Pricing Vulnerable Options and Good Deal Bounds. A Structural Model

Agatha Murgoci,
Department of Finance,
Stockholm School of Economics,
Box 6501,
SE-113 83 Stockholm,
SWEDEN
e-mail: agatha.murgoci@hhs.se

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Abstract

We price vulnerable options - i.e. options where the counterparty may default. Default is modeled in a structural framework. The technique employed for pricing is Good Deal Bounds.

Key words: Incomplete markets, good deal bounds, Vulnerable options.

JEL code: ??

Contents

1 Introduction

Vulnerable options are options that bear counterparty risk - in other words, the writer of the option may not deliver the underlying.
The main reason for having a counterparty risk is the fact that these options are traded over-the-counter (OTC). If traded on an organized exchange, the counterparty risk associated with the option disappears due to the presence of the market maker. According to BIS, the OTC equity-linked option gross market value in the first half of 2006 was of USD 523 bln, with notional amount outstanding of USD 5 361 bln. Thus, there is a necessity to have fair pricing of equity linked options traded OTC, taking into account the counterparty risk nonexistent in organized exchanges.
Previous literature prices vulnerable options under the important assumption that they are traded in complete markets, i.e. that in addition to S, also Y is the price of a traded asset. Papers pricing vulnerable options in a complete market setup include [10], [9], [8], [11]. In real life, vulnerable options are traded mainly over-the-counter, and the assets of the counterparty are not traded assets on the market. Thus, we are in a classical case of incomplete markets. The first to notice this inconsistency were [7]
[7] have priced the vulnerable options using the structural model set up by [11] and using "good deal bounds".

The good deal bounds were first introduced by [4] and constitute a "hybrid" between no-arbitrage pricing and utility-based pricing. They narrow the wide bands of possible prices obtained with no arbitrage pricing, while avoiding the model-sensitivity implied by utility-based pricing. The method imposes a new restriction in the arbitrage free model, by setting upper bounds on the Sharpe ratio-s of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. [3] translate the stochastic discount problem to an equivalent martingale problem and use martingale methods to solve the optimization problem. Thus, the calculations are more tractable and the price processes can be characterised by point processes, besides the traditional Wiener framework.
In this paper, we extend the pricing framework of vulnerable options proposed by [7] and we:

- Use the martingale framework proposed by [3] which allows for higher degree of tractability, and in consequence,
- Streamline earlier results in literature;
- Extend the results for European calls options to options with homogenous payoff functions.

2 The Complete Market case - The Klein Model

2.1 Setup

The main purpose of this paper is presenting a unified framework for pricing vulnerable options, which are options where the writer of the option may default.
Traditionally, vulnerable options were analysed in a structural framework, i.e. a model for credit risk that takes into account the value of the assets of the option writer(counterparty) in order to define default. The main ingredients for such a framework are the dynamics of the stock and the dynamics of the assets of the counterparty. The current paper starts from the traditional framework of pricing vulnerable options in complete markets and extends it to incomplete markets. This section takes the setup proposed by [11] and calculates the price of a vulnerable option by applying a different method, the change of numeraire. This method will alow for a better tractability of the old results, as well as for extending the results for a call option to other vanilla products, such as min or exchange options.
The option is written on the stock S and has maturity T and strike K. For simplicity, we will assume through the entire paper that the stock S is traded on the market. The case of an option written on an untraded stock is a straightforward extension for the incomplete market case analysed in subsequent sections. Since the option to be priced is traded over-the-counter, there is also counterparty risk to be taken into account.

