Linear Subspaces of Special Varieties

Introduction

$(\mathbb{K} = \overline{\mathbb{K}})$

Thm (Cayley–Salmon 1849) Every smooth cubic surface $X \subset \mathbb{P}^3$ contains exactly 27 lines.

Definition $X \subset \mathbb{P}^n$ projective variety. $k \in \mathbb{N}$.

$$F_k(X) = \left\{ \begin{matrix} k\text{-dim} \\ \text{linear} \end{matrix}\ L \subset \mathbb{P}^n \;\middle|\; \begin{matrix} L \text{ contained} \\ \text{in } X \end{matrix} \right\}$$

$\rightsquigarrow$ "kth Fano scheme of X"

We'll see $F_k(X)$ has structure of proj. variety/scheme.

Ex: Cayley–Salmon $\Longleftrightarrow \# F_k\left(\begin{matrix}\text{cubic} \\ \text{surface}\end{matrix}\right) = 27$

Ex: $X = V(xy - zw) \subset \mathbb{P}^3$

lines $\{sx = tz,\ ty = sw\}$

and

$\{sx = tw,\ ty = sz\}$

are contained in X. We'll see these are all!

$\rightsquigarrow F_1(X) = \mathbb{P}^1 \cup \mathbb{P}^1.$

"Classical" Motivation

1) Enumerative geometry: 27 lines on cubic surface, 2875 lines on quintic threefold, etc.

2) $F_k(X)$ gives important info concerning intrinsic <u>and</u> embedded geometry of X.

<u>Famous example:</u> (Fano, Clemens-Griffiths 1972)

<u>Thm</u> Any smooth cubic threefold X (in $\mathbb{P}^4$) is not rational.

Proof uses $F_1(X)$ and "Intermediate Jacobians"

$F_1(X) \leftarrow$ smooth surface

Grassmannians

General Reference: 3264 and all that by Eisenbud + Harris

$$\mathbb{G}(k,n) = \left\{ \text{Linear } L \subset \mathbb{K}^{n+1} \;\middle|\; \dim L = k+1 \right\}$$

$$\mathbb{G}(k,n) = G(k+1, n+1)$$

$$= \left\{ [L] \subset \mathbb{P}^n \mid \dim [L] = k \right\}$$

We will see: This is a smooth projective variety of dimension $(k+1)(n-k)$

<u>Ex:</u> $\mathbb{G}(0,n) = \mathbb{P}^n$

- Any $L \in G(k,n)$ can be represented as rowspan of $(k+1)\times(n+1)$ matrix $M_L = \begin{array}{cc} & n+1 \\ k+1 & \begin{bmatrix} - & \cdots & - \\ - & \cdots & - \end{bmatrix} \end{array}$
  - M_L only determined up to action by GL_{k+1} (i.e. row ops)
  - M_L must have rank $k+1$, i.e. have a non-vanishing $(k+1)\times(k+1)$ minor P_I

<u>Fact</u>: $\varphi : G(k,n) \longrightarrow \mathbb{P}^{\binom{n+1}{k+1}-1}$ $\quad (I \subset \{0,\dots,n\})$

$L \longmapsto$ minors P_I of M_L

is a closed embedding

(φ is injective and has closed image)

- φ is well defined: minors invariant under SL_{k+1} action, one is non-vanishing

"Plücker Embedding"

<u>Example/Exercise</u>: $G(1,3) \subset \mathbb{P}^5$ cut out by $P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12}$

<u>Local Picture</u>: If $P_I \neq 0$, can act by GL_k so that $(M_L)_I = Id_k$

e.g. if $I = \{0, \dots k\}$,

$$M_L = \begin{pmatrix} 1 & & 0 & a_{0k} & \cdots & a_{0n} \\ & \ddots & & & & \\ 0 & & 1 & a_{kk} & & a_{kn} \end{pmatrix}$$

$\rightsquigarrow$ the subset U_I of $\mathbb{G}(k,n)$ with $p_I \neq 0$ is isomorphic to $\mathbb{A}^{(k+1)(n-k)}$

hence has dimension $(k+1)(n-k)$ (and is smooth).

Fano Schemes

$X \subset \mathbb{P}^n$ $\quad J = I(X)$

We have that $\underset{\substack{\uparrow \\ \text{for now, a set.}}}{F_k(X)} \subset \mathbb{G}(k, \underset{\substack{\uparrow \\ \text{Variety}}}{n})$

We will see that this is a <u>closed</u> subset.

