1} \end{array}\right. \qquad (j_p - i_p \text{ - times})$$

For the k+1 -th row, if added, we have only one possibility

$$\boxed{1}\ \cdots\ \boxed{1}\ ,$$

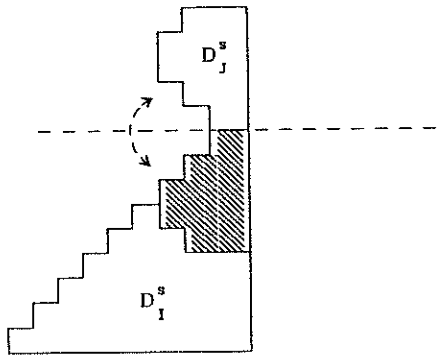
because of (2) in Theorem 4.1(i). This gives a (factor) contribution 1 or 2 to $f(I,(r);J)$ for every row numbered from 1 to k , in the case $j_{k+1} \ne 0$. On the other hand if $j_{k+1} = 0$, then the k-th row of $D^s_{J/I}$ looks like

$$\boxed{1}\ \cdots\ \boxed{1}$$

i.e. it gives a (factor) contribution 1 to $f(I,(r);J)$. We see that $f(I,(r);J)$ from Theorem 4.1(i) agrees with Pieri's formula in Proposition 2.7.

Before passing to the next example we need the following definition.

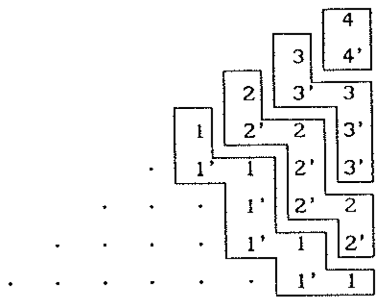
We say that strict partitions $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_r)$ are <u>ρ_n - complementary</u> if $k+r = n$ and $\{i_1,\dots,i_k,j_1,\dots,j_r\} = \{1,2,\dots,n\}$. Pictorially, in terms of shifted diagrams, it means that D^s_I, D^s_J placed in the following way :



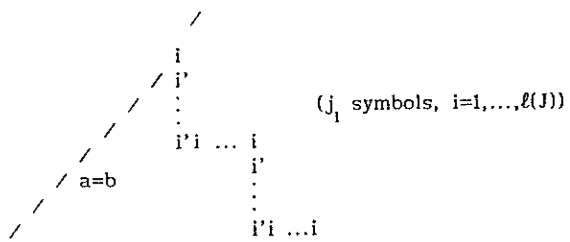
fill up the $D^s_{\rho_n}$.

For example (6431) and (8752) are ρ_8-complementary.

(5) $f(I,J;\rho_n) = 1$ iff I and J are ρ_n-complementary. For example to see that $f((6431),(8752);\rho_8) \geq 1$, we consider the skew shifted tableau :



In general, we construct such a shifted skew shape ρ_n/I with j_1 symbols 1 or 1', j_2 symbols 2 or 2' etc. by adding succesively border rims of the form (by a rim we mean a skew diagram without 2x2 square)



This is the only one skew shifted tableau of skew diagram $D^s_{\rho_n/I}$ which satisfies the requirements of Theorem 4.1(i)(2). Indeed, the $\ell(J)$ places on the line a=b with coordinates $a = \ell(I)+1, \ell(I)+2, \ldots, \ell(I)+\ell(J) = n$, must be occupied by $1,2,\ldots,\ell(J)$. A rim, filled up with symbols i and i', starts from the place $(\ell(I)+i, \ell(I)+i)$. Since such a rim must occupy j_i places, we see succesively for $i=1,2,\ldots,\ell(J)$, that all these rims are uniquely determined. This also shows that for partition $J' \neq J$, there is no skew shifted tableau of skew shape ρ_n/I which is filled up with j'_1 symbols 1 or 1', j'_2 symbols 2 or 2' etc., and satisfies the requirements of Theorem 4.1(i). We left to the reader a verification that the above skew shifted tableau has the lattice property.

(6) $P_{32}(A) = s_{32}(A) + s_{311}(A) + s_{221}(A)$, and the summands stems from

```
                 2            2
 2 2             2'           1 2'
 1 1 1   ,       1 1 1   ,    1' 1
```

(7) $g(I,I) = 1$ if I is strict, and it stems from shifted tableau T such that $T(a,b) = b$ for $b=1,\ldots,\ell(I)$. Moreover if $I = \rho_k$, $g(I,J) = 0$ for $J \neq I$ .Therefore $P_{\rho_k}(A) = s_{\rho_k}(A)$ in accordance with Lemma 2.3.

5. A GLIMPSE INTO ELIMINATION THEORY; SOME GENERALIZATIONS OF THE RESULTANT OF TWO POLYNOMIALS.

Let us fix integers $m,n > 0$ and $r \geq 0$. Consider two sequences

$$c.(A) = (c_1(A),\ldots,c_n(A)) \ , \ c.(B) = (c_1(B),\ldots,c_m(B))$$

of indeterminates such that $c.(A)c.(B)$ is a sequence of independent variables . Let $\mathbb{Z}[c.(A)c.(B)] = \mathbb{Z}[c_1(A),\ldots,c_n(A),c_1(B),\ldots,c_m(B)]$ be a polynomial $\mathbb{Z}$-algebra, where $\deg c_i(A) = \deg c_i(B) = i$. Let

$$A(x) = x^n + \sum_{i=1}^{n}(-1)^i c_i(A)\, x^{n-i} \quad , \quad B(x) = x^m + \sum_{j=1}^{m}(-1)^j c_j(B)\, x^{m-j}$$

be two polynomials in $\mathbb{Z}[c.(A)c.(B)][x]$ with "generic" coefficients.

Let $\mathcal{T}_r \subset \mathbb{Z}[c.(A)c.(B)]$ be the set of all polynomials T such that for every ring homomorphism $f:\mathbb{Z}[c.(A)c.(B)] \longrightarrow K$ (a field) <u>if</u> the polynomials

$$f[A(x)] = x^n + \sum_{i=1}^{n} (-1)^i f(c_i(A))\, x^{n-i} \quad \text{and} \quad f[B(x)]$$

have r+1 roots in common, <u>then</u> $f(T) = 0$.

In parallel with the general case we will consider the following variant. Let $\mathbb{Z}[c.(A)] = \mathbb{Z}[c_1(A),\dots,c_n(A)]$ be a polynomial $\mathbb{Z}$-algebra graded as above.

Let $\mathcal{T}_r{}' \subset \mathbb{Z}[c.(A)]$ (resp. $\mathcal{T}_r{}'' \subset \mathbb{Z}[c.(A)]$, r-even) be the set of all polynomials T such that for every ring homomorphism $f: \mathbb{Z}[c.(A)] \longrightarrow K$ (a field of characteristic $\neq 2$) <u>if</u> $f[A(x)]$ and $f[A(-x)]$ have r+1 roots in common (resp. r+1 nonzero common roots), <u>then</u> $f(T) = 0$. (Note that if char(K)=2 then every root of $f[A(x)]$ is also a root of $f[A(-x)]$; therefore we eliminate such fields of specializations).

Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$ be two sequences of indeterminates such that AB is a sequence of variables . Then the assignment

$$c_k(A) \longmapsto \text{(k-th elementary symmetric polynomial in A)}$$

defines an isomorphism $\mathbb{Z}[c.(A)] \simeq \mathcal{S}ym(A)$. A similar assignment for B allows us to treat $\mathbb{Z}[c.(A)c.(B)] \simeq \mathbb{Z}[c.(A)] \otimes \mathbb{Z}[c.(B)]$ as the ring $\mathcal{S}ym(A) \otimes \mathcal{S}ym(B) \simeq \mathcal{S}ym(A,B)$ of the polynomials over $\mathbb{Z}$ which are symmetric in A and B . In particular we can treat the ideals $\mathcal{T}_r$, $\mathcal{T}_r{}'$ and $\mathcal{T}_r{}''$ as ideals in $\mathcal{S}ym(A,B)$ (respectively in $\mathcal{S}ym(A)$).

The interpretation of $\mathcal{T}_r \subset \mathcal{S}ym(A,B)$, can be described as follows. $\mathcal{T}_r$ is the ideal of all polynomials P in $\mathcal{S}ym(A,B)$ such that for every specialization $f:\mathbb{Z}[a_1,\dots,a_n,b_1,\dots,b_m] \longrightarrow K$ (a field), if card $\{f(a_1),\dots,f(a_n)\} \cap \{f(b_1),\dots,f(b_n)\} \geq r+1$,then $f(P) = 0$ (see also 7.2). Sometimes, to emphasis the dependance of the ideals $\mathcal{T}_r$ and $\mathcal{T}_r{}'$, $\mathcal{T}_r{}''$ on different sets of independent variables, we will denote the above ideals in $\mathcal{S}ym(A,B)$ and $\mathcal{S}ym(A)$ respectively by $\mathcal{T}_r(A,B)$ and $\mathcal{T}_r{}'(A)$, $\mathcal{T}_r{}''(A)$.

Let $\mathcal{J}_r \subset \mathcal{S}ym(A,B)$ be the ideal generated by $s_I(A-B)$ where I runs over all partitions such that $I \supset (m-r)^{n-r}$. Let $\mathcal{J}_r{}' \subset \mathcal{S}ym(A)$ be the ideal generated by $P_I(A)$ where I runs over all (strict) partitions such that $I \supset \rho_{n-r}$, and let $\mathcal{J}_r{}'' \subset \mathcal{S}ym(A)$ be the ideal generated by $P_I(A)$ where I runs over all (strict) partitions such that $I \supset \rho_{n-r-1}$.

<u>Lemma 5.1</u> (i) $\mathcal{T}_0 \subset \mathcal{S}ym(A,B)$ is a principal ideal generated by $s_{(m)^n}(A-B)$ (which corresponds in $\mathbb{Z}[c.(A)c.(B)]$ to the resultant of A(x) and B(x)).

(ii) $\mathcal{T}_0{}' \subset \mathcal{S}ym(A)$ is a principal ideal generated by $P_{\rho_n}(A)$.

(iii) $\mathcal{T}_0{}'' \subset \mathcal{S}ym(A)$ is a principal ideal generated by $P_{\rho_{n-1}}(A)$.

<u>Proof.</u> (i) This follows from classical algebra (for the last statement see Lemma 2.3(i).

(ii) Since $P_{\rho_n}(A) = s_{(1)^n}(A) \cdot s_{\rho_{n-1}}(A)$ (see Lemma 2.3(iii)) and both $s_{(1)^n}(A)$, $s_{\rho_{n-1}}(A)$ are irreducible in $\mathbb{Z}[c.(A)]$ it suffices to prove that every $T \in \mathcal{T}_0{}'$ is divisible by $s_{(1)^n}(A)$ and $s_{\rho_{n-1}}(A)$. The first divisibility follows from the fact that T vanishes if A(x) has a root equal to zero. The second divisibility is a consequence of the vanishing of T if all roots $a_1,\dots,a_n$ of A(x) are nonzero , but there exist $i \neq j$ such that $a_i + a_j = 0$.