In a first setting, default will depend on the assets of the counterparty - the writer of the option. They are denoted by Y. As stated in [11], the assets of the counterparty are defined such that they include all assets of the counterparty, marked to market, as well as all derivative positions.
To begin with, we consider the assets of the counterparty traded on the market. This assumption implies we are in a complete market setup. The total value of the claims against the counterparty is denoted by D and we are not concerned with modelling D. We assume a riskless bond B with interest rate r is traded on the market.
We proceed by giving the main features of the market:

Assumption 2.1

1. *Let $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ be given; $\underline{\mathcal{F}}$ is the internal filtration generated by the Wiener process $\tilde{W}$, which will be defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dY_t &= \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t \\ dS_t &= \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t \\ dB_t &= B_t r dt \end{aligned}$$

   where Y_t denotes the assets of the counterparty underwriting the option, S_t the price of the stock on which the option is contracted and B_t the bank account. The assets of the counterparty are defined such that they include all assets of the counter-party, marked to market, as well all derivative positions.

3. *μ_t and α_t are scalar deterministic functions of time, $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are (1,2) row vector deterministic functions of time, specified as follows:*

$$\begin{aligned} \bar{\gamma}_t &= (\, \gamma_t, \quad 0\,) \\ \bar{\sigma}_t &= (\, \sigma_t \rho, \quad \sigma_t \sqrt{1-\rho^2}\,) \end{aligned}$$

4. *Let $\tilde{W}$ be a two dimensional P-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2 \end{pmatrix}$$

   with $\tilde{W}^1$ and $\tilde{W}^2$ independent scalar P-Wiener processes.

5. *Assume that both the assets of the counterparty underwriting the option and the stock are traded on the market.*

6. *The payoff of a vulnerable European call option,* $X = \Phi(S_T, Y_T, T)$*, is given by*

$$X = \Phi(S_T, Y_T, T) = \max(S_T - K, 0)I(Y_T \geq D) + \mathcal{R}I(Y_T < D)$$

where D *is the value of the total value of the claims against the counter-party.*

Before proceeding, we need to clarify a few things. First, note that while $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are defined as row vector coefficients of the diffusion terms, sometimes it is more convienient to work with the scalar values of σ_t and γ_t. However, it is straightforward when one uses scalar notation instead of vector notation and switching to the vector notation does not pose any technical difficulty.
Next, we are going to give an intuition for the payoff function of a vulnerable option. The general payoff function for vulnerable options has two components - the payoff of the claim in case of no default and the recovery payoff, i.e. the payoff in case the counterparty defaults. We denote the general payoff by X and the recovery payoff by $\mathcal{R}$. If there is no default, the payoff of the claim is the standard option payoff, i.e. $\max(S_T - K, 0)$; if the counterparty defaults, the payoff is the recovery payoff $\mathcal{R}$, which is to be defined in each section of the paper, due to modelling differences. In this section, the default occurs if the value of the assets of the counterparty at time T, Y_T, falls below the value of the claims writtena gainst the counterparty, D. All payments are done at time T.

For the complete market setup, the value of the recovery payoff $\mathcal{R}$ is given by:

$$\mathcal{R} = (1 - \beta)\frac{Y_T}{D}\max(S_T - K, 0)$$

The logic behind the above formula is straightforward. One gets a proportional part of the value of the claim, corresponding to how much the assets of the counter-party have fallen below the value of the claim. However, there are some deadweight costs associated to the bankrupcy procedure. These costs are captured by the β parameter. This recovery specification is very close to the specification for recovery of treasury.

Assumption 2.2 *Let the recovery payoff be given by:*

$$\mathcal{R}I(Y_T < D) = (1 - \beta)\frac{Y_T}{D}\max[S_T - K, 0]I\{Y_T < D\}$$

Having defined the main assumptions of the model, we will now proceed to price the vulnerable option in the complete markets setup, by using the change of numeraire technique.