<u>Working locally:</u> (with e.g. $I = \{0, \dots k\}$)

$L \in U_I$

$$M_L = \begin{pmatrix} 1 & & 0 & \\ & \ddots & & a_{ij} \\ 0 & & 1 & \end{pmatrix}$$

$L \subset X \iff \forall\, s \in \mathbb{K}^{k+1}$ and $\forall$ homog. $f \in J$,

f vanishes at $[(s_0, \dots s_k) \cdot M_L] \in \mathbb{P}^n$

$\Longleftrightarrow$ $\forall$ homogeneous $f \in J$, the coefficients of $f(s \cdot M_L) \in k[a_{ij}][s_i]$

$c_\lambda \in k[a_{ij}]$ vanish.

<u>Example</u>: $X = V(\underbrace{x_0 x_1 - x_2 x_3}_{= f})$ $\qquad F_1(X)$

$$\underbrace{(s_0 \;\; s_1) \cdot \begin{pmatrix} 1 & 0 & a_{02} & a_{03} \\ 0 & 1 & a_{12} & a_{13} \end{pmatrix}}$$

$$f(\quad) = s_0 s_1 - (a_{02} s_0 + a_{12} s_1)(a_{03} s_0 + a_{13} s_1)$$

$$= -a_{02} a_{03} s_0^2 - a_{12} a_{13} s_1^2 + (1 - a_{02} a_{13} + a_{12} a_{03}) s_1 s_2$$

$\rightsquigarrow$
$$c_{(2,0)} = -a_{02} a_{03}$$
$$c_{(1,1)} = 1 - a_{02} a_{13} - a_{12} a_{03}$$
$$c_{(0,2)} = -a_{12} a_{13}$$

$\rightsquigarrow$ $a_{02} = a_{13} = 0$ <u>or</u> $a_{03} = a_{12} = 0$

We obtain the two families of lines we have seen.

<u>Defn</u> On a chart U_I, $F_K(X)$ is the ~~variety~~ scheme defined by the ideal $\{c_\lambda\} \subset \mathbb{K}[a_{ij}]$

Two minutes on schemes:

For us, "scheme" means we remember the ideal I (if e.g. $X = V(I)$)

- <u>Ex:</u> $V(x) = V(x^2)$ as varieties, but not as schemes

- For $I \subset \mathbb{K}[x_1, \ldots x_n]$, $X = V(I)$, then the <u>coordinate ring</u> of X is $\mathbb{K}[X] := \mathbb{K}[x_1, \ldots x_n] / I$

- Just as Affine Varieties $\longleftrightarrow$ f.g. integral $\mathbb{K}$-algebras

we have

Affine schemes (of finite type over $\mathbb{K}$) $\longleftrightarrow$ f.g. $\mathbb{K}$-algebras

<u>Rem:</u> On intersections $U_I \cap U_{I'}$, the descr. of $F_K(X)$ agree!

Fano Schemes of Hypersurfaces

Example $f \in \mathbb{K}[x_0, \dots x_n]_d \quad X = V(f) \subset \mathbb{P}^n.$

Locally, $F_k(X)$ defined by

coefficients of $s_0^{\lambda_0} \cdots s_k^{\lambda_k} \quad (\sum \lambda_i = d)$

in $f(s \cdot M_L) \in \mathbb{K}[a_{ij}][s_i]$

$\Longrightarrow F_k(X)$ cut out by $\binom{d+k}{k}$ equations

$$\binom{d+k}{k} = \binom{d+k}{d}$$

Prop $\dim F_k(X) \geq \underbrace{(k+1)(n-k) - \binom{d+k}{k}}_{= \varphi(n,d,k)}$ "expected dimension"

If equality holds, then $F_k(X)$ is locally a complete intersection.

When does equality hold?

Exercise: $X \subset \mathbb{P}^4$ smooth quadric threefold. $\varphi(4,2,2) = 0$ but $F_2(X) = \emptyset$.

Thm 1) If $\varphi(n,d,k) < 0$, then a general deg d hypersurface X has

$$F_k(X) = \emptyset.$$

2) If $\varphi(n,d,k) \geq 0$ and a general deg d hypersurface contains a k-plane, then (a) every deg d hypersurface contains k-planes

(b) for general X, $\dim F_k(X) = \varphi(n,d,k)$

Rem: Hypothesis of 2) satisfied unless $k>1$ and $d=2$

Pf:

$$\Phi = \left\{ (X,L) \;\middle|\; \begin{array}{ll} X \subset \mathbb{P}^n & \text{hypersurface of deg } d \\ L \subset X & \text{linear of dim } k \end{array} \right\} \subset \mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{G}(k,n)$$

$$\begin{array}{ccc} \Phi & \xrightarrow{\pi} & \mathbb{G}(k,n) \\ \downarrow & & \\ \mathbb{P}^{\binom{n+d}{d}-1} & & \end{array}$$