(iii) The proof of this statement is even easier; it suffices to remark that $P_{\rho_{n-1}}(A)$ is (generically) irreducible in $\mathbb{Z}[c.(A)]$. ■

<u>Lemma 5.2</u> $\mathcal{J}_r \subset \mathcal{T}_r$, $\mathcal{J}_r{}' \subset \mathcal{T}_r{}'$ and $\mathcal{J}_r{}'' \subset \mathcal{T}_r{}''$

<u>Proof.</u> (i) Let $A = A' \dot{\cup} C$, $B = B' \dot{\cup} C$ be the sets of roots of $f(A(x))$, $f(B(x))$ where $f: \mathbb{Z}[c.(A)c.(B)] \longrightarrow K$ is a ring homomorphism, K being a field and card C = r+1 . Then $s_I(A-B) = s_I(A'-B')$. Finally, if $I \supset (m-r)^{n-r}$, then $s_I(A'-B') = 0$ by the factorization formula, because card A' = n-r-1 and card B' = m-r-1 .

(ii) Let $A = A' \dot{\cup} C$ where card C = r+1 be the set of roots of $f[A(x)]$ (f as above), satisfying $(C)=(C^-)$. Then $P_I(A) = P_I(A')$. Now, if $I \supset \rho_{n-r}$, then $P_I(A') = 0$ by the factorization formula, because card A' = n-r-1 .

(iii) In this case, we assume that $A = A' \dot{\cup} C$ where card C = r+2 is even, $C=(c_1,c_2,\dots,c_{r+1},c_{r+2})$ and $c_{2i-1} = -c_{2i}$ i=1,2,... . If $I \supset \rho_{n-r-1}$, then $P_I(A) = P_I(A') = 0$ by the factorization formula because card (A) = n-r-2 .■

The main result of this Section is the following

<u>Theorem 5.3</u> (i) $\mathcal{T}_r = \mathcal{J}_r$,

(ii) $\mathcal{T}_r{}' = \mathcal{J}_r{}'$,

(iii) $\mathcal{T}_r{}'' = \mathcal{J}_r{}''$ (r-even).

<u>Proof.</u> Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$ be two sequences of indeterminates such that AB is a sequence of independent variables.

(i) We will give two different proofs of (i). The first proof has already appeared in $[P]_2$. We take this opportunity to make smooth some points of exposition of $[P]_2$. To give the first proof we need the following

<u>Claim</u> No nonzero $\mathcal{S}ym(B)$ - combination of the $s_I(A-B)$'s , with all I such that I does not contain $(m-r)^{n-r}$, belongs to $\mathcal{T}_r(A,B)$.

We prove the claim by induction on r . For r=0 suppose that there exists a nonzero element $\sum_I \alpha_I(B)\, s_I(A-B)$ in $\mathcal{T}_0(A,B)$, where $\alpha_I(B) \in \mathcal{S}ym(B)$, and I does not contain $(m)^n$ if $\alpha_I(B)\neq 0$. It follows from Lemma 5.1(i) that

$$\sum_I \alpha_I(B)\, s_I(A-B) = s_{(m)^n}(A-B) \left[\sum_J \beta_J(B)\, s_J(A) \right]$$

for some nonzero $\beta_J(B)$'s in $\mathcal{S}ym(B)$. Let $\bar{J}$ be the partition such that $\beta_{\bar{J}}(B) \neq 0$ and which dominates other J's for which $\beta_J(B) \neq 0$ with respect to the "reverse lexicographic ordering" of [M]: for $J \neq \bar{J}$ the first non-vanishing difference $\bar{j}_h - j_h$ is positive, h=1,2,... . Then in the expression of the right hand side as a $\mathcal{S}ym(B)$-combination the summand $\beta_{\bar{J}}(B)\cdot s_{(m)^n+\bar{J}}(A)$ appears. Indeed, it stems from $s_{(m)^n}(A)\cdot\beta_{\bar{J}}(B)\cdot s_{\bar{J}}(A)$ by applying the factorization formula. It cannot be cancelled with any other summand by the Littlewood-Richardson rule (see [M] I.9) applied to the summands $s_K(A)\ s_J(A)\ \beta_J(B)$, in the right hand side (recall that $(a_1,\dots,a_n,b_1,\dots,b_m)$ is a sequence of independent variables). On the contrary, on the left hand side such an summand cannot appear (there is no I , $\alpha_I(B) \neq 0$, such that $I \supset (m)^n$).

To perform the induction step $r-1 \longrightarrow r$ suppose that there is a nonzero element of the form $\sum \alpha_I(B)\, s_I(A-B)$ in $\mathcal{T}_r(A,B)$, where $\alpha_I(B) \in \mathcal{S}ym(B)$ and $\alpha_I(B) \neq 0$ implies I does not contain $(m-r)^{n-r}$. Let us use the following specialization $A = A'\dot{\cup} C$, $B = B'\dot{\cup} C$, where A',B',C are three transcendental over each other sets of variables and card(C)=r. Ob-

serve that by our inductive assumption the $\mathcal{S}ym(B)$ - combination $\sum_I \alpha_I(B)\, s_I(A-B)$ in question does not belong to $\mathcal{T}_{r-1}(A,B)$ (in fact for a I such that $\alpha_I(B) \neq 0$, I does not contain $(m-r+1)^{n-r+1}$). This implies th after the above specialization $\sum_I \alpha_I(B',C)\, s_I(A'-B') \in \mathcal{S}ym(A',B',C)$ is nc zero, because C consists of independent indeterminates. Since, by the fundamental theorem on symmetric polynomials, $\mathcal{S}ym(C)$ is a polynomial alge bra over $\mathbb{Z}$, we conclude that there exists a specialization $\phi: \mathcal{S}ym(C) \longrightarrow$ such that the induced ring homomorphism

$$\Phi:\ \mathcal{S}ym(A',B',C)=\mathcal{S}ym(A',B')\otimes \mathcal{S}ym(C) \xrightarrow{1\otimes\phi} \mathcal{S}ym(A',B')$$

satisfies $\sum_I \Phi[\alpha_I(B',C)]\ s_I(A'-B') \neq 0$. Moreover, by construction this el ement vanishes when A' and B' are specialized to the sets of roots of two polynomials of degree n-r and m-r such that card $(A' \cap B') \geq 1$. In other words, with the notation before Lemma 5.1,

$$0 \neq \sum_I \Phi[\alpha_I(B',C)]\ s_I(A'-B') \in \mathcal{T}_0(A',B') .$$

Therefore this $\mathcal{S}ym(B')$-combination contradicts the preliminary step r=0 when A and B are replaced by A' and B' respectively. The claim is proved.

Every element in $\mathcal{S}ym(A,B)$ can be written as $\sum_I \alpha_I(B)\, s_I(A-B)$, where $\alpha_I(B) \in \mathcal{S}ym(B)$ (use the linearity formula). Therefore the equality $\mathcal{T}_r = \mathcal{J}$ now follows from the claim and Lemma 5.2(i) .

Now, we want to present an alternative, method of proving Theorem 5. which will work also in cases (ii) and (iii). We first prove the following

<u>Lemma 5.4</u> Let R be a commutative ring . Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_m)$ be two sequences such that AB is a sequence of independent indeterminates over R. Assume that $n \leq m$.

(i) Denote by $\mathcal{S}ym(A,B)_R$ the ring of polynomials symmetrical in A and separately with coefficients in R . Then the ideal of polynomials $P \in \mathcal{S}ym(A,B)_R$ such that $P(A',B')=0$ if A',B' are specializations of A and satisfying $\{A'\} \subset \{B'\}$, is equal to the ideal ($s_I(A-B)$, $I \supset (m-n+1)$).

(ii) Assume that $R \supset \mathbb{Z}$. Denote by $\mathcal{S}ym(A)_R$ the ring of polynomials symmetrical in A with coefficients in R. Then the ideal of polynomials $P \in \mathcal{S}ym(A)_R$ such that $P(A') = 0$ if A' is a specialization of A in a field of characteristic $\neq 2$, satisfying $\{A'\} = \{A'^{-}\}$, is equal to the ideal $(P_I(A), |I|\geq 1)$.

<u>Proof.</u> (i) It follows from the theory of symmetric polynomials (see [M] chap.I) that $\mathcal{S}ym(A,B)_R \simeq R[s_1(A),\dots,s_n(A),s_1(A-B),\dots,s_m(A-B)]$, where $s_1(A),\dots s_n(A),s_1(A-B),\dots,s_m(A-B)$ are algebraically independent over R. Consider the specialization $A'=A$, $B'=A\cup C$, where C is a sequence of algebraically independent elements over $R[A]$, $\mathrm{card}(C)=m-n$. If $P(A,B) \in \mathcal{S}ym(A,B)_R$ is not in $(s_I(A-B), I\supset(m-n+1)) = (s_i(A-B), i=m-n+1,\dots,m)$ (see Proposition 5.8), then it contains a nontrivial R-combination of monomials in $s_1(A),\dots,s_n(A),s_1(A-B),\dots,s_{m-n}(A-B)$. Under the above specialization this combination gives rise to a nonzero element in $\mathcal{S}ym(A,C)_R$ which contradicts $P(A',B')=0$. The assertion follows.

(ii) Since every polynomial $P_I(A)$ has the property that $P_I(A)=0$ if $\{A'\}=\{A'^{-}\}$, it suffices to prove that if $P(A')=0$ for any specialization of A' of A in a field of characteristic $\neq 2$ satisfying $\{A'\}=\{A'^{-}\}$, then $P(A)\in(P_I(A),|I|\geq 1)$. Let $n=\mathrm{card}\,A=2m$ (resp. $n=2m+1$). Consider the specialization $A'=(b_1,\dots,b_m,-b_1,\dots,-b_m)$ (resp. $A'=(b_1,\dots,b_m,-b_1,\dots,-b_m,0)$) where $b_1,\dots,b_m$ are independent over R. We have $\psi_{2i+1}(A)=0$ and $\psi_{2i}(A')=2(b_1^{2i}+\dots+b_m^{2i})$ for every i. From the theory of symmetric polynomials (see [M],chap.I) $\mathcal{S}ym(A)_R\otimes\mathbb{Q} = R[\psi_1(A),\dots,\psi_n(A)]\otimes\mathbb{Q}$. If a polynomial $P(A)\in\mathcal{S}ym(A)_R\otimes\mathbb{Q}$ does not belong to $(\psi_1(A),\psi_3(A),\dots,\psi_{2[n/2]-1}(A))$, then it contains a nontrivial combination of monomials in $\psi_2(A),\psi_4(A),\dots,\psi_{2[n/2]}(A)$. Under the above specialization, this combination gives rise to a nonzero element in $\mathcal{S}ym(b_1^2,\dots,b_m^2)_R\otimes\mathbb{Q}$, which contradicts $P(A')=0$. Finally, since $\psi_{2i+1}(A)\in(P_I(A),|I|\geq 1)$ (see Proposition 2.5), we get the assertion. ∎

The next Lemma, the proof of which we postpone to the Appendix, plays a crucial role in the proof of (ii),(iii).

<u>Lemma 5.5</u> Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$ be two sequences of indeterminates such AB is a sequence of independent variables. Write $A_r=(a_1,\dots,a_r)$, $A^{n-r}=(a_{r+1},\dots,a_n)$, $B_r=(b_1,\dots,b_r)$, $B^{m-r}=(b_{r+1},\dots,b_m)$.