2.2 Pricing the vulnerable options by change of numeraire

2.2.1 Change of numeraire for the case with zero recovery

First, I will calculate the value of the claim, for the case of zero recovery. The assumption of zero recovery is adopted only for the duration of the current section. However, starting from this simplifying case gives a better clarity in exposition and calculations are more tractable. The payoff function becomes $X = \max[S_T - K, 0]I[Y_T \geq D]$ and we obtain:

$$
\begin{aligned}
X &= \max[S_T - K, 0]I[Y_T \geq D] = (S_T - K)I[S_T \geq K]I[Y_T \geq D] && (1)\\
&= S_T I[S_T \geq K]I[Y_T \geq D] - KI[S_T \geq K]I[Y_T \geq D] && (2)
\end{aligned}
$$

In order to price the claim with payoff X, we will apply the change of numeraire technique. For details, see [2]. For the first term,$S_T I[S_T \geq K]I[Y_T \geq D]$, we do a change of measure to Q^S, the measure corresponding to S_t as numeraire. For the second term,$KI[S_T \geq K]I[Y_T \geq D]$, the change of measure is to the forward measure, Q^T. In the end, the zero-recovery pricing problem is reduced to calculating four probabilities: the probability the option is exercised $(S_T - K \geq 0)$ under Q^S and Q^T and the probability of no default under Q^S and Q^T.

Denoting by Π the price of the vulnerable option, we will start from the following pricing expression:

$$
\Pi = S_0 Q^S[S_T \geq K; Y_T \geq D] - Kp(0,T)Q^T[S_T \geq K; Y_T \geq D] \tag{3}
$$

and we will first attack the second term, Q^T, and then the first, Q^S. The calculations are detailed below.

- Under the forward measure, Q^T, we need to calculate:

$$
Q^T(Y_T \geq D, S_T \geq K) = Q^T\left(\frac{Y_T}{p(T,T)} \geq D, \frac{S_T}{p(T,T)} \geq K\right) \tag{4}
$$

  since $p(T,T) = 1$.

  We denote $Z_Y(t) = \frac{Y_t}{p(t,T)}$ and $Z_S(t) = \frac{S_t}{p(t,T)}$ and want to calculate $Q^T(Z_Y(T) \geq D, Z_S(T) \geq K)$. Under the forward measure, $Z_Y(t)$ and $Z_S(t)$ are martingales (as the price of asset with the forward price as numeraire).

$$
\begin{aligned}
dZ_Y(t) &= Z_Y(t)\bar{\sigma}_t dW_t^T\\
dZ_S(t) &= Z_S(t)\bar{\gamma}_t dW_t^T
\end{aligned}
$$

  The solutions to the above equations are:

$$
\begin{aligned}
Z_Y(T) &= Z_Y(0)\exp\left\{-\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T \bar{\sigma}_t dW_t^T\right\}\\
Z_S(T) &= Z_S(0)\exp\left\{-\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^T\right\}
\end{aligned}
$$

We notice that the equations above have a very similar structure: the exponent is the sum of a time integral and a stochastic integral with a deterministic integrand, which leads to the entire exponent being normally distributed. The variance of the exponent is $\int_0^T \|\bar{\sigma}_t\|^2 dt$ for the first equation, respectively $\int_0^T \|\bar{\gamma}_t\|^2 dt$ for the second equation.
Now, we need to transform the two lognormal variables into standard-normal variables. In order to perform this easy transformation, we use the following string of equivalent inequalities:

$$Y_T \geq D$$

$$Z_Y(T) \geq D$$

$$Z_Y(0) \exp\left\{-\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T (\bar{\sigma}_t) dW_t^T\right\} \geq D$$

$$\frac{\ln Z_Y(T) - \ln Z_Y(0) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \geq \frac{\ln D - \ln Z_Y(0) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$\xi \geq \underbrace{\frac{\ln \frac{Dp(0,T)}{Y(0)} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}}_{b_2}$$

where ξ is standard-normally distributed.
Following the same steps, we have:

$$S_T \geq K \Leftrightarrow Z_S(T) \geq K$$

and by writing explicitely $Z_S(T)$, taking logs and suitably transforming the resulting normal variable, we obtain the equivalent inequality:

$$\eta \geq \underbrace{\frac{\ln \frac{Kp(0,T)}{S(0)} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}}_{a_2}$$

where η is standard normally distributed.