Fiber at X is $F_k(X)$

Fibers are linear spaces:

(consider kernel of)

$$\mathbb{K}[\mathbb{P}^n]_d \longrightarrow \mathbb{K}[L]_d \to 0$$

$\rightsquigarrow$ Fibers of π have dimension

$$\binom{n+d}{d} - \binom{k+d}{d} - 1$$

$\Rightarrow$ Φ has dimension $(k+1)(n-k) + \binom{n+d}{d} - \binom{k+d}{d} - 1$

$$= \varphi(n,d,k) + \binom{n+d}{d} - 1$$

$\Rightarrow$ If generic fiber is non-empty, it has dimension

$$\varphi(n,d,k) \quad \blacksquare$$

Open problem: Find <u>explicit</u> family $X_{n,d,k} \subset \mathbb{P}^n$ s.t. $\dim F_k(X_{n,d,k}) = \varphi(n,d,k)$.

Many other results for generic hypersurfaces, e.g. ~~(Barth, Van den Ven)~~ (Similar results for $k>1$ by Langer)

Thm: Let $X \subset \mathbb{P}^n$ be a hypersurface of deg d. $(\mathbb{K} = \mathbb{C})$

1) $\varphi(n,d,1) \geq 0 \Rightarrow F_1(X)$ smooth
2) $\varphi(n,d,1) \geq 2 \Rightarrow F_1(X)$ connected

What is this course about?

We will consider $F_K(X)$ for special X,

e.g. $X = V(\det_n)$ or X toric.

Questions:
- How many irred. components?
- Dimensions?
- Connectedness?
- Smoothness?

$\vdots$

Motivations:

1) We'll give a "combinatorial" proof of Cayley-Salmon

2) $\mathrm{pr}(\det_3) = 5$, i.e.

$$\det_3 \neq \sum_{i=1}^{4} l_{i1} l_{i2} l_{i3} \qquad l_{ij} \text{ linear}$$

3) $F_K(\det_n)$ can shed light on components of orbit closure of $\det_n$

4) $X \subset \mathbb{P}^n$ irred variety

$\rightsquigarrow X^\vee = \left\{ H \in (\mathbb{P}^n)^* \;\middle|\; \begin{array}{l} \exists\, x \in X_{smooth} \text{ s.t.} \\ T_x X \subset H \end{array} \right\}$

"dual variety"

$\delta :=$ dual defect $= \operatorname{codim} X^\vee - 1$

<u>Fact</u>: If $\delta \geq k$, then X is covered by k-planes.

<u>Special case</u>: $A \subset \mathbb{Z}^n \rightsquigarrow X_A \subset \mathbb{P}^{\#A-1}$ projective toric variety

If $\delta = 0$, then $X_A^\vee$ is hypersurface $V(\Delta_A)$ A-discriminant

This controls the singularities of the A-hypergeometric system!

The <u>dual defect</u> can be determined by analyzing $F_k(X_A)$. Done by Furukawa and Ito '16

5) Hilbert Schemes

$P(t) \in \mathbb{Q}[t]$
$X \subset \mathbb{P}^n$

$$\mathrm{Hilb}_X^P = \{ Y \subset X \mid h_Y = P \}$$

These can be very bad, but:

<u>Thm</u> (Hartshorne) $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is connected.

What other Hilbert schemes are connected (or irred, or smooth, etc).

Projective Toric Varieties

$M = \mathbb{Z}^n$ $\quad \mathcal{A}$ finite subset of M

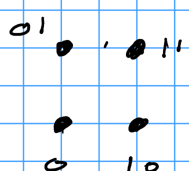
$$\langle \mathcal{A} \rangle := \langle u - v \mid u, v \in \mathcal{A} \rangle$$

Assume $\langle \mathcal{A} \rangle = M$

$$\Phi_{\mathcal{A}} : (\mathbb{K}^*)^n \longrightarrow \mathbb{P}^{\#\mathcal{A}-1}$$

$$(t_1, \ldots, t_n) \longmapsto (t^u)_{u \in \mathcal{A}} \qquad t^u = \prod_i t_i^{u_i}$$

$$X_{\mathcal{A}} := \overline{\operatorname{Im} \Phi_{\mathcal{A}}}$$

Ex: 