(i) The map $\varphi:\mathcal{S}ym(A_r,A^{n-r},B_r,B^{m-r})\longrightarrow\mathcal{S}ym(A,B)$ given by

$$P=P(A_r,A^{n-r},B_r,B^{m-r})\longrightarrow\sum u\times v\left[P\cdot s_{(m-r)^{n-r}}(A^{n-r}-B^{m-r})\right],$$

where the sum is over all pairs (u,v), $u\in S_n/S_r\times S_{n-r}$, $v\in S_m/S_r\times S_{m-r}$,

induces an isomorphism

$$\bar{\varphi}:\mathcal{S}ym(A_r,A^{n-r},B_r,B^{n-r})/(s_I(A_r-B_r),|I|\geq 1)\simeq \mathcal{I}_r/\mathcal{I}_{r-1}.$$

(ii) The map $\varphi:\mathcal{S}ym(A_r,A^{n-r})\longrightarrow\mathcal{S}ym(A)$ given by

$$P(A_r,A^{n-r})\longrightarrow\sum_{w\in S_n/S_r\times S_{n-r}} w\left[P(A_r,A^{n-r})\cdot P_{\rho_{n-r}}(A^{n-r})\right] \qquad (\#)$$

induces an isomorphism

$$\bar{\varphi}:\mathcal{S}ym(A_r,A^{n-r})/(P_I(A_r),|I|\geq 1)\simeq \mathcal{I}'_r/\mathcal{I}'_{r-1}$$

(iii) The map $\varphi:\mathcal{S}ym(A_r,A^{n-r})\longrightarrow\mathcal{S}ym(A)$ given by

$$P(A_r,A^{n-r})\longrightarrow\sum_{w\in S_n/S_r\times S_{n-r}} w\left[P(A_r,A^{n-r})\cdot P_{\rho_{n-r-1}}(A^{n-r})\right]$$

(r–even), induces an isomorphism

$$\bar{\varphi}:\mathcal{S}ym(A_r,A^{n-r})/(P_I(A_r),|I|\geq 1)\simeq \mathcal{I}''_r/\mathcal{I}''_{r-2}.$$

We come back to the proof of Theorem 5.3. We will prove the statemer (ii). First, we show that for every r we have $\mathcal{I}'_r/\mathcal{I}'_{r-1}\hookrightarrow\mathcal{T}'_r/\mathcal{T}'_{r-1}$. For r=0 the assertion is true by Lemma 5.1. Assume r≥1. By Lemmas 5.5 and 5.2 we infer that $\varphi(P_I(A_r),|I|\geq 1)\subseteq\mathcal{I}'_{r-1}\subseteq\mathcal{T}'_{r-1}$. Therefore φ induces a morphism

$$\bar{\varphi}:\mathcal{S}ym(A_r,A^{n-r})/(P_I(A_r),|I|\geq 1)\longrightarrow\mathcal{S}ym(A)/\mathcal{T}'_{r-1}.$$

We have $\mathrm{Im}\,\varphi\subseteq\mathcal{T}'_r$. Indeed, if we specialize A in a field of characteristic ≠2 by putting $A=B\dot\cup C$, where card B = r+1 and $\{B\}=\{B^{-}\}$, then in any summand w[ ] appearing in (#), one of the factors $w(a_1)$ or $w(a_i+a_j)$ of $w[P_{\rho_{n-r}}(A^{n-r})]$ must vanish. Thus we have the map

$$\bar{\varphi}:\mathcal{S}ym(A_r,A^{n-r})/(P_I(A_r),|I|\geq 1)\longrightarrow\mathcal{T}'_r/\mathcal{T}'_{r-1}.$$

We claim that $\bar{\varphi}$ is a monomorphism. To see it apply Lemma 5.4 (ii) with $R=\mathcal{S}ym(A^{n-r})$. If $\bar{\varphi}(P(A_r))\in\mathcal{T}'_{r-1}$ (where $P(A_r)\in\mathcal{S}ym(A_r)_R$), then for every specialization A' of A_r in a field of characteristic ≠2, satisfy ing $\{A'\}=\{A'^{-}\}$, we have $P(A')=0$. Thus, we infer from Lemma 5.4 (ii) that $P(A_r) \in (P_I(A_r), |I|\geq 1)$ and the assertion, that for every $r=1,\dots,n-1$ we have $\mathcal{I}'_r/\mathcal{I}'_{r-1} \hookrightarrow \mathcal{I}'/\mathcal{I}'_{r-1}$, now follows.

Applying Lemma 5.4(ii) once again we get that $\mathcal{I}'_{n-1} = \mathcal{I}'_{n-1}$. We use a descending induction on r. Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}'_{r-1} & \longrightarrow & \mathcal{I}'_r & \longrightarrow & \mathcal{I}'_r/\mathcal{I}'_{r-1} & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & \mathcal{I}'_{r-1} & \longrightarrow & \mathcal{I}'_r & \longrightarrow & \mathcal{I}'/\mathcal{I}'_{r-1} & \longrightarrow & 0 \end{array} ,$$

where α and β are the inclusions. Assume by induction that β is the identity. Then, since γ is injective, a simple diagram chase shows that α is also the identity, and the induction step is proved.

The proof of (i) and (iii) for n even can be done mutatis mutandis. To prove (iii) for $n=2m+1$, say, we need to know that $\mathcal{I}''_{2(m-1)}$ is the ideal generated by $P_I(A)$, $I\supset(2,1)$. We know already that $\mathcal{I}'_{2m-1}$ is generated by $P_I(A)$, $I\supset(2,1)$. Suppose that $P(A)$ is in $\mathcal{I}''_{2(m-1)} \setminus \mathcal{I}'_{2m-1}$. Then $P(B)=0$ for a specialization $B=(b_1,\dots,b_{2m},0)$ of A such that b_i are nonzero complex numbers and $\{B\} = \{B^-\}$. But then by letting some of the b_i's tend to zero, we see that in fact $P(A)$ must lie in $\mathcal{I}'_{2m-1}$. Thus $\mathcal{I}''_{2(m-1)} = \mathcal{I}'_{2m-1}$ and the proof of Theorem 5.3 (iii) for n odd can be performed in a similar way as in the above cases. ■

Let us note (see also 7.4) :

<u>Corollary 5.6</u> The polynomial $s_{(m-r)^{n-r}}(A-B)$ is invariant under the substitution $a_i \longmapsto a_i+t$, $b_j \longmapsto b_j+t$, $i=1,\dots,n$, $j=1,\dots,m$, t being a variable independent of A and B .

<u>Proof.</u> Indeed, this polynomial is the generator of the component of the minimal degree of the ideal $\mathcal{I}_r$. $\mathcal{I}_r$ is preserved by any automorphism of $\mathbb{Z}[c.(A)c.(B)][x,t]$, and in particular - by the automorphism which is the identity on $\mathbb{Z}[c.(A)c.(B)][t]$, and which sends x onto x+t . The assertion follows. ■

Now we pass to some algebraic properties of the ideals $\mathcal{I}_r$, $\mathcal{I}'_r$, $\mathcal{I}''_r$.

<u>Lemma 5.7</u> (S) For every partition I , where $\ell(I) \leq n-r$, $s_{(m-r)^{n-r}+I}(A-B)$ is in the ideal generated by $s_{(m-r)^{n-r}+I'}(A-B)$, where $I' \subset (r)^{n-r}$.

(S') For every partition J , where $\ell(J^\sim) \leq m-r$, $s_{(m-r)^{n-r},J}(A-B)$ is in the ideal generated by $s_{(m-r)^{n-r},J'}(A-B)$, where $J' \subset (m-r)^r$.

(Q) For every partition I , where $\ell(I) \leq n-r$, and $k=n-r$ or $k=n-r-1$, $P_{\rho_k+I}(A)$ is in the ideal generated by $P_{\rho_k+I'}(A)$, where $I' \subset (r)^{n-r}$.

<u>Proof.</u> (S) We can assume that A and B consist of independent indeterminates. Thus we can identify A and B as the sets of Chern roots of vector bundles E and F in the following situation. Let $X = G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty)$, $E = (R^n)_X$ and $F = (R^m)_X$, where R^n (resp. R^m) is the tautological vector bundle on $G_n(\mathbb{C}^\infty)$ (resp. on $G_m(\mathbb{C}^\infty)$). Theorem 3.3 remains true if X is a topological paracompact space, E is a complex vector bundle on X, the Chern classes are located in $H^\cdot(X,\mathbb{Z})$ and $H.(-,\mathbb{Z})$ plays the role of $A.(-)$. In particular this theorem holds for X and E defined above. Let $\pi: G=G_r(E) \longrightarrow X$ be the Grassmannian bundle of r-subbundles in E . Let $g = s_{(m)^{n-r}+I}(Q-F_G)$ in $H^\cdot(G,\mathbb{Z})$. By the factorization formula

$$g = s_I(Q)\, s_{(m)^{n-r}}(Q-F_G)$$

From the Schubert Calculus for vector bundles (cf. [F] chap.14) we infer that $s_I(Q)$ is a combination $\sum \alpha_{I'} s_{I'}(Q)$, where $I' \subset (r)^{n-r}$ and $\alpha_{I'}$ is a polynomial in the Chern classes of E. In particular $\alpha_{I'} \in \mathcal{S}ym(A,B)$. We have

$$s_{I'}(Q)\, s_{(m)^{n-r}}(Q-F_Q) = \sum_{I'} \alpha_{I'} s_{(m)^{n-r}+I'}(Q-F_G)$$

by the factorization formula once again. Finally by Theorem 3.3(i) we see that

$$s_{(m-r)^{n-r}+I}(E-F) = \pi_*\left[s_{(m)^{n-r}+I}(Q-F_G) \right] =$$

$$= \pi_*\Big[\sum_{I'} \alpha_{I'} s_{(m)^{n-r}+I'}(Q-F_G) \Big] = \sum_{I'} \alpha_{I'} s_{(m-r)^{n-r}+I'}(A-B)$$

where $\alpha_{I'} \in \mathcal{S}ym(A,B)$, as wanted. The proof of (S') is analogous.

(Q) Assume the above notation. Let now $X = G_n(\mathbb{C}^\infty)$ and $E = (R^n)_X$. Let

$$g = c_{top}(R\otimes Q)\, P_{\rho_k+I}(Q) \qquad (\text{ in } H^{\cdot}(G,\mathbb{Z}))\ .$$

By the factorization formula

$$g = c_{top}(R\otimes Q)\, s_{\rho_k}(Q)\, s_I(Q)\ .$$

From the Schubert Calculus for vector bundles, we have $s_I(Q) = \sum \alpha_{I'} s_{I'}(Q)$, where $I' \subset (r)^{n-r}$ and $\alpha_{I'} \in \mathcal{S}ym(A)$. Therefore

$$g = \sum_{I'} \alpha_{I'} c_{top}(R\otimes Q)\, s_{\rho_k}(Q)\, s_{I'}(Q) = \sum_{I} \alpha_{I'} c_{top}(R\otimes Q)\, P_{\rho_k+I'}(Q)$$

Finally, by Theorem 3.3(ii) we have

$$P_{\rho_k+I}(E) = \pi_*\Big[c_{top}(R\otimes Q)\, P_{\rho_k+I}(Q) \Big]$$

$$= \pi_*\Big[\sum_{I'} \alpha_{I'}\, c_{top}(R\otimes Q)\, P_{\rho_k+I'}(Q) \Big] = \sum_{I'} \alpha_{I'}\, P_{\rho_k+I'}(A)$$

where $\alpha_{I'} \in \mathcal{S}ym(A)$, as needed. ■

From the above Lemma we infer

Proposition 5.8 (i) $\mathcal{I}_r$ is generated by $s_{(m-r)^{n-r}+I}(A-B)$,where $I \subset (r)^{n-r}$.

(ii) $\mathcal{I}_r{}'$ is generated by $P_{\rho_{n-r}+I}(A)$, where $I \subset (r)^{n-r}$.

(iii) $\mathcal{I}_r{}''$ is generated by $P_{\rho_{n-r-1}+I}(A)$, where $I \subset (r)^{n-r}$ (r–even) .