Summing up the calculations above, one obtains:

$$Q^T(Y_T \geq D, S_T \geq K) = Q^T(\eta \geq a_2, \xi \geq b_2, \rho_1) \tag{5}$$

where η and ξ are standard normal, ρ_1 is the correlation coefficient between η and ξ. The constants a_2 and b_2 are given above.
The first step in clarifying the right-handside term in (5) is calculating the correlation coefficient ρ_1. We formulate the result as a lemma.

Lemma 2.1 *Given assumptions 2.1 and η and ξ defined as in (5), the correlation coefficient between η and ξ is given by:*

$$\rho_1 = \frac{\rho \int_0^T \sigma_t \gamma_t \prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}. \tag{6}$$

Proof. We know that

$$\xi = \frac{\ln\left(\frac{Z_Y(T)}{Z_Y(0)}\right) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$\eta = \frac{\ln\left(\frac{Z_S(T)}{Z_S(0)}\right) + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

The only stochasticity in the formulas comes from the expressions A_1 and A_2 defined below:

$$A_1 = \ln\left(\frac{Z_Y(T)}{Z_Y(0)}\right) = -\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T \bar{\sigma}_t dW_t^T$$

$$A_2 = \ln\left(\frac{Z_S(T)}{Z_S(0)}\right) = -\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^T$$

Simplifying even further, we obtain that:

$$\rho_1 = Corr[A_1; A_2] = Corr\left[\int_0^T \bar{\sigma}_t dW_t^T; \int_0^T \bar{\gamma}_t dW_t^T\right]$$

where $\bar{\sigma}_t = (\, \sigma_t\rho, \quad \sigma_t\sqrt{1-\rho^2}\,)$ and $\bar{\gamma}_t = (\, \gamma_t, \quad 0\,)$ and $W^T = (\, W^{1;T}, \quad W^{2;T}\,)'$, with $W^{1;T}$ and $W^{2;T}$ independent T-Wiener processes[1]. By direct calculation, it follows that

$$\rho_1 = \frac{Cov\left[\int_0^T \bar{\sigma}_t dW_t^T; \int_0^T \bar{\gamma}_t dW_t^T\right]}{\sqrt{Var\left[\int_0^T \bar{\sigma}_t dW_t^T\right] Var\left[\int_0^T \bar{\gamma}_t dW_t^T\right]}} = \frac{\rho \int_0^T \sigma_t \gamma_t \prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}.$$

If σ_t and γ_t are constant, we have $\rho_1 = \rho$. □

At this point, we need to introduce some new notation.

[1] The independence of $W^{1;T}$ and $W^{2;T}$ follows from the independence of the P-Wiener processes $\tilde{W}^1$ and $\tilde{W}^2$ (see assumption 2.1). If we denote by φ^T the Girsanov kernel between the objective probability measure P and the equivalent martingale measure Q^T, it is easy to see that $dW_t^{1;T} W_t^{2;T} = (-\varphi^T dt + d\tilde{W}_t^1)(-\varphi^T dt + d\tilde{W}_t^2) = 0$

Definition 2.1 *Let $\mathcal{N}(a, b, \mathbf{r})$ be the probability $P[X \leq a; Y \leq b]$, where X and Y are standard normal variables with correlation coefficient $\mathbf{r}$.*

We are going to use this notation to express the right handside of (5) in terms of CDF-s. In order to do this, we need to study the behaviour of the bivariate standard normal distribution, when the two variable are correlated, which will be done by means of characteristic functions.
Let $\Phi(t_1, t_2)$ be the characteristic function for the bivariate normal distribution with correlation coefficient $\mathbf{r}$, $N[\mu_1, \mu_2, \sigma_1, \sigma_2, \mathbf{r}]$:

$$\Phi(t_1, t_2) = \exp\left\{ i[t_1\mu_1 + t_2\mu_2] - \frac{1}{2}[t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2\mathbf{r}\sigma_1\sigma_2 t_1 t_2] \right\} \tag{7}$$