$\longrightarrow \quad (s,t) \longmapsto (1, s, t, st)$

$$X_{\mathcal{A}} = V(x_0 x_3 - x_1 x_2)$$

Exercise: $I(X_{\mathcal{A}}) = \left\{ \prod x_u^{a_u} - \prod x_u^{b_u} \;\middle|\; \begin{array}{l} \sum a_u u = \sum b_u u \\ \sum a_u = \sum b_u \end{array} \right\}$

The torus $T=(\mathbb{K}^*)^n$ acts on $\mathbb{P}^{\#A-1}$:

$$t\cdot(\,x_u\,) = (\,t^u\cdot x_u\,)$$

$$\rightsquigarrow \quad X_A = \overline{T\cdot(1,\dots,1)}$$

<u>Main question:</u> What is $F_{\mathbb{K}}(X_A)$?

$P=\operatorname{conv} A$. <u>Faces</u> of A are sets of the form $A\cap F$ for F a face of P

$$\rightsquigarrow I \prec A$$

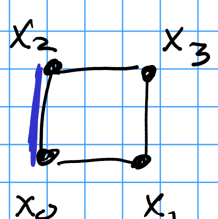
This induces a closed embedding

$$X_I \hookrightarrow X_A \quad \text{via}$$

$$\mathbb{K}[X_A] \twoheadrightarrow \mathbb{K}[X_I]$$

$$x_u \longmapsto \begin{cases} x_u & u\in I \\ 0 & \text{else}\end{cases}$$

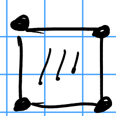
<u>Ex:</u>



$$X_I \subset X_A$$

$X_I \cong \mathbb{P}^1$, $X_I = V(x_1,x_3)$, $X_A = V(x_0x_3 - x_1x_2)$

Thm: Torus orbits $\mathcal{O}$ of X_A correspond to faces $\Sigma_{\mathcal{O}}$ of A, with $\overline{\mathcal{O}} = X_{\Sigma_{\mathcal{O}}}$

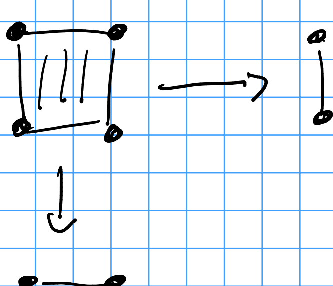
Ex:  has 9 orbits

Cayley Structures

$$\Delta_\ell := \{e_0, \dots, e_\ell\} \subset \mathbb{Z}^{\ell+1}$$

A length ℓ Cayley Structure on A is a surjection $\pi : A \longrightarrow \Delta_\ell$ preserving affine relations (i.e. $\sum a_u \cdot u = \sum b_u u$, $\sum a_u = \sum b_u$ $\Rightarrow \sum a_u \pi(u) = \sum b_u \pi(u)$)

Two such structures are equivalent if they differ by permutation of e_i.

Ex:  has two non-equivalent Cayley structures

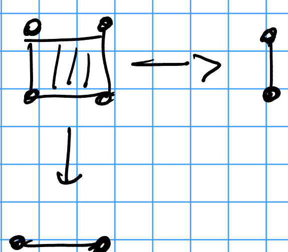
We define a partial order on Cayley structures on faces of A as follows:

$$\pi : \Gamma \longrightarrow \Delta_\ell$$
$$\pi' : \Gamma' \longrightarrow \Delta_{\ell'}$$

$\pi \prec \pi'$ iff $\Gamma \prec \Gamma'$, and there exists comm. diagram

$$\begin{tikzcd} \Gamma' \arrow[r, "\pi'"] & \Delta_{\ell'} \arrow[d] \\ \Gamma \arrow[u, phantom, "\cup"] \arrow[r, "\pi"] & \Delta_\ell \end{tikzcd}$$

Ex:



Are the two <u>maximal</u> Cayley structures

Ex: $A = \{3\times3 \text{ Permutation matrices}\}$

u_1, u_2, u_3 – even ones

v_1, v_2, v_3 – odd ones

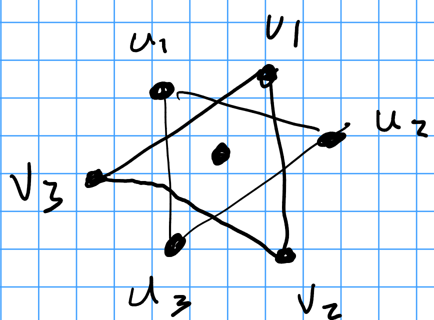
$u_1 + u_2 + u_3 = v_1 + v_2 + v_3$ (only relation)