Let us notice also

Proposition 5.9 The polynomials $s_{I_k}(A-B)\, s_{J_k}(A)$, where $I_k \supset (m-k)^{n-k}$ and I_k does not contain $(m-k+1)^{n-k+1}$, $\ell(J_k) \le k$, $k=0,1,\dots,r$, form a $\mathbb{Z}$–basis of $\mathcal{I}_r$. Another $\mathbb{Z}$–basis of $\mathcal{I}_r$ is given by $s_{I_k}(A-B)\, s_{J_k}(B)$ for the same I_k, J_k , $k=0,1,\dots,r$.

The proof depends heavily on the isomorphism

$$\mathcal{I}_r/\mathcal{I}_{r-1} \simeq \mathcal{S}ym(A_r,A^{n-r},B_r,B^{n-r})/(s_I(A_r-B_r), |I|\ge 1)$$

described in Lemma 5.5(i). For details see $[P]_3$, Proposition 6.2.

Remark 5.10 One can prove that $\mathcal{I}_r$ is prime, $\mathcal{I}_r{}'$ is radical and $\mathcal{I}_r{}''$ (r–even) is prime. For example, writing $\mathbb{Z}[AB]$ for the polynomial ring $\mathbb{Z}[a_1,\dots,a_n,b_1,\dots,b_m]$ the primeness of $\mathcal{I}_r$ follows from an interpretation of $\mathcal{I}_r$ as the kernel of the map

$$\mathbb{Z}[c.(A)c.(B)] \hookrightarrow \mathbb{Z}[AB] \longrightarrow \mathbb{Z}[AB]\big/(a_1-b_1,\dots,a_{r+1}-b_{r+1}),$$

where the first inclusion is given by $c_k(A) \longmapsto$ (k-th elementary symmetric polynomial in A) and likewise for the $c_k(B)$'s (see $[P]_2$).

At the end of this Section we discuss related criterion giving conditions when two equations have r+1 roots in common ($r \ge 0$). Let

$$A(x) = x^n + \sum_{i=1}^{n} (-1)^i c_i(A)\, x^{n-i}\ , \quad B(x) = x^m + \sum_{j=1}^{m} (-1)^j c_j(B)\, x^{m-j}$$

be two polynomials with the coefficients in the ring R which is a domain. Let S be the algebraic closure of the field of quotients of R . Then A(x) and B(x) factorize in S[x]

$$A(x) = \prod_{i=1}^{n} (x-a_i) \quad , \quad B(x) = \prod_{j=1}^{m} (x-b_j)$$

where a_i , $i=1,\dots,n$; b_j , $j=1,\dots,m$ are in S. Let $A = (a_1,\dots,a_n)$ $B = (b_1,\dots,b_m)$.

Proposition 5.11

(i) In the above situation A(x) and B(x) have at least r+1 roots in common iff

$$s_{(m-i)^{n-i}}(A-B) = 0$$

for $i = 0,1,\dots,r$.

(ii) Assume that char(S)≠2. Then A(x) and A(−x) have at least r+1 roots in common iff

$$P_{\rho_{n-i}}(A) = 0$$

for $i = 0,1,\dots,r$.

(iii) Assume that char(S)≠2 and A(0)≠0. Then, for even r, A(x) and A(-x) have at least r+1 common roots iff

$$P_{\rho_{n-2i-1}}(A) = 0$$

for $i = 0,1,\dots,r/2$.

<u>Proof.</u> The proofs of (i)-(iii) are similar. Let us show, for example (iii). It follows from the assumptions that A(x) and A(-x) have r+2 roots in common. If $P_{\rho_{n-1}}(A) = \prod_{i<j}(a_i+a_j)$ is zero then we can assume that $a_{n-1}=-a_n$ after some change of indices. Then $P_{\rho_{n-3}}(A) = P_{\rho_{n-3}}(a_1,\dots,a_{n-2})$ and its vanishing means $a_{n-3} = -a_{n-2}$ after some change of indices. By re-repeating succesively the above argument, we get the assertion.■

<u>Remark 5.12</u> The assertion (i) was proved for the first time by Trudi in [T] by using different language and different methods.

<u>Remark 5.13</u> In [Po], the author proved an another criterion, when two equations have a prescribed number of roots in common. In the above notation A(x) and B(x) have at least r+1 roots in common iff the elements

$$s_{(m-r)^{n-r}+(i)}(A-B)$$

of R vanish for $i=0,1,\dots,r$. Pomey gives also the following expression for the greatest common divisor of two polynomials in one variable (see [Po] and for a modern approach [L]; the proof of Lemma 4.3). Assume that C(x) - the greatest common divisor of A(x) and B(x) is of degree r. Then, up to the factor $s_{(m-r)^{n-r}}(A-B)$,

$$C(x) = \sum_{i=0}^{r} s_{r-i}(x-A)\, s_{(m-r)^{n-r}+(i)}(A-B) \quad ,$$

in the natural "λ-ring" notation .

6. Q-POLYNOMIALS AND SCHUBERT CALCULUS FOR GRASSMANNIANS OF ISOTROP SUBSPACES.

Let V be a 2n-dimensional vector space over $\mathbb{C}$, and let $\Phi: V\times V \longrightarrow \mathbb{C}$ be a nondegenerate symplectic form on V . Denote

$$X = \left\{ L \subset V : L \text{ is n-dimensional subspace of } V, \text{ isotropic with respect to } \Phi \right\}$$

<u>Proposition 6.1</u> (i) X is an algebraic variety of dimension n(n+1)/2, Pic(X)=$\mathbb{Z}$, and the generator of Pic(X) is given by the following composition

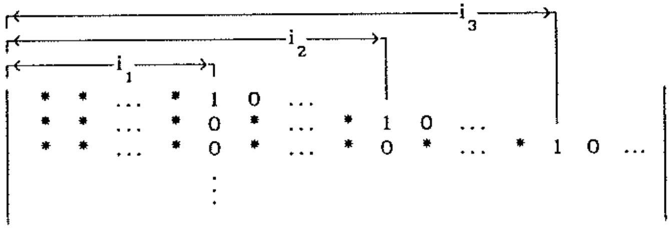
$$X \xrightarrow[\text{standard}]{} G_n(V) \xrightarrow[\text{Plucker}]{} \mathbb{P}_{\mathbb{C}}^{\binom{2n}{n}-1} .$$

(ii) X can be interpreted as the homogenous space $Sp(2n)/P_\alpha$,where α is the "right end root" in the standard basis Σ of the root system of type C and P_α is the corresponding maximal parabolic subgroup.

(iii) X can be interpreted as the homogenous space Sp(n)/U(n) (in the notation of [He]).

For a proof of (i),(ii) - see [L-Se]; as for details concerning (iii) - see [He].

The additive structure of the Chow ring A(X) is a part of a general theory about cell-decompositions of the homogenous spaces G/P_α (see [D], [B-G-G]). In particular one knows that X has a cell-decomposition, where the cells are affine spaces and are parametrized by elements of W/W_α , where W is the Weyl group of type C_n and W_α is the subgroup of W generate by the simple reflections corresponding to roots in $\Sigma-\{\alpha\}$. In this concrete case one can obtain it as follows. Let $e_1,\dots,e_n,f_n,\dots,f_1$ be a basis of V such that $\Phi(e_i,e_j)= \Phi(f_i,f_j)=0$, and $\Phi(e_i,f_j)= -\Phi(f_j,e_i)= \delta_{i,j}$ $i,j=1,\dots,n$. Every n-plane L can be represented as a nx2n matrix where the rows are sequences of coordinates of vectors spanning L with respect to the above basis. We can choose this matrix in the following form

(#) 

i.e. depending on $\sum i_k - k(k+1)/2$ parameters for some sequence $0 \le i_1 < \dots < i_n \le 2n$. Conversely, every such a matrix represents a certain n-plane. If L is isotropic, however, then using orthogonality relations between rows of (#) one sees that the sequence $I=(i_1,\dots,i_n)$ arising in the above way must satisfy the condition

(##) $\qquad 2n+1-i_p \ne i_q \quad$ for every $p,q = 1,\dots,n$.

Moreover, we have relations between the entries of (#). First, from the orthogonality of two rows with nontrivial parameters where the length of the first row is ≤ n :

$$\begin{array}{l} \qquad\quad |\leftarrow a \rightarrow|\leftarrow a \rightarrow| \\ (\dots x \ \dots \ 1 \ 0 \ \dots \ 0 \ ; \ 0 \ \ . \ . \ . \ . \ . \ . \ \ 0\ ) \\ (\ \ . \ . \ . \ . \ . \ . \ \ ; \ \ . \ . \ . \ \ y \ \dots \ 1 \ 0 \ \dots \ 0\ ), \\ |\leftarrow b \rightarrow|\leftarrow b \rightarrow| \end{array}$$

we obtain $x=y$ (x and y are nonzero because of (##)). Secondly, from the orthogonality of two rows with nontrivial parameters where the length of both of them are ≥ n

$$\begin{array}{l} \qquad\quad |\leftarrow a \rightarrow|\leftarrow a \rightarrow| \\ (\dots x \ \ . \ . \ . \ \ ; \ \ . \ . \ . \ . \ \ 1 \ 0 \ \ . \ .. \ 0) \\ (\ \ . \ . \ . \ \ y \ \ . \ . \ . \ . \ \ ; \ \ . \ . \ . \ \ 1 \ 0 \dots 0) \\ \quad |\leftarrow b \rightarrow|\leftarrow b \rightarrow| \end{array}$$

we get $x=y$.

For example if $n=3$ we obtain the following sequences satisfying (##) : (123), (124), (135), (145), (236), (246), (356), (456). The following matrices are associated respectively with (135) (246) and (456).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & 1 & 0 & 0 & 0 \\ 0 & x_{32} & 0 & x_{34} & 1 & 0 \end{bmatrix}, \quad \text{where } x_{22} = x_{34} ,$$

$$\begin{bmatrix} x_{11} & 1 & 0 & 0 & 0 & 0 \\ x_{21} & 0 & x_{23} & 1 & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 & x_{35} & 1 \end{bmatrix}, \quad \text{where } x_{11} = x_{35} ,\ x_{21} = x_{33} ,$$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 1 \end{bmatrix}, \quad \text{X is symmetric with respect to the antidiagonal.}$$

In general for a given sequence $I=(i_1,\dots,i_n)$, $1 \le i_1 < \dots < i_n < 2n$ sa tisfying (##), define k to be the greatest number such that $i_k \le n$. Th the corresponding matrix depends on

$$\sum_{p=1}^{k} i_p + (n+1)(n-k) - n(n+1)/2$$

independent parameters. In the above example, we obtain successively num bers 0, 1, 2, 3, 3, 4, 5, 6 . Of course every isotropic n-space corresp nding to the sequence $I=(i_1,\dots,i_n)$ belongs to the following set

(###) $$\left\{ L \in X : \dim(L \cap V_{i_p}) \ge p \quad p=1,\dots,n \right\}$$

where V_i $i=1,\dots,2n$ is the vector space spanned by the first i vector in the sequence $e_1,\dots,e_n,f_n,\dots,f_1$. Note that the sequence $I=(i_1,\dots,$ $1 \le i_1 < \dots < i_n \le 2n$ satisfying (##) is uniquely specified by $i_1,\dots,i_k$. Therefore we can rewrite the set (###) as

$$\Omega(i_1,\dots,i_k) = \left\{ L \in X : \dim(L \cap V_{i_p}) \ge p ,p=1,\dots,k \right\}$$

The equations of $\Omega(i_1,\dots,i_k)$ in X are written in an explicit way in [L-Se], where the following fact is also proved.