By replacing the values for μ_i and σ_i $i \in \{1, 2\}$, we obtain the characteristic function $\Phi_1(t_1, t_2)$ for the standard normal bivariate distribution with correlation coefficient $\mathbf{r}$, $N[0, 0, 1, 1, \mathbf{r}]$:

$$\Phi_1(t_1, t_2) = \exp\left\{ -\frac{1}{2}[t_1^2 + t_2^2 + 2\mathbf{r}t_1 t_2] \right\} \tag{8}$$

Since the characteristic function $\Phi_1(t_1, t_2)$ is a real function, we know the distribution is symmetric (see [5]), or

$$P[X \geq a; Y \geq b] = P[X \leq -a; Y \leq -b] = \mathcal{N}[-a; -b; \mathbf{r}], \tag{9}$$

where X, Y are standard normal distributed and have correlation coeficient $\mathbf{r}$.
Then, we try to transform $P[X \geq a; Y \leq b]$ into a CDF. We know that if (X, Y) $N[0, 0, 1, 1, \mathbf{r}]$, then $(-X, Y)$ $N[0, 0, 1, 1, -\mathbf{r}]$. Hence, we conclude that:

$$P[X \geq a; Y \leq b] = P[-X \leq -a; Y \leq b] = \mathcal{N}[-a; b; -\mathbf{r}] \tag{10}$$

Note the change of signs in the correlation coefficient.
Thus, we obtain $Q^T(\eta \geq a_2, \xi \geq b_2, \rho_1) = \mathcal{N}(-a_2, -b_2, \rho_1)$.

Going back to (5), we conclude that:

$$Q^T(Y_T \geq D, S_T \geq K) = \mathcal{N}(-a_2, -b_2, \rho_1) \tag{11}$$

where ρ_1 is given by lemma (2.1) and:

$$a_2 = \frac{\ln \frac{Kp(0,T)}{S(0)} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

$$b_2 = \frac{\ln \frac{Dp(0,T)}{Y(0)} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

- We need to calculate $Q^S(Y_T \geq D, S_T \geq K)$ before being able to give a solution to the the initial pricing equation (3).

  To this purpose, we start by identifying the dynamics of the assets under the new probability measure Q^S. The dynamics of Y_t under Q^S are given by standard theory as:

  $$dY_t = Y_t(\mu_t + \bar{\sigma}_t \varphi_t^S)dt + Y_t \bar{\sigma}_t dW_t^S$$

  where φ^S is the Girsanov kernel for the transformation $P \to Q^S$. The Girsanov kernel φ^S is obtained by imposing the martingale condition under Q^S for $\frac{Y_t}{S_t}$:

  $$\mu_t - \alpha_t - \sigma_t \gamma_t \rho + 2\gamma_t^2 + (\bar{\sigma}_t - \bar{\gamma}_t)\varphi_t^S = 0$$

  and for $\frac{B_t}{S_t}$:

  $$r - \alpha_t + 2\gamma_t^2 - \bar{\gamma}_t \varphi_t^S = 0.$$

  Since we have a system of two equations with two unknowns, φ^S is completely identified and it is given by:

  $$\varphi_t^S = \left(\frac{r-\alpha_t+2\gamma^2}{\gamma_t}, \quad \frac{\gamma_t(\mu_t - r) + \sigma_t \rho (r - \alpha_t)}{\sigma_t \gamma_t \sqrt{1-\rho^2}} \right)'$$

  The solution to SDE describing the dynamics of Y_t is:

  $$Y_T = Y_0 \exp\left(\int_0^T (\mu_t + \varphi_t^S \bar{\sigma}_t)dt - \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T \bar{\sigma}_t dW_t^S \right) \quad (12)$$

  Since the exponent part from (12) is formed by a time integral and a stochastic integral with a deterministic integrand, it is clear that the exponent of Y_t is normally distributed with variance $\int_0^T \|\bar{\sigma}_t\|^2 dt$.