Facets: $\Gamma_{ij} = A \setminus \{u_i, v_j\}$

9+6 maximal Cayley structures:

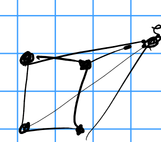
(3×2) $\quad A \longrightarrow \Delta_2$ by projecting onto a row/column

3×3 $\quad \Sigma_{ij} \longrightarrow \Delta_3$ (Σ_{ij} is already a simplex)



(Ilten-Zotine, Furukawa-Ito)

<u>Main Theorem</u> There is a bijection between irreducible components of $F_K(X_A)$ and maximal Cayley structures of length at least K. (need $K \geq 1$) Ex:



<u>Components of $F_K(X_A)$</u>

$\pi : \Sigma \longrightarrow \Delta_\ell$ Cayley structure

$\rightsquigarrow$ surjective homomorphism

$$\mathbb{K}[X_A] \twoheadrightarrow \mathbb{K}[X_\Sigma] \twoheadrightarrow \mathbb{K}[s_0, \dots, s_\ell]$$

(19)

$\rightsquigarrow$ linear space $L_\pi \subset X_A$ (of dimension d)

In equations: $L_\pi = \left(\begin{cases} x_u = 0 & \text{for } u \notin I \\ x_u = x_v & \text{for } \pi(u) = \pi(v) \end{cases} \right)$

As a rowspan:

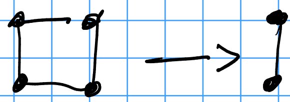
$$\begin{array}{cccc} \pi(u)=e_0 & \pi(u)=e_1 & & u \notin I \\ \left[\begin{array}{c|c|c|c} 1\ 1 \cdots 1 & 0 & & \\ & 1 \cdots 1 & \cdots & 0 \\ 0 & 0 & & \end{array}\right] \end{array}$$

Defn: $Z_\pi = \overline{T . [L_\pi]} \subset F_\ell(X_A)$

(This is a toric variety!)

What is $T . L_\pi$?

$$\begin{array}{cccc} \pi(u)=e_0 & \pi(v)=e_1 & & \\ \left[\begin{array}{c|c|c|c} \cdots t^u \cdots & & & \\ & \cdots t^v \cdots & \cdots & 0 \\ & & & \end{array}\right] \end{array}$$

Ex:  L_π: $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$$t \cdot L_\pi = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 0 & t & st \end{pmatrix} \sim \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 0 & 1 & s \end{pmatrix}$$

In closure, obtain $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Longrightarrow \quad Z_\pi = \mathbb{P}^1$$

$$Z_{\pi,k} = \left\{ [L'] \;\middle|\; \begin{array}{l} L' \dim k,\ L' \subset L \\ [L] \in Z_\pi \end{array} \right\}$$

(locally toric)

Precise statement of theorem: Components of $F_k(X_A)$ given by $Z_{\pi,k}$ for π maximal

Proof sketch

1) Show that $Z_{\pi,k} \subset Z_{\pi',k} \iff \pi \le \pi'$

2) Now it suffizes to show that every

$[L] \in F_k(X_A)$ is contained in <u>some</u> $Z_{\Pi, k}$

$$L = \begin{pmatrix} & & c_{0u} & & \\ \cdot & \cdots & & & \cdots \\ & & c_{ku} & & \end{pmatrix} \qquad y_u = \sum c_{iu} y_i \in \mathbb{K}[y_0, \ldots y_k]$$

Rem: $L \subset X_A \iff \forall \sum a_u u = \sum b_u u,$

$$\prod y_u^{a_u} = \prod y_u^{b_u}$$

$$\Gamma := \{ u \in A \mid y_u \neq 0 \}$$

This is a face of A!

$$\Gamma \longrightarrow \mathbb{P}(\mathbb{K}[y]_1)$$

$$u \longmapsto [y_u]$$

This is a Cayley structure of length $l \geq k$:

If $\sum a_u u = \sum b_u u$, then

$\prod y_u^{a_u} = \prod y_u^{b_u}$. Claim follows from unique facterization.

Now, show that $[L]$ satisfies local equations for $Z_{\pi, K}$ (tedious, but straightforward).

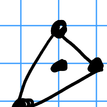
What about:

- Smoothness? If X_A smooth in codim K, then $Z_{\pi, K}$ (in reduced structure) are smooth.

- Connectedness?

$$Z_{\pi, K} \cap Z_{\pi', K} = \bigcup_{\pi'' < \pi, \pi'} Z_{\pi'', K}$$

Non-Reduced Structure

Ex:  has three maximal Cayley structures of length ≥ 1 (three edges)

$\rightsquigarrow F_1(X_A)$ "is" 3 points.