<u>Lemma 6.2</u> The subscheme $\Omega(i_1,\dots,i_k)$ is a schematic intersection of the Schubert variety

$$\left\{ L \in G_n(V) \mid \dim(L \cap V_{i_p}) \ge p , \ p=1,\dots,n \right\}$$

with standard structure of subscheme of $G_n(V)$, with X .

Denote also by $\Omega(i_1,\dots,i_k)$ the cycle in the Chow ring A(X) determined by the cycle (###) (it does not depend on the choice of the basis $e_1,\dots,e_n,f_n,\dots,f_1$). Thus we get an element

$$\Omega(i_1,\dots,i_k) \in A_{i_1+\dots+i_k+(n+1)(n-k)-n(n+1)/2}(X) ,$$

and in fact the previous considerations shows that the cycles $\Omega(i_1,\dots,i_k)$ are closures of cells of the cell-decomposition of X determined by the collection of matrices described above. Using standard arguments (see [F] Ex.19.1.11) we infer that

$$A_\cdot(X) = \bigoplus_{\substack{1\le i_1<\dots<i_k\le n \\ k=0,\dots,n}} \mathbb{Z}\,\Omega(i_1,\dots,i_k) .$$

The following notation is better suited to codimensions. Associate to a strict partition $I=(i_1,\dots,i_k) \subset \rho_n$ the class

$$\sigma(I) = \Omega(n+1-i_1,n+1-i_2,\dots,n+1-i_k) \in A_{n(n+1)/2-|I|}(X) \cong A^{|I|}(X).$$

The Schubert cycles $\sigma(r)$, $r = 1,\dots,n$ (i.e. $k=1$, $i_1=r$) are called special (they parametrize isotropic n-planes L such that $\dim(L \cap V_{n+1-r}) \ge 1$).

By the above (see also [B-G-G] Theorem 5.5) we have :

Corollary 6.3 The set $\{\sigma(I) : I\text{-strict},\ I \subset \rho_n\}$ is a $\mathbb{Z}$-basis of $A^\cdot(X)$.

The following fact was proved in [H-B].

Theorem 6.4 (Hiller-Boe) Let $I=(i_1,\dots,i_k)$ be a strict partition of length k and let $1 \le r \le n$. Then

$$\sigma(I)\cdot\sigma(r) = \sum d(I,(r);J)\ \sigma(J)$$

where the sum is over all strict partitions $J=(j_1,\dots,j_{k+1})$ such that $i_{p-1} \ge j_p \ge i_p$ ($i_0=n$, $i_{k+1}=0$) , $|J| = r+|I|$, and $d(I,(r);J) = 2^{\bar{d}(I,(r);J)}$, $\bar{d}(I,(r);J) = \mathrm{card}\{1 \le p \le k : j_{p+1}<i_p<j_p\}$.

Let us fix n independent variables $a_1,\dots,a_n$ and write Q_I instead of $Q_I(a_1,\dots,a_n)$.

Recall the definition of the numbers $e(I,J;K)$ (see Section 4):

$$Q_I \cdot Q_J = \sum e(I,J;K)\ Q_K$$

Comparing Proposition 2.7. and Theorem 6.4 we see that for any strict partitions $I,J \subset \rho_n$, $1 \le r \le n$, we have $e(I,(r);J) = d(I,(r);J)$.

Lemma 6.5 For any strict partition $I \subset \rho_n$, and any polynomial $\mathcal{P}$ in "n variables" if

$$\mathcal{P}(q_1,\dots,q_n)\cdot Q_I = \sum \alpha_J\ Q_J$$

and

$$\mathcal{P}(\sigma(1),\dots,\sigma(n))\cdot\sigma(I) = \sum \beta_J\ \sigma(J) ,$$

then $\alpha_J = \beta_J$ for every strict partition $J \subset \rho_n$.

Proof. It is sufficient to assume that $\mathcal{P}$ is a monomial and induct on the number of variables in the monomial. This reduces the question to the case of multiplication by a single $\sigma(r)$ and $q_r(A)$. The assertion now follows from the equality $e(I,(r);J) = d(I,(r);J)$ by taking into an account that the both families $\{ Q_I\!: I\text{-strict},\ I \subset \rho_n\}$, $\{ \sigma(I)\!:I\text{-strict},\ I \subset \rho_n\}$ are linearly independent over $\mathbb{Z}$. ■

Proposition 6.6 ("A Giambelli - type" formula) For every strict partition $I=(i_1,\dots,i_k) \subseteq \rho_n$ where k is even (we can assume that this is always the case by putting $i_k= 0$ if necessary) we have

$$\sigma(I) = \mathrm{Pfaffian}\left[\ \sigma(i_p,i_q)\ \right] , \quad 1 \le p < q \le k.$$

where for $i > j > 0$,

$$\sigma(i,j) = \sigma(i)\sigma(j) + 2\sum_{p=1}^{j}(-1)^p\ \sigma(i+p)\sigma(j-p) \qquad (\#)$$

and where $\sigma(i,0) = \sigma(i)$ ($\sigma(i)=0$ for $i>n$, $i<0$).

Proof. It suffices to check (#) as well as the equalities:

(##) If $I=(i_1,\dots,i_k)$ is a strict partition and $\ell(I)=k$ is even, then

$$\sigma(i_1,\dots,i_k) = \sum_{p=2}^{k}(-1)^p\ \sigma(i_1,i_p)\sigma(i_2,\dots,\hat{i}_p,\dots,i_k)$$

(###) If $I=(i_1,\dots,i_k)$ is strict and $\ell(I)=k$ is odd, then

$$\sigma(i_1,\dots,i_k) = \sum_{p=1}^{k}(-1)^{p-1}\,\sigma(i_p)\sigma(i_1,\dots,\hat{i}_p,\dots,i_k)$$

(#) Assume that

$$\sigma(i)\cdot\sigma(j) + 2\sum_{p=1}^{j}(-1)^p\,\sigma(i+p)\sigma(j-p) = \sum \alpha_I\,\sigma(I) \quad ,$$

for some $\alpha_I \in \mathbb{Z}$. Then, invoking Lemma 6.5 and the first part of the "determinantal" definition of a Q-polynomial from Section 1. , we see that, $\alpha_{(i,j)} = 1$ and $\alpha_I = 0$ for $I \neq (i,j)$. As for (##) and (###) - argue in a similar way using induction on $\ell(I)$ and the second part of the "determinantal" definition of the $Q_I(A)$'s .■

Remark 6.7 This polynomial expression of $\sigma(i_1,\dots,i_k)$ in terms of the $\sigma(i)$'s gives an explicit answer to the first question in [H-B] p.63. Note that even in the case of the "usual" Grassmannians we have not seen in the literature such a derivation of the determinantal formula of the Schubert Calculus, which stems from a comparaison of Pieri formulas for S-polynomials and Schubert cycles, and, from the determinantal presentation of Schur S-polynomial.

Proposition 6.8 For $I,J \subset \rho_n$, the following equality holds in $A^\cdot(X)$:

$$\sigma(I)\;\sigma(J) = \sum_K e(I,J;K)\;\sigma(K) \;,$$

where the sum is over all strict partitions $K \subset \rho_n$.

Proof. Develop $\sigma(J)$ as a polynomial in the special Schubert varieties $\sigma(r)$, with the help of Proposition 6.6. Then apply Lemma 6.5. ■

The following fact is a consequence of Example 4.2(5) and the previous Proposition.

Corollary 6.9 For strict partitions $I,J \subset \rho_n$, $\sigma(I)$ is Poincaré dual of $\sigma(J)$ iff I is ρ_n-complementary of J.

Remark 6.10 The L-R rule for shifted tableaux from $[\mathrm{Ste}]_2$ gives a combinatorial interpretation of the e(I,J;K)'s (see Section 4). Thus the L-R rule for shifted tableaux answers the second question in [B-H] p.63.

Remark 6.11 There is no vector bundle E on X for which $\sigma(I) = Q_I(E)$. On the other hand, the following fact is shown in [H-B]. Let, in the notatio of loc.cit. (and in the terminology of [D]), $c\colon S.(X(T))^{W_\alpha} \longrightarrow A^\cdot(X)$ be the characteristic homomorphism, where $X(T)$ is the group of characters of the maximal torus $T \subset Sp(2n)$. Note that in our situation, $W_\alpha = S_n$ and $S.(X(T)) = \mathbb{Z}[a_1,\dots,a_n]$ where $a_1,\dots,a_n$ are identified with the Chern roots of the tautological vector bundle on $G_n(V)$. Then, one has (see [H-B Lemma 2.13') $c(s_{(1)^p}(a_1,\dots,a_n)) = \sigma(p)$, $1 \le p \le n$. Denoting by R the restriction to X of the tautological vector bundle on $G_n(V)$, this means that $c_p(\check{R}) = \sigma(p)$, $1 \le p \le n$. Thus in terms of the Chern classes of the tautological bundle R we have for $i > j > 0$

$$\sigma(i,j) = c_i(\check{R})\,c_j(\check{R}) + 2\sum_{p=1}^{j}(-1)^p\,c_{i+p}(\check{R})\,c_{j-p}(\check{R})$$

and similarly for $\sigma(I)$ corresponding to partitions with more parts. For example

$$\sigma(i,1) = s_{2,(1)^{i-1}}(\check{R}) - c_{i+1}(\check{R}) \;.$$

We will now sketch similar results for the space

$$Y = \{\, L \subset W : L \text{ is n-dimensional subspace of } W ,\ \text{isotropic with respect to } \Psi \,\}$$

where W is 2n+1 dimensional vector space over $\mathbb{C}$ and $\Psi\colon W\times W \longrightarrow \mathbb{C}$ is a nondegenerate orthogonal form on W .

Proposition 6.12 (i) Y is an algebraic variety of dimension $n(n+1)/2$, $\mathrm{Pic}(Y)=\mathbb{Z}$.

(ii) Y can be interpreted as the homogenous space $SO(2n+1)/P_\alpha$,where α is the "right end root" in the standard basis Σ of the root system of type B_n and P_α is the corresponding maximal parabolic subgroup.

(iii) Y can be interpreted as the homogenous space $SO(2n+1)/U(n)$.

(references as in Proposition 6.1 ; note that the composition

$$Y \xhookrightarrow[\text{standard}]{} G_n(W) \xhookrightarrow[\text{Plucker}]{} \mathbb{P}_{\mathbb{C}}^{\binom{2n+1}{n}-1}$$

is not a generator of Pic(Y). For explicit equations defining Y in a projective space via the generator of Pic(Y) see [Ca]).

Let $e_1,\dots,e_n,g,f_n,\dots,f_1$ be a basis of W in which the form Ψ satisfies $\Psi(e_i,e_j) = \Psi(f_i,f_j) = \Psi(e_i,g) = \Psi(f_j,g) = 0$, $\Psi(e_i,f_j)=\Psi(f_j,e_i)=\delta_{i,j}$ $i,j=1,2,\dots,n$, $\Psi(g,g)=1$.

A $n\times(2n+1)$ matrix (#) corresponding to the sequence $1 \le i_1 < \dots$ $\dots < i_n \le 2n+1$ represents in the above basis an isotropic n-plane iff $i_p \ne 2n+2-i_q$ for every $p,q = 1,\dots,n$. In particular $i_p \ne n+1$ for every p.