  We need to transform the lognormal variables into standard normal variables:

  $$Y_T \geq D$$
  $$Y(0) \exp\left(\int_0^T (\mu_t + \varphi_t^S \bar{\sigma}_t)dt - \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T (\bar{\sigma}_t) dW_t^S \right) \geq D$$

$$\xi \geq \underbrace{\frac{\ln \frac{D}{Y(0)} - \int_0^T (\mu_t + \varphi_t^S \bar{\sigma}_t)dt + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}}_{b_1}$$

where ξ is standard normally distributed.
For S_T we reformulate:

$$S_T \geq K \Leftrightarrow \frac{1}{S_T} \leq \frac{1}{K} \Leftrightarrow \frac{p(T,T)}{S_T} \leq \frac{1}{K}$$

We know that $\frac{p(t,T)}{S_t}$ is a martingale under Q^S. The dynamics for $\frac{p(t,T)}{S_t}$ can be calculated by Ito formula. Since, in this model, the interest rate is deterministic and the price of a riskless bond is given by $p(t,T) = \exp\{-r(T-t)\}$, the dynamics are:

$$d\left[\frac{p(t,T)}{S_t}\right] = -\frac{p(t,T)}{S_t}\bar{\gamma}_t dW_t^S$$

The solution to the above equation is:

$$\frac{p(T,T)}{S_T} = \frac{p(0,T)}{S_0}\exp\left(-\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt - \int_0^T \bar{\gamma}_t dW_t^S\right)$$

The exponent is normally distributed with variance $\int_0^T \|\bar{\gamma}_t\|^2 dt$.
We have:

$$S_T \geq K$$

$$\frac{p(T,T)}{S_T} \leq \frac{1}{K} \Leftrightarrow \log\frac{p(T,T)}{S_T} \leq \log\frac{1}{K}$$

$$\eta \leq \underbrace{\frac{\log \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}}_{a_1}$$

Now, we have:

$$Q^S(Y_T \geq D, S_T \geq K) = Q^S(\eta \leq a_1, \xi \geq b_1, \mathbf{r}) \tag{13}$$

where η and ξ are standard normal variables; $\mathbf{r}$ is the correlation coefficient between the two standard normal variables; a_1 and b_1 are defined above.

Following the same steps as before, it is easy to show by direct computation that $\mathbf{r} = \rho_1$.
Also, the properties of the bivariate normal distribution derived before allow us to change the generic probability into a CDF. Thus, we obtain:

$$Q^S(Y_T \leq D, S_T \geq K) = \mathcal{N}(a_1, -b_1, -\rho_1) \tag{14}$$

where ρ_1 is given by lemma (2.1) and

$$\begin{aligned} a_1 &= \frac{\log \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\ b_1 &= \frac{\ln \frac{D}{Y_0} - \int_0^T (\mu_t + \varphi_t^S \bar{\sigma}_t) dt + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}. \end{aligned}$$

Up to this point, we have performed all calculations necessary to obtain a pricing expression for vulnerable options, in the case of complete markets, for the zero recovery payoff. We needed to calculate the probabilities that appear in the transformation of the general risk neutral pricing formula:

$$\Pi(0, X) = S_0 Q^S[S_T \geq K; Y_T \geq D] - Kp(0,T) Q^T[S_T \geq K; Y_T \geq D]$$

Now we can gather the results from the last calculations and obtain a pricing expression similar to the Black Scholes equation.