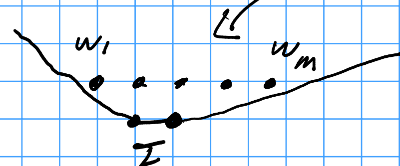
<u>But</u>: $X_A = V(xyz - w^3)$ a cubic surface.

Where is the number 27?

There is a combinatorial formula to determine <u>scheme</u> structure of $F_k(X_A)$ when
- X_A normal
- $k = \dim X_A - 1$

<u>Special case</u>: X_A projectively normal surface

<u>Locally</u>:

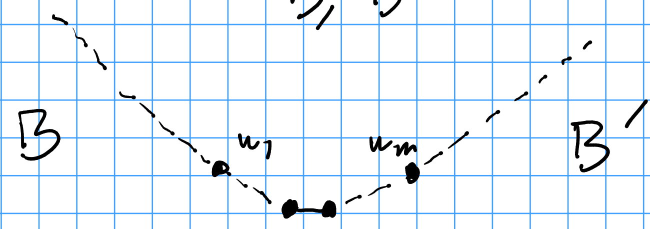


points at "height one"

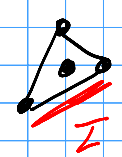
<u>Prop</u>: In neighborhood of $U\, L_I$, $F_1(X_A)$ has

$$\mathbb{K}[\;U\;] = \begin{cases} \mathbb{K} & m > 2 \\ \mathbb{K}[t]/t^{\min \delta, \delta'} & m = 2 \\ \mathbb{K}[s,t]/(s^{\delta}, t^{\delta'}) & m = 1 \end{cases}$$

where δ, δ' first height of $u \in \mathcal{A}$ in regions $\mathcal{B}, \mathcal{B}'$



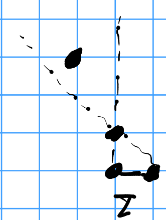
<u>Example:</u>



$$\rightsquigarrow \begin{array}{c} X \\ \shortparallel \\ V(xyz - w^3) \subset \mathbb{P}^3 \end{array}$$

singular cubic surface

Change of coordinates:



$\delta, \delta' = 3$

$\Rightarrow$ Locally, $\mathbb{K}[U] = K[s,t]/_{s^3, t^3}$

has dimension 9

$\Rightarrow \operatorname{Deg} F_1(X) = 27$

Open problems:

- Describe scheme structure of $F_k(X_\mathcal{A})$ in general.
- Conjecture: $X_\mathcal{A}$ smooth $\Rightarrow$ $F_k(X_\mathcal{A})$ smooth

Proof of Cayley's Theorem

Lemma (important later): $X \subset \mathbb{P}^n$ w/ coord ring S

$L \subset X$ k-dim linear space with ideal $I \subset S$.

Then $T_{[L]} F_k(X) \cong \mathrm{Hom}_S(I, S/I)_0$

Exercise: X smooth cubic surface, $L \subset X$ a line $\Longrightarrow$ $\mathrm{Hom}_S(I, S/I)_0 = 0$

As a corollary, we obtain that for a smooth cubic surface X,

$F_1(X)$ is the union of f.m. reduced points.

How many?

Have universal Fano scheme Φ

$$\Phi \xrightarrow{\ \varphi\ } W$$

space of cubics $\longrightarrow W$

For $[X] \in W$, $\varphi^{-1}([X]) = F_1(X)$.

At $X = V(xyz - w^3)$, $\varphi^{-1}([X])$ has expected dimension! Consequences:

1) φ is surjective

2) φ is <u>flat</u> over open locus containing all $[X]$ with $\dim F_1(X) = 0$

<u>Degree</u> is constant in flat proj. families $\Rightarrow$ Degree $F_1(X) = 27$ for all smooth X.

But $F_1(X)$ reduced $\Rightarrow$ $\# F_1(X) = 27$.

$\blacksquare$

Fano Schemes of $\det_n$

- $\det_n = \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$ $\quad X_n = V(\det_n) \subset \mathbb{P}^{n^2-1}$

- $F_k(X_n)$ "=" $\left\{ \begin{array}{l} k+1\text{-dim lin. spaces} \\ \text{of singular matrizes} \end{array} \right\}$

- <u>Questions:</u>
  - When is $F_k(X_n) \neq \emptyset$?
  - What are (some) irred components of $F_k(X_n)$?
  - When is $F_k(X_n)$ connected?