A collection of matrices satisfying the above conditions gives a cell-decomposition of Y into affine spaces of dimension $\sum i_p+(n+1)(n-k)-$ $-n(n+1)/2$ (the sum is over $p = 1,\dots,k$), where k is the greatest number for which $i_k \le n$. Denote by $\Omega\langle i_1,\dots,i_k\rangle$ the class in A.(Y) of the cycle

$$\left\{ L\in Y:\dim(L\cap V_{i_p}) \ge p \quad , \quad p=1,\dots k, \right\}$$

where V_i is the subspace in W spanned by the first i vectors in the sequence $e_1,\dots,e_n,g,f_n,\dots,f_1$. If $I=(i_1,\dots,i_k) \subset \rho_n$ is a strict partition, let $\sigma\langle I\rangle$ denote the element of $A^{|I|}(Y)$ which is equal to the class $\Omega\langle n+1-i_1,\dots,n+1-i_k\rangle$. For $k=1$, $i_1=r$, $\sigma\langle r\rangle$ is called <u>special</u> ($r = 1,\dots,n$).

<u>Theorem 6.13</u> ([H-B]) In the notation of Corollary 2.8 ,for every strict partition $I \subset \rho_n$, $1 \le r \le n$,

$$\sigma\langle I\rangle\ \sigma\langle r\rangle = \sum f(I,(r);J)\ \sigma\langle J\rangle$$

where the sum is over all strict partitions $J \subset \rho_n$.

From it, arguing as above one obtains

<u>Proposition 6.14</u> For every strict partition $I=(i_1,\dots,i_k) \subseteq \rho_n$,k even, we have

$$\sigma\langle I\rangle = \mathrm{Pfaffian}\left[\ \sigma\langle i_p,i_q\rangle\right] \quad , \quad 1 \le p < q \le k$$

where for $i > j > 0$,

$$\sigma\langle i,j\rangle = \sigma\langle i\rangle\sigma\langle j\rangle + 2\sum_{p=1}^{j-1}(-1)^p\ \sigma\langle i+p\rangle\sigma\langle j-p\rangle + (-1)^j\sigma\langle i+j\rangle.$$

and where $\sigma\langle i,0\rangle = \sigma\langle i\rangle$ ($\sigma\langle i\rangle=0$ for $i>n$, $i<0$).

<u>Proposition 6.15</u> In the notation of Section 4, for any strict partitions $I,J \subset \rho_n$, in A'(X),

$$\sigma\langle I\rangle\ \sigma\langle J\rangle = \sum f(I,J;K)\ \sigma\langle K\rangle,$$

where the sum is over all strict partitions $K \subset \rho_n$.

<u>Remark 6.16</u> Let, in the notation of [H-B], $c\colon S.(X(T))^{W_\alpha} \longrightarrow A^\cdot(Y)$ be the characteristic homomorphism, where X(T) is the group of characters of the maximal torus $T \subset SO(2n+1)$. In the present situation $W_\alpha = S_n$, and $S.(X(T)) = \mathbb{Z}[a_1,\dots,a_n]$ where $a_1,\dots,a_n$ are identified with the Chern roots of the tautological vector bundle on $G_n(W)$. Then, one has $c(s_{(1)^p}(a_1,\dots,a_n)) = 2\sigma(p)$, $1\le p\le n$ (see [H-B] Lemma 2.13). Thus, denoting by R the restriction to Y of the tautological vector bundle on $G_n(W)$ we have $c_p(\check{R}) = 2\sigma\langle p\rangle$, $1\le p\le n$. Thus one can write $2^{l(I)}\sigma\langle I\rangle$ as a polynomial in the $c_p(\check{R})$'s. For example, for $i > j > 0$,

$$4\sigma\langle i,j\rangle = c_i(\check{R})\ c_j(\check{R}) + 2\sum_{p=1}^{j}(-1)^p\ c_{i+p}(\check{R})\ c_{j-p}(\check{R}) .$$

We can summarize the above considerations in the following theorem.

<u>Theorem 6.17.</u>(i) In A'(X), under the identification $q_i \longmapsto \sigma(i)$, $i = 1,\dots,n$, $\sigma(I)$ corresponds to Q_I for every strict partition $I \subset \rho_n$ More precisely, the assignment $Q_I \longmapsto \sigma(I)$ defines a ring homomorphism and allows one to identify A'(X) with the ring of Q-polynomials modulo the ideal $\oplus_I \mathbb{Z}Q_I$, where I runs over all partitions not contained in ρ_n.

(ii) In A'(Y), under the identification $P_i \longmapsto \sigma\langle i\rangle$, $i=1,\dots,n$, $\sigma\langle I\rangle$ corresponds to P_I for every strict partition $I \subset \rho_n$. More precisely, the assignment $P_I \longmapsto \sigma\langle I\rangle$ defines a ring homomorphism and allows one to identify A'(Y) with the ring of P-polynomials modulo the ideal $\oplus_I \mathbb{Z}P_I$, where I runs over all partitions not contained in ρ_n.

(In the above, $Q_I=Q_I(A)$, $P_I=P_I(A)$, where $A=(a_1,\dots,a_n)$ is a sequence of independent variables.)

It is known (see [B-G-G] for example) that the characteristic homomorphism c described in Remark 6.11 induces an isomorphism of A'(X) with $A^\cdot(Sp(2n)/B)^{S_n}$, where B is a Borel subgroup in Sp(2n). Since $A^\cdot(Sp(2n)/B) = \mathbb{Z}[A]/(P(a_1^2,\dots,a_n^2)|\ P$ symmetric, deg P>0) (see [D] 4.6.(a) for instance), the Theorem and Remark 6.11 imply that the assignment $s_{(1)^i}(A) \longmapsto q_i(A)$, $i=1,\dots,n$, induces an ring isomorphism of $\mathcal{S}ym(A)/(\ P(a_1^2,\dots,a_n^2)\ |\ P$ symmetric, deg $P>0$) with the ring of Q-polynomials modulo the ideal $\oplus\ \mathbb{Z}Q_I$, where I runs over all partitions not contained in ρ_n.

Let now U be a 2n-dimensional vector space endowed with a nondegenerate orthogonal form $\xi : U\times U \longrightarrow \mathbb{C}$. Consider

$$Z = \{\ L\subset U : L \text{ is n-dimensional subspace of U isotropic with respect to } \xi\ \}$$

It is known that Z has two connected components, which are isomorphic to $Z'=SO(2n)/P_\alpha$ where α are the "right end roots" in the root system of type D_n. Moreover Z' is an algebraic variety of dimension $n(n-1)/2$. Let $e_1,\dots,e_n,f_n,\dots,f_1$ be a basis of U such that $\xi(e_i,e_j)=\xi(f_i,f_j)=0$, $\xi(e_i,f_j)=\xi(f_j,e_i)=\delta_{i,j}$. Let V_i be the vector space spanned by the first i vectors of the above basis.Then the Schubert varieties in $G_n(U)$ (determined by the flag $V_1\subset\dots\subset V_{2n}$) which give rise to the Schubert varieties in Z' are indexed by the the sequences $(i_1,\dots,i_n)$, where $i_p \neq 2n+1-i_q$ for $p,q=1,\dots,n$, and if k denotes the largest number such that $i_k\le n$, then n-k is even (for details we refer to [L-Se]). Let us denote by $\Omega[i_1,\dots,i_k]$ the class in A.(Z') determined by this Schubert variety. It can be shown (see [L-Se]) that

$$\dim \Omega[i_1,\dots,i_k] = i_1+\dots+i_k+n(n-k)-n(n+1)/2$$

Moreover, we have (see for instance [G-Z] Lemma 8)

<u>Lemma 6.18</u> Let U' be a subspace of U such that dim U'=2n-1 and U' is non-degenerate with respect to ξ. Let x_o be a vector orthogonal to U' such that $\xi(x_0,x_0)=1$. Let Y be the set of all isotropic (n-1)-subspaces in U'. Then Z is isomorphic to the space of pairs (L,x), where L$\in$ Y and x is a vector in U', orthogonal to L, such that $\xi(x,x)=1$. An isomorphism is given explicitly as follows : a pair (L,x) corresponds to the isotropic subspace $L\oplus\mathbb{C}(x+ix_0)$. Moreover the map $Z \longrightarrow Y$ given by $L\longmapsto L\cap U'$ gives an isomorphism of each of connected components of Z onto Y.

By taking into an account this Lemma and considering in U a flag of isotropic subspaces $U_1\subset\dots\subset U_{n-1}\subset U_n\subset U$ such that dim $U_i=i$, $U_{n-1}\subset U'$, one

proves easily the following

<u>Proposition 6.19</u> Under the isomorphism of the Chow groups A.(Z') and A.(Y) induced by the map from Lemma 6.18, the cycle $\Omega\langle i_1,\dots,i_k\rangle\in$ A.(Y) corresponds to $\Omega[i_1,\dots,i_k]\in$ A.(Z') if n and k are of the same parity, to $\Omega[i_1,\dots,i_k,n]$ if n and k are of different parity.

Let $I=(i_1,\dots,i_k)$ be a sequence such that $1\le i_1<\dots<i_k\le n-1$. Let $\sigma[I]$ denote the element of $A^{|I|}(Z')$ which is equal to the class $\Omega[n-i_1,\dots,n-i_k]$ if n and k are of the same parity and to $\Omega\,[n-i_1,\dots,n-i_k,n]$ if n and k are of different parity. Then we have the following complement of Theorem 6.17.

<u>Theorem 6.17'</u> In A'(Z'), under the identification $P_i \longmapsto \sigma[i]$ $i=1,\dots,n-1$, $\sigma[I]$ corresponds to P_I for every strict partition $I\subset\rho_{n-1}$. More precisely, the assignment $P_I \longmapsto \sigma[I]$ defines a ring homomorphism ar allows us to identify A'(Z') with the ring of P-polynomials modulo th ideal $\underset{I}{\oplus}\ \mathbb{Z}P_I$, where I runs over all partitions not contained in ρ_{n-1}. (Here, $P_I = P_I(a_1,\dots,a_{n-1})$ denotes the P-polynomial in n-1 variables.)

The material of this Section relies heavily on the calculations in [H-B]. After the previous version of this work was written, I was informed that D.Peterson (Vancouver) created an another approach to the results descibed in this Section.

7. MISCELLANY

7.1 Note that by the factorization formula for Q-polynomials, for every partition I

$$s_I(A) = Q_{\rho_{n-1}+I}(A)\ /\ Q_{\rho_{n-1}}(A)\ .$$

Combining this fact with Proposition 2.6 we see that every symmetric polynomial in A is a rational function in $\psi_k(A)$, k-odd. This is a classical result of Laguerre.

7.2 Consider the following situation. Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_n)$ be two sequences of indeterminates, such that AB is a sequence of independent variables. Let $\mathcal{F}\subset\mathbb{Z}[AB]$ be the ideal of all polynomials $P\in$

$\mathbb{Z}[AB]$ such that for every ring homomorphism $f: \mathbb{Z}[AB] \longrightarrow K$ (a field) if $\{f(a_1),\dots,f(a_n)\} = \{f(b_1),\dots,f(b_n)\}$, then $f(P) = 0$. In [Fi] the author proved that $\mathcal{F}$ is generated by $s_{(1)^i}(A) - s_{(1)^i}(B)$ $i=1,\dots,n$. It turns out that with the help of results of Section 5 of the present work one can describe explicitly the following more general ideals. Fix integers m, $n > 0$ and $r \geq 0$. Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_m)$ be two sequences of indeterminates such that AB is a sequence of independent variables. Let $\mathcal{F}_r \subset \mathbb{Z}[AB]$ be the ideal of all polynomials $P \in \mathbb{Z}[AB]$ such that for every ring homomorphism $f: \mathbb{Z}[AB] \longrightarrow K$ (a field) if

$$\text{card } \{f(a_1),\dots,f(a_n)\} \cap \{f(b_1),\dots,f(b_m)\} \geq r+1 ,$$

then $f(P) = 0$. It turns out that $\mathcal{F}_r = \mathcal{J}_r\mathbb{Z}[AB]$, where $\mathcal{J}_r$ is the ideal considered in Section 5. For details see a forthcoming paper $[L\text{-}P]_2$.