Proposition 2.2 *Under the assumptions 2.1, the price for a vulnerable option with maturity T, strike price K, and zero recovery, Π_0^1, is given by:*

$$\Pi_0^1 = S_0 \mathcal{N}(a_1, -b_1, -\rho_1) - Kp(0,T)\mathcal{N}(-a_2, -b_2, \rho_1) \tag{15}$$

where

$$\begin{aligned} a_1 &= \frac{\log \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\ b_1 &= \frac{\ln \frac{D}{Y_0} - \int_0^T (r + \bar{\gamma}_t \bar{\sigma}_t') dt + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\ a_2 &= \frac{\ln \frac{Kp(0,T)}{S_0} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\ b_2 &= \frac{\ln \frac{Dp(0,T)}{Y_0} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \end{aligned}$$

$$\rho_1 = = \frac{\rho\int_0^T \sigma_t\gamma_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

2.2.2 Change of numeraire for the recovery payoff

Now, we can go back to the original case and price the recovery payoff for complete markets. We recall from the assumption 2.2 that the recovery payoff is given by:

$$\begin{aligned}
\mathcal{R}I(Y_T < D) &= (1-\beta)\frac{Y_T}{D}\max[S_T - K, 0] I\{Y_T < D\} \\
&= (1-\beta)\frac{Y_T}{D}(S_T - K) I\{S_T \geq K\} I\{Y_T < D\} \\
&= \frac{1-\beta}{D} Y_T S_T I\{S_T \geq K\} I\{Y_T < D\} - \frac{1-\beta}{D} K Y_T I\{S_T \geq K\} I\{Y_T < D\}
\end{aligned}$$

Since the calculations about to follow are cumbersome, it is for the benefit of the reader to split the above formula in 2 parts, to be analysed separately:

$$R_1 = \frac{1-\beta}{D} Y_T S_T I\{S_T \geq K\} I\{Y_T < D\} \quad (16)$$

$$R_2 = \frac{1-\beta}{D} K Y_T I\{S_T \geq K\} I\{Y_T < D\}. \quad (17)$$

The recovery payoff is given by the equality $\mathcal{R}I(Y_T < D) = R_1 - R_2$. We will use the general risk neutral pricing formula for a general claim $\mathcal{X}(T, Z_T)$:

$$\Pi_t(\mathcal{X}) = e^{\{r(T-t)\}} E^Q\left[\mathcal{X}(T, Z_T) | \mathcal{F}_t\right]$$

where $\Pi_t(\mathcal{X})$ denotes the price of a claim $\mathcal{X}$ at time t.
Then, we apply the technique of change of numeraire. For R_2, I will use the martigale measure which has the Y_t as numeraire. For R_1, the situation is not so straight forward. It seems that the most appropriate numeraire would be S_tY_t. However, this is not the price of a traded asset. Hence, one cannot say that we apply the traditional change of numeraire. Even if S_tY_t is not a traded asset, one can still perform an appropriate change of measure in order to calculate easier $e^{\{r(T-t)\}}E^Q\left[\mathcal{X}(T, Z_T)| \mathcal{F}_t\right]$. More details upon the exact change of measure will follow in the paper.

- We will start calculations with R_2. We apply the risk-neutral pricing formula and a change of numeraire from the bank account to Y_t. We should calculate $Q^Y(S_T \geq K, Y_T < D)$ and we will use the following chain of inequalities:

$$Y_T < D \Leftrightarrow \frac{1}{Y_T} > \frac{1}{D} \Leftrightarrow \frac{p(T,T)}{Y_T} > \frac{1}{D}$$

Under Q^Y, $\frac{p(t,T)}{Y_t}$ is a martingale with dynamics:

$$d\frac{p(t,T)}{Y_t} = -\frac{p(t,T)}{Y_t}\bar{\sigma}dW_t^Y$$

Since p(t,T) is deterministic, we obtain:

$$\frac{p(T,T)}{Y_T} = \frac{p(0,T)}{Y_0}\exp\left(-\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt - \int_0^T \bar{\sigma}_t dW_t^V\right)$$

The exponent is normally distributed with variance $\int_0^T \|\bar{\sigma}_t\|^2 dt$. We have:

$$\begin{aligned}
&Y_T < D \\
&\frac{p(T,T)}{Y_T} > \frac{1}{D} \\
&\log\frac{p(T,T)}{Y_T} > \log\frac{1}{D} \\
&\xi > \underbrace{\frac{\log\frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}}_{b_4}
\end{aligned}$$

where ξ is standard normally distributed.