Compression Spaces

$V, W \subset \mathbb{K}^n$, $\dim V = s+1$, $\dim W = s$

$\{ M : \mathbb{K}^n \to \mathbb{K}^n \mid MV \subset W \}$ is called an

<u>s-compression space</u>.

<u>Ex:</u> $n=3$ $\quad V = \langle e_1, e_2 \rangle$, $\quad W = \langle e_1 \rangle$ $\quad \rightsquigarrow \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$

<u>Rem</u>

- this is a linear space of singular matrices
- Has dimension

$$K(s) := n^2 - (s+1)(n-s) - 1 \text{ in } \mathbb{P}^{n^2-1}$$

$$C_{n,k}(s) = \left\{ L \in G(k, n^2-1) \;\middle|\; \begin{array}{l} L \text{ contained in} \\ \text{an } s\text{-compression} \\ \text{space} \end{array} \right\}$$

$$\delta(s) = s(n-s) + (s+1)(n-s-1)$$

<u>Thm</u> (Chan-I)

1) The $C_{n,k}(s)$ are distinct irred comp's of $F_k(X_n)$ $\quad s = 0, \dots, n-1$

2) $$\operatorname{Dim} C_{n,k}(s) = \delta(s) + (k+1)(K(s) - k)$$

<u>Rem</u> In some cases, these are Grassmannian bundles over products of Grassmannians

<u>Rem</u> X_n is covered by $n(n-1)-1$ dim planes (take e.g. $C_{n,k}(0)$).

Non-Emptiness of $F_K(X_n)$

Thm (Diedonné) $F_K(X_n) \neq \emptyset \iff$

$K < n(n-1)$.

Proof: 1) Consider $T = (\mathbb{K}^*)^n \times (\mathbb{K}^*)^n$ - action

on $\mathbb{P}^{n^2-1}$ (via scaling of rows/columns)

This preserves X_n and induces action on $F_K(X_n)$.

2) $L \in G(K, n^2-1) \longleftrightarrow n \times n$ matrix

filled with forms in $s_0, \ldots, s_K$

L is T-fixed $\iff$ entries are s_i or 0

(any linear relation among entries can only involve 1 term)

3) Fact: Any projective T-variety X contains a torus fixed point.

4) Exercise: Any T-fixed point of

$G(k, n^2-1)$ contained in X_n must be a "standard compression space".
(Use Hall's marriage theorem) ∎

Connectedness

Thm (Chan-I) $F_k(X_n)$ is <u>disconnected</u> iff

$$n^2 - 2n < k < K(0) \quad \text{or} \quad \exists\, 0 < s < n-1$$

s.t.

$$K(s) - \min(n-s-1, s) < k \le K(s)$$

Pf: 1) The criterion is fullfilled $\iff$ the union of the $C_{n,k}(s)$ is connected

Ex: $n = 3$

$k \le 3$

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$$

need $k \le 3$ $\qquad$ $k \le 3$

Use that intersection occur at torus fixed points!

2) When criterion is <u>not</u> fulfilled, compute tangent space dimension at torus fixed points to show $C_{n,k}(s)$ is smooth (hence doesn't intersect other comps)

<u>Ex:</u> $n=3$ $k=4$ $\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$ has ideal

x_{22} x_{23}

x_{32} x_{33}

$\mathrm{Hom}(I, S/I)_0$ $\quad x_{ij} \longmapsto y_{ij} \in \mathbb{k}[x_{11}, \ldots]$

<u>Relation</u> $\cancel{x_{11} x_{22} x_{33}} + x_{12} \underline{x_{23}} x_{31} + x_{13} x_{21} \underline{x_{32}}$

$- x_{21} x_{12} \underline{x_{33}} - x_{31} x_{13} \underline{x_{22}}$

y_{33}: $\alpha_1 x_{31} + \alpha_2 x_{13}$

y_{23} $\alpha_1 x_{21} + \alpha_3 x_{13}$ $\Rightarrow$ $\underline{\dim = 4}$

y_{32}: $\alpha_2 x_{12} + \alpha_4 x_{31}$

y_{22}: $\alpha_3 x_{12} + \alpha_4 x_{21}$

<u>Example:</u>

$n=$	2	3	4	5	6
Connected: $k \leq$	–	3	8	13	21 ...
or $k=$					24

Application: product rank

Defn: The **product rank** of $f \in \mathbb{K}[x_0, \dots, x_n]_d$ is the smallest $r \in \mathbb{N}$ s.t.

$$f = \sum_{i=1}^{r} l_{i1} \cdots l_{id} \qquad l_{ij} \in \mathbb{K}[\underline{x}]_1$$

(Denoted by $pr(f)$).