7.3 In his fundamental paper [Sch], I.Schur used Q-polynomials to describe the irreducible projective characters of the symmetric groups. This is equivalent to the description of the family of (linear) characters χ of the spinsymmetric group $T(n)$, satisfying $\chi(c)=-1$. Here,

$$T(n) = \langle c, t_1, \dots, t_{n-1} \mid c^2=1,\ ct_k=t_kc,\ t_k^2=1,\ t_kt_{k+1}t_k=t_{k+1}t_kt_{k+1},$$
$$t_\ell t_k=ct_kt_\ell ,\quad |\ell-k|>1 \rangle .$$

Let $T(\infty)$ denote the inductive limit of $T(1) \subset T(2) \subset \dots$ .In [N], the author gives, with the help of the Q-polynomials, the description of central positive-definite functions of the group $T(\infty)$. For the infinite symmetric group the same problem was solved before in [Th] and [K-V] with the help of Schur S-polynomials "in a difference of alphabets" (see [Th],[K-V] and [N] for all needed definitions and details).

7.4 We conjecture that the property of the Schur S-polynomials associated with the partition $(m-r,\dots,m-r)$ ($n-r$ times), established in Corollary 5.5, in fact, provides a characterization of the partitions of this shape. Note, that after showing to A.Lascoux and I.G.Macdonald the cited corollary, they found two independent proofs of it.

7.5 Let $A = (a_1,\dots,a_n)$ be a sequence of independent indeterminates. (n can be the infinity). For a given polynomial $P(A) \in \mathcal{S}ym(A)$, denote by $[P(A)]$ its class in the ring $\mathcal{S}ym(A)_{\mathbb{Q}} / (\psi_k(A),\ k\text{-even})$. Let now J be a partition and let ϕ be the polynomial such that $[s_J(A)] = [\phi(\psi_1(A),\psi_3(A),\dots)]$. Put $\bar{s}_J = [\phi(2\psi_1(A),2\psi_3(A),\dots)]$. For example, it follows easily from [M],(2.14') and the formula in Proposition 2.6 that

$$\bar{s}_i(A) = \bar{s}_{(1)^i}(A) = [q_i(A)]$$

for every i. Associate now to a given strict partition $I = (i_1,\dots,i_k)$ $k = \ell(I)$, the partition

$$J = (i_1-1,\dots,i_k-1 \mid i_1,\dots,i_k)$$

in Frobenius notation (see [M] I.1). In [Y], the author proved, by usin differential equations, the following equality:

$$\bar{s}_J = 2^{-\ell(I)}\,[Q_I(A)]^2$$

As I.G.Macdonald points out (personal communication, Oberwolfach, July 1988), the above equality can be obtained as a consequence of the Giambelli formula, expressing a Schur S-function as a determinant of hook Schur functions (see [M] I.3.9), and the definition of Schur Q-functions with the help of pfaffians (see I.I(Q)). For a generaliza tion of the above equality to skew Schur Q-functions see [J-P].

7.τ We take this opportunity to correct two defects in the (second) proo of Theorem 7.2, given in Section 7 of $[P]_3$. Firstly, the argument on page 443/444 proving that $\mathrm{Im}(i'_*)$ is generated by $p^*(\mathrm{Im}\ i''_*)$ goes properly as follows. After deleting the right hand side square on page 443[20] read:" ...where $p: X' \longrightarrow X''$ is the Grassmannian bundle $G_{n-r}(V_{X''}/(R^r)_{X''})$. Thus $A(X')$ is a free $A(X'')$-module via p^*. Moreover, there exist elements $\Omega_k = s_{I_k}(Q)$ (Q is the tautological bundle on X'), such that $A(X')$ $= \oplus_k A(X'')\cdot\Omega_k$, and denoting by Ω_k^D the operator $s_{I_k}(i'^*Q)$ on $A(D_{r-1}(\varphi'))$ we have $A(D_{r-1}(\varphi')) = \oplus_k \Omega_k^D \cap A(D_{r-1}(\varphi''))$ - see [F] Proposition 14.6.5 . Consider... ". Then read on page 444[1] :" We have $i_*(\sum_k \Omega_k^D \cap p_D^*(d_k)) =$ $= \sum i'_*\, p_D^*(d_k)\cdot\Omega_k = \sum p^* i''_*(\Omega_k^D \cap d_k)$ in $A(X')$.... ".

Secondly, Lemma 7.5 (and its proof) remains correct when it reads: " If $v \gg 0$, then multiplication

$$\cdot Q_{n-r}(C) : \Lambda^{l}(D_r - D_{r-1}) \otimes \mathbb{Z}[1/2] \longrightarrow \Lambda^{l+\gamma^s}(D_r - D_{r-1}) \otimes \mathbb{Z}[1/2]$$

is a monomorphism for $i \le N(n,r)$." This is because $Q_{\rho_{n-r}}(A) = 2^{n-r} s_{\rho_{n-r}}(A)$.

Then the arguments given on p.440–444 show that $\mathcal{I}^s_r \subseteq \mathcal{P}^s_r \subseteq \mathcal{I}^s_r \otimes \mathbb{Z}[1/2]$. To prove that actually $\mathcal{P}^s_r \subseteq \mathcal{I}^s_r$ it suffices to show that for a strict partition $I \supset \rho_{n-r}$, $2^e P_I(A) \in \mathcal{P}^s_r$ iff $e \ge n-r$. Consider the vector bundle morphism $\varphi' = \varphi \oplus id : E^{\vee} \oplus 1^r \longrightarrow E \oplus 1^r$ on X, where (X,φ) is the construction (24) in $[P]_3$ with rank E = n−r, and 1^r stands for the trivial rank r-bundle. Then, since $D_r(\varphi') = D_0(\varphi)$, the required assertion follows because $2^e P_I(a_1,\dots,a_{n-r}) \in \mathcal{P}^s_0$ iff $e \ge n-r$. The second proof of the analogous theorem in the generic and antisymmetric cases works without this additional argument.

Moreover, the following misprints were noticed in $[P]_3$:

page 420_8 – read: "... for the multiset $\{a_1,\dots,a_n\}$ ",
page 421_1 and page 422^{13} – read "... = $s_{J-n+1,I}(E-H)$ ..."
page 432^{11} – read: "... m=n, s=n−1 ...",
page $445^{13,18}$ – read: "... $\alpha \in H^{\cdot}(X,\mathbb{Z})$ ...",
page 451^{16} – read : "... $s_{(m-r)^{n-r}+I}(A-B)$ ...",

in the bottom part of page 447 the Laplace expansions for pfaffians should be endowed with signs; finally, whenever we speak on the Chern numbers of $D_r(\varphi)$, we assume φ to be holomorphic.

APPENDIX (Proof of Lemma 5.5).

We prove (for example) the case (iii), by using methods of $[P]_3$. Let V be a vector space over $\mathbb{C}$ of dimension v . Let $G_n(V)$ be the Grassmannian of n-subspaces in V and let $R^{(n)}$ be the tautological vector bundle on $G_n(V)$. Let $X_v = \Lambda^2 R^{(n)}$ and $E = R^{(n)}_{X_v}$. Then, on X_v we have a tautological morphism $\varphi: E^{\vee} \longrightarrow E$ and the degeneracy locus $D_r(v) = D_r(\varphi)$ (r≥0, r–even). Consider the exact sequence $(\gamma = (n-r-1)(n-r)/2)$

$$\Lambda^{k-\gamma}(D_r(v) - D_{r-2}(v)) \xrightarrow{\alpha} \Lambda^k(X_v - D_{r-2}(v)) \longrightarrow \Lambda^k(X_v - D_r(v)) \longrightarrow 0 \qquad (\#)$$

If v tends to the infinity, then this sequence becomes

$$\mathcal{S}ym(A_r, A^{n-r})/(P_I(A_r), |I| \ge 1) \xrightarrow{\alpha} \mathcal{S}ym(A)/\mathcal{I}''_{r-2} \longrightarrow \mathcal{S}ym(A)/\mathcal{I}''_r \longrightarrow 0$$

where $A = (a_1,\dots,a_n)$ is the set of the Chern roots of E , $A_r = (a_1,\dots,a_r)$ is the set of the Chern roots of the tautological rank r-bundle $Fl_{r,n}(V)$, which serves as a base space of a fibration of $D_r(v) - D_{r-2}(v)$ with the fiber being the space of nondegenerate r×r antisymmetric matrices. To see it, one argues as in the proof of Proposition 6.2 in $[P]_3$; this way one gets also that α is a monomorphism. It remains to prove that α is induced by

$$P(A_r, A^{n-r}) \longrightarrow \sum w \left[P(A_r, A^{n-r}) \cdot P_{\rho_{n-r-1}}(A^{n-r}) \right] ,$$

where the sum is over all $w \in S_n / S_r \times S_{n-r}$.

Note first that for a canonical desingularization

$$Z = Zeros(E^{\vee} \to E \to Q) \subset G_r(E) \xrightarrow{\pi} X_v$$

of $D_r(v)$ (see [J-L-P]), $Z \cap \pi^{-1}(X_v - D_{r-2}(v))$ is isomorphic to $D_r(v) - D_{r-2}(v)$. Secondly, arguing as in [J-L-P] we have $Z = c_{top}(\Lambda^2 Q + R \otimes Q)$, where $0 \longrightarrow R \longrightarrow E_{G_r(E)} \longrightarrow Q \longrightarrow 0$ is the tautological sequence on $G_r(E)$, $c.(R) = \prod_{a \in A_r}(1+a)$, $c.(Q) = \prod_{a \in A^{n-r}}(1+a)$, and $R^{\vee} \simeq R$ on $D_r(v) - D_{r-2}(v)$. Then the assertion follows from $c_{top}(\Lambda^2 Q) = P_{\rho_{n-r-1}}(A^{n-r})$ and Lemma 3.1(ii) (where, however, the sequence of the Chern roots of E has the reverse order).

The proof of (i) can be performed mutatis mutandis with the help of Proposition 6.2 in $[P]_3$.

The proof of (ii) is a bit more subtle. It suffices to prove that for $\beta : \mathcal{S}ym(A_r, A^{n-r}) \longrightarrow \mathcal{S}ym(A)$ defined by

$$\beta(P(A_r, A^{n-r})) = \sum w [P \cdot P_{\rho_{n-r}}(A^{n-r})] ,$$

where the sum is over all $w \in S_n / S_r \times S_{n-r}$, the following properties hold

1) $\mathrm{Im}\, \beta = \mathcal{I}'_r$.
2) $\beta(P_I(A_r)) \subset \mathcal{I}'_{r-1}$ for every strict partition I.
3) The induced map

$$\bar{\beta} : \mathcal{S}ym(A_r, A^{n-r})/(P_I(A^{n-r}), |I| \ge 1) \longrightarrow \mathcal{S}ym(A)/\mathcal{I}'_{r-1}$$

is a monomorphism.

