REALIZATION OF HYPERGEOMETRIC MOTIVES IN PRODUCTS OF FERMAT CURVES

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A generalized hypergeometric differential operator of order r with parameters $\vec{\alpha}, \vec{\beta} \in \mathbb{C}^r$ is the operator on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by

$$\prod_{i=1}^{r}\Big(s\frac{d}{ds} - \alpha_i\Big) - s\prod_{i=1}^{r}\Big(s\frac{d}{ds} - \beta_i\Big). \tag{1}$$

As we will learn soon, hypergeometric differential equations are rigid and the eigenvalues of local monodromies around $s = 0$ and $s = \infty$ are given by $\{\exp(2\pi i\alpha_j)\}_{j=1}^r$ and $\{\exp(2\pi i\beta_k)\}_{k=1}^r$ respectively. A necessary condition for a differential operator to be of geometric origin is that the local monodromies are quasi-unipotent, which in this case gives $\vec{\alpha}, \vec{\beta} \in \mathbb{Q}^r$. It is due to Katz [1] that this condition is also sufficient for rigid local systems on $\mathbb{P}^1$. Thus for every $\vec{\alpha}, \vec{\beta} \in \mathbb{Q}^r$ there should exist a 1-parametric family of algebraic varieties such that the Gauss–Manin connection on a rank r subquotient in its cohomology is equivalent to the local system of solutions of the hypergeometric differential operator (1). Such objects are called *hypergeometric motives* (see e.g. [2]). We shall be looking at their particular realizations.

Fix an integer $N > 1$ and let $\mathcal{C}$ be the Fermat curve $X^N + Y^N = Z^N$. Fix a primitive Nth root of unity ζ_N. The group $\mathbb{Z}/N \times \mathbb{Z}/N$ acts on $\mathcal{C}$ by $[X : Y : Z] \mapsto [\zeta_N^a X : \zeta_N^b Y : Z]$. Take $r \geq 1$ and consider the following family of hypesurfaces in the r-fold product of $\mathcal{C}$'s

$$X_t = \{X_1 \dots X_r = tZ_1 \dots Z_r\} \subset \mathcal{C} \times \cdots \times \mathcal{C}\,.$$

On X_t we then have action of the subgroup $G \subset (\mathbb{Z}/N \times \mathbb{Z}/N)^r$ consisting of $(a_1, b_1, \dots, a_r, b_r)$ with $\sum a_i = 0$. Consider the splitting of the middle cohomology of this family by characters $\chi \in \widehat{G}$:

$$H^{r-1}(X_t) = \bigoplus_{\chi \in \widehat{G}} H^{r-1}(X_t)^\chi.$$

Problem 1. Show that after the substitution $s = t^N$ the Gauss–Manin connection on $H^{r-1}(X_t)^\chi$ is hypergeometric. Give an explicit relation between the classical hypergeometric data $(\vec{\alpha}, \vec{\beta})$ and data (N, r, χ).

Our guess that the Gauss–Manin connection on $H^{r-1}(X_t)^\chi$ should be hypergeometric comes from the Euler integral representation of generalized hypergeometric functions, see e.g. [3]. For $r = 2$ varieties X_t are smooth when $t \notin \{0, \sqrt[N]{1}\}$. I attach a note [4] in which Problem 1 is solved for this case. When $r > 2$ one needs to substitute X_t above by a smooth variety (to resolve singularities).

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Problem 2. Compute Hodge numbers of $H^{r-1}(X_t)^\chi$.

If Problems 1 and 2 are solved successfully, this would give a computation of the Hodge numbers of hypergeometric motives. A formula for their Hodge numbers was conjectured by Golyshev–Corti [5] and proved recently by Fedorov [6].

References

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