Ex:
- $pr(\det_2) = 2$
- $pr(\det_3) \le \cancel{6}$ 5 by Derksen

Thm (Teitler–I) $pr(\det_3) = 5$

To prove this, first a detour...

Generic Sums of Products

$d \geq 2$ | $X_r^d = V\left(\sum_{i=1}^{r} \prod_{j=1}^{d} y_{ij}\right) \subset \mathbb{P}^{rd-1}$

What can we say about $F_k(X_r^d)$?

Conjecture ($S_i\beta$ - I) $\quad r$ even, $d \geq 3$

Assume that $k \geq \frac{rd}{2} - 1$

Then every k-plane of X_r^d is contained in $V(y_{ij})$ for some ij or

$$V\left(y_{i1} \cdots y_{id} - y_{j1} \cdots y_{jd}\right) \text{ for some } i \neq j.$$

Thm ($S_i\beta$-I)

Conjecture is true if $d = 4$ or $r \leq 6$

Ex: $\boxed{r=4, d=3}$ What are 5-dim linear spaces of X_4^3?

$$5 \geq \tfrac{4 \cdot 3}{2} - 1 \;\checkmark$$

Type 1: Contained in $V(y_{ij})$

toric!

Type 2: Contained in $V(y_{i1}y_{i2}y_{i3} + y_{j1}y_{j2}y_{j3})$

$\rightsquigarrow$ contained in $V(y_{ij})$

or, up to permutation, given by rowspan of

$$\begin{array}{cccccccccccc}
y_{11} & y_{12} & y_{13} & y_{21} & \cdots & & y_{31} & \cdots & & y_{41} & \cdots & \\
\end{array}$$
$$\left(\begin{array}{ccc|ccc|ccc|ccc}
1 & & & p & & & & & & & & \\
& 1 & & & q & & & & & & & \\
& & -pq & & & 1 & & & & & & \\
& & & & & & 1 & & & p' & & \\
& & & & & & & 1 & & & q' & \\
& & & & & & & & -p'q' & & & 1
\end{array}\right)$$

Proof that $pr(\det_3) = 5$.

<u>Key step:</u> Show that $pr(\det_3) \neq 4$.

Facts that we need:

1) $V(\det_3) \subset \mathbb{P}^8$ covered by a 2-dim family of 5-planes

(given $M \in V(\det_3)$, take $v \neq 0$, $v \in \ker M$

Consider $\{ M' \mid M' \cdot v = 0 \}$)

2) The linear span of this family is $\mathbb{P}^8$

3) $\det_3$ is <u>concise</u>

Assume $\det_3 = l_{11} l_{12} l_{13} + \dots + l_{41} \cdots l_{43}$

this gives linear map (injective, since $\det_3$ concise)

$$\mathbb{K}^9 \xrightarrow{\varphi} \mathbb{K}^{12}$$
$$x \longmapsto (l_{ij}(x))$$

(Affine cone over) $V(\det_3)$ is $\varphi^{-1}(X_4^3)$

$L := \varphi(L)$

$\rightsquigarrow \quad X_4^3 \cap L \cong V(\det_3)$

$\Longrightarrow$ X_4^3 must contain a 2d family F of (proj) 5-planes whose span is L

5 planes are either of Type I or II (see above)

F of Type I $\Longrightarrow$ $L \subset V(y_{ij})$

$\Longrightarrow V(\det_3) \cong V\left(\sum_{i=1}^{3} l_{i1} \cdots l_{i3}\right)$

$\Rightarrow \mathrm{pr}(\det_3) \leq 3.$

F of type II: of p, q, p', q', 2 must be non-constant

Cannot be e.g. p, p' since then span is ≥ 9-dim (projectively)

$\rightsquigarrow$ WLOG p, q non-constant, p', q' constant

L contained in
$$p' y_{31} = y_{41}$$
$$q' y_{32} = y_{42}$$
$$y_{33} = -p'q' y_{43}$$

$$\Rightarrow \det_3 = l_{11} l_{12} l_{13} + l_{21} l_{22} l_{23} + l_{31} l_{32} l_{33}$$
$$- p'q' \frac{1}{p'q'} l_{31} l_{32} l_{33}$$
$$= l_{11} l_{12} l_{13} + l_{21} l_{22} l_{23}$$

$\Rightarrow \mathrm{pr}(\det_3) \leq 2$ //

other arguments show $\mathrm{pr}(\det_3) \neq 1, 2, 3$ $\blacksquare$