The assertion 1) follows from the fact proved in the symmetric morphism case in $[P]_3$ Section 7 : $\mathrm{Im}\,(2^{n-r}\beta) = (Q_I(A),\ I\supset\rho_{n-r})$. The assertion 2) can be obtained from (iii) proved above as follows. If r is even, consider

$$\beta' : \mathcal{S}ym(A_r, A^{n-r+1}) \longrightarrow \mathcal{S}ym(A')$$

where $A^{n-r+1}=(a_{r+1},\dots,a_n,a)$, $A'=(a_1,\dots,a_n,a)$, defined by

$$\beta'(P) = \sum w\ [P\cdot P_{\rho_{n-r}}\ (A^{n-r+1})]$$

where the sum is over all $w\in S_{n+1}/S_r xS_{n-r+1}$. By (iii) we know that $\beta'(P_I(A_r)) \subset \mathcal{I}''_{r-2}(A') = (P_I(A'),\ J\supset\rho_{n-r-2})$. If we specialize a=0 , this implies $\beta(P_I(A_r)) \subset (P_I(A),\ I\supset\rho_{n-r-2}) = \mathcal{I}'_{r-1}(A')$. We use the fact that if $\beta'(P_I(A_r,A^{n-r+1})) = \sum_J \alpha_J(A')P_J(A')$, where $\alpha_J(A') = \sum_K \gamma_K s_K(A')$ with $\gamma_K\in\mathbb{Z}$, then also $\beta(P_I(A_r,A^{n-r})) = \sum_J \alpha_J(A)P_J(A)$ where $\alpha_J(A) = \sum_K \gamma_K s_K(A)$ with the same coefficients as above. This follows from the universal character of the formula for Gysin pushforward stated in $[P]_3$ Proposition 7.8 , which is a geometric translation of β (see Section 3) , and from the universal character of the expresion of $P_I(A)$ in terms of the $s_I(A)$'s (see Section 4). The expression "universal character" means here that the coefficients of the formulas in question do not depend on the cardinality of the sets of indeterminates .

For r odd we consider $A_{r+1}=(a_1,\dots,a_r,a)$ and $A'=(a_1,\dots,a_n,a)$. Define β'' : $\mathcal{S}ym(A_{r+1},A^{n-r}) \longrightarrow \mathcal{S}ym(A')$ by

$$\beta''(P) = \sum w\ [\ P\cdot P_{\rho_{n-r}}\ (A^{n-r})]$$

where the sum is over all $w\in S_{n+1}/S_{r+1}xS_{n-r}$. By the above we know that $\beta''(P_I(A_{r+1})) \subset \mathcal{I}'_r(A') = (P_J(A'),\ J\supset\rho_{n+1-r})$. Again the specialization a=0 gives us $\beta(P_I(A_r)) \subset (P_J(A),\ J\supset\rho_{n-r+1}) = \mathcal{I}'_{r-1}$.

The assertion 3) then can be proved in the same way as in the case (i) and (ii). ∎

REFERENCES

[B-G-G] I.N.Bernstein, I.M.Gel'fand, S.I.Gel'fand, Schubert cells and cohomology of the spaces G/P, Russian Math.Surv.28, 1-26 (1973).

[B-R] A.Berele, A.Regev, Hook Young diagrams with applications to combin torics and to representation theory of Lie superalgebras, Adv. in Math., 64,118-175 (1987).

[Ca] E.Cartan, *The Theory of Spinors* ,Hermann (1966).

[D] M.Demazure, Désingularisation des variétés de Schubert géneralisées, Ann. scient. Éc. Norm. Sup. t.7, 53-88 (1974).

[Fi] K.G.Fischer, Symmetric polynomials and Hall's theorem, Discrete Math., 69,225-234 (1988).

[F] W.Fulton, *Intersection Theory*, Springer Verlag (1984).

[G-Z] I.M.Gel'fand, A.V.Zelevinsky, Models of representations of classical groups, Functional Analysis and Its Applications, vol.18, no.3, 14-31 (1984).

[H-T] J.Harris, L.Tu, Chern numbers of kernel and cokernel bundles, Inv. Math. 75, 467-475 (1984).

[He] S.Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press (1962).

[H-B] H.Hiller, B.Boe, Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n , Adv. in Math. 62, 49-67 (1986).

[H-H] P.N.Hoffman, J.F.Humhreys, a forthcoming book on Projective Representations of Symmetric Groups.

[J-P] T.Józefiak, P.Pragacz, A determinantal formula for skew Q-functions to appear.

[J-L-P] T.Józefiak, A.Lascoux, P.Pragacz, Classes of determinantal varieties associated with symmetric and antisymmetric matrices, Izwiestja AN SSSR 45,no 9, 662-673, (1981).

[K] R.C.King, S-functions and characters of Lie algebras and superalgebras, preprint, University of Southampton (1988).

[K-V] S.Kerov,A.Vershik,The characters of the infinite symmetric group, Siam.J.Alg.Disc.Math., 7, 116-123 (1986).

[L-Se] V.Lakshmibai,C.S.Seshadri, Geometry of G/P II,Proc.Indian Acad.Sci. A 87,1-54 (1978).

[L] A.Lascoux, La résultante de deux polynômes, Séminaire d'Algèbre Dubreil-Malliavin , Springer Lecture Notes 1220, 56-72 (1986).

[L-P]$_1$ A.Lascoux, P.Pragacz, S-function series, J.Physics A Math. Gen. 21, 4105–4114 (1988).

[L-P]$_2$ A.Lascoux, P.Pragacz, Schubert polynomials and ideals generated by symmetric functions, in preparation.

[L-S]$_1$ A.Lascoux, M.P.Schützenberger, *Formulaire raisonné de fonctions symétriques*, prepublication de l'Universite Paris 7 (1985).

[L-S]$_2$ A.Lascoux, M.P.Schützenberger, Symmetrization operators on polynonomial rings, Functional Analysis and Its Applications, 21, no.4, 77–78 (1987).

[M] I.G.Macdonald,*Symmetric functions and Hall polynomials*, Oxford Univ. Press (1979).

[Mo]$_1$ A.Morris, On Q-functions, J.London Math.Soc. 37,no.148, 445–455 (1962).

[Mo]$_2$ A.Morris, A note on the multiplication of Hall functions, J.London Math.Soc. 39,481–488 (1964).

[Mu] T.Muir, *History of determinants*, 5 volumes, Dover reprints (1960).

[N] M.A.Nazarov, Factor-representations of the infinite spinsymmetric group, preprint, Moscow State University (1987).

[Po] E.Pomey, Sur le plus grand common diviseur de deux polynômes, Nouv. Ann. Math. (3) 7, 66–90, 407–427 (1888).

[P]$_1$ P.Pragacz, Determinantal varieties and symmetric polynomials,Functional Analysis and Its Applications,21,no.3,89–90 (1987).

[P]$_2$ P.Pragacz, A note on elimination theory, Indagationes Math. 49 (2), 215–221 (1987).

[P]$_3$ P.Pragacz, Enumerative geometry of degeneracy loci, Ann.Éc.Norm.Sup. vol.3, 413–454 (1988).

[R] J.B.Remmel, The combinatorics of (k,ℓ)-hook Schur functions, Contemporary Mathematics, 34, 253–287 (1984).

[Sa] B.Sagan, Shifted tableaux,Schur Q-functions and a conjecture of Stanley, J.Combin.Theory Ser. A 45, 62–103 (1987).

[Sch] I.Schur, Uber die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, J.Reine Angew. Math. 139, 155–250 (1911).

[Se]$_1$ A.Sergeev, Representations of Lie superalgebras gl(n/m) and Q(n) in the space of tensors, Functional Analysis and Its Applications t.18. no.1, 80–81 (1984).

[Se]$_2$ A.Sergeev, Letter to the author, January 9 (1988).

[So] G.Sokolov, Die Kunst der Fuge durch Herrn Johann Sebastian Bach ehemahligen Capellmeister und Musikdirector zu Leipzig , Melodia, Moscow (1982).

[St] R.P.Stanley, Problem 4 in Problem Session, Combinatorics and Algebra Contemporary Mathematics,vol.34 (1984).

[Ste]$_1$ J.Stembridge, A characterization of supersymmetric polynomials J. Algebra, 95, 439–444 (1985).

[Ste]$_2$ J.Stembridge, Shifted tableaux and the projective representations of the symmetric group, Adv. Math. to appear.

[Th] E.Thoma, Die unzerlegbaren positiv-definieren Klassenfunktionen der abzahlbar unendlichen symmetrischen Gruppe, Math. Zeit., 85, 40–61 (1964).

[T] N.Trudi, *Teoria de Determinanti*, Napoli (1862).

[W] D.R.Worley,A theory of shifted Young tableaux,Ph.D. thesis, M.I.T. (1984).

[Y] Y.You, Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups, preprint, M.I.T. (1988).

Piotr Pragacz
Mathematical Institute,
Polish Academy of Sciences,
Chopina 12,
87-100 Toruń, Poland

Errata

Theorem 3.3 (ii) is stated incorrectly. (If $k = q$ then it is correct together with the given proof.) In general one has

$$(\pi_E)_*\big(c_{top}(R_E \otimes Q_E) \cdot P_J(R_E) \cdot P_I(Q_E) \cap \pi_E^*\alpha\big) = d\ P_{I,J}(E) \cap \alpha$$

where d is zero if $(q-k)(n-q-h)$ is odd and $(-1)^{(q-k)r}\binom{[(n-k-h)/2]}{[(q-k)/2]}$ - otherwise.

The proof goes as follows. At first one shows the assertion for $J = \emptyset$ and any I. We have a commutative diagram

$$\begin{array}{ccccc}
Fl^{k,k-1,\dots,1}(Q) = Fl^{q,k,k-1,\dots,1}(E) & & = & & G^{q-k}(K) \\
\Big\downarrow \tau_Q^k & & & & \Big\downarrow \pi' \\
G^q(E) & \xrightarrow{\ \pi_E\ } & X & \xleftarrow{\ \tau_E^k\ } & Fl^{k,k-1,\dots,1}(E)
\end{array}$$

where $K = Ker(E \to Q^k)$. We have:

$$\begin{aligned}
&(\pi_E)_*\big(c_{top}(R_E \otimes Q_E) \cdot P_I(Q_E) \cap \pi_E^*\alpha\big) = \\
&\quad = (\pi_E\tau_Q^k)_*\Big(a_1^{i_1}\dots a_k^{i_k} \prod_{\substack{1\le i<j\le q \\ i\le k}} (a_i + a_j) \prod_{\substack{1\le i\le q \\ q<j\le n}} (a_i + a_j) \cap (\pi_E\tau_Q^k)^*\alpha\Big) \\
&\quad = (\tau_E^k)_*\pi'_*\Big(c_{top}(R_K \otimes Q_K) \cdot a_1^{i_1}\dots a_k^{i_k} \prod_{\substack{1\le i<j\le n \\ i\le k}} (a_i + a_j) \cap (\tau_E^k\pi')^*\alpha\Big) \\
&\quad = \pi'_*c_{top}(R_K \otimes Q_K) \cdot P_I(E) \cap \alpha.
\end{aligned}$$

Now, it follows easily from Theorem 3.3(i) that $\pi'_*c_{top}(R_K \otimes Q_K)$ equals

$$card\{I \subset (q-k)^{n-q} : |I|\text{-even}\} - card\{I \subset (q-k)^{n-q} : |I|\text{-odd}\}$$

which is the requested multiplicity d in this case.

Passing to the dual Grassmannian, we get the formula for $I = \emptyset$ and any J. If $|I| \geq 1$ (and J is arbitrary) we proceed as on pages 154-155 with the following changes:

p.155 l. -6 should read: " $\dots = dP_{I',J}(R') \cap \pi_2^*\alpha$ ",

p.155 l. -4 should read: " $\dots = \pi_{2*}\big[d \cdot c_2 P_{I',J}(R') \cdot \xi^{i_i} \cap \pi_2^*\alpha\big] = dP_{I,J}(E) \cap \alpha$ ".

(Observe that the multiplicity "d" associated with $P_{I',J}(R')$ is the same as the one associated with $P_{I,J}(E)$.)