Now, we turn to the first part of the probability to compute. The dynamics of S_t under Q^Y are given by:

$$dS_t = (\alpha_t + \bar{\gamma}_t\varphi_t^y)S_t dt + S_t\bar{\gamma}_t dW_t^Y$$

The Girsanov kernel is obtained by imposing the martingale condition to the dynamics of the asset on the market, expressed in the new numeraire, Y_t. The assets in case are $\frac{S_t}{Y_t}$ and $\frac{B_t}{Y_t}$ and the derived conditions are:

$$\begin{aligned}
\alpha_t - \mu_t - \sigma_t\gamma_t\rho + 2\sigma_t^2 + [\bar{\gamma}_t - \bar{\sigma}_t]\varphi_t^y &= 0 \\
r - \mu_t + 2\sigma_t^2 - \bar{\sigma}_t\varphi_t^y &= 0.
\end{aligned}$$

Since we have two equations with two unknowns, the Girsanov kernel is completely identified:

$$\varphi_t^y = \left(\frac{r-\alpha_t+\sigma_t\gamma_t\rho}{\gamma_t}, \quad \frac{\gamma_t(r-\mu_t+2\sigma_t^2)-\sigma_t\rho(r-\alpha_t+\sigma_t\gamma_t\rho)}{\sigma_t\gamma_t\sqrt{1-\rho^2}}\right)$$

Then, we proceed to solve the stochastic differential equation above, which yields:

$$S_T = S_0\exp\left(\int_0^T(\alpha_t + \bar{\gamma}_t\varphi_t^y)dt - \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^Y\right) \quad (18)$$

The exponent of S_t is normally distributed with variance $\int_0^T \|\bar{\gamma}_t\|^2 dt$. As before, we need to transform the lognormal variable into a standard normal variable:

$$
\begin{aligned}
&S_T \geq K \\
&S(0)\exp\left(\int_0^T (\alpha_t + \varphi_t^y \bar{\gamma}_t)dt - \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^Y\right) \geq K \\
&\eta \geq \underbrace{\frac{\ln\frac{K}{S(0)} + \int_0^T [r + \bar{\sigma}_t \bar{\gamma}_t']dt + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}}_{a_4}
\end{aligned}
$$

where η is standard normally distributed. Thus,

$$Q^Y(S_T \geq K, Y_T < D) = Q^Y(\eta \geq a_4, \xi > b_4, \rho_2)$$

where η and ξ are standard normal and

$$
\begin{aligned}
a_4 &= \frac{\ln\frac{K}{S_0} + \int_0^T [\alpha_t + \varphi_t^y \bar{\gamma}_t]dt + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_4 &= \frac{\log\frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}
\end{aligned}
$$

By ρ_2, we denote the correlation coefficient between the two standard normal variables. As in the previous subsection, by direct calculation and following the same steps, it is straightforward to show

$$\rho_2 = \frac{\rho\int_0^T \sigma_t \gamma_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} = \rho_1$$

. Using the properties of the bivariate normal distribution derived before, we obtain:

$$Q^Y(S_T \geq K, Y_T < D) = \mathcal{N}(-a_4, -b_4, \rho_1)$$

- Calculation for R_1 are detailed below. We take each part of the calculation separetely for a better exposition. As before, the starting point for the calculations is a change of measure. However, since no trivial numeraire leads to esier computations, we will use a change of measure rather than a change of numeraire.
  Starting from a general expression for the value of R_1 at time t, which we will denote by $\Pi_t[R_1]$:

$$
\begin{aligned}
\Pi_t[R_1] &= E^Q\left[e^{-\int_t^T r_s ds} S_T Y_T I[S_T \geq K, Y_T < D] \middle| \mathcal{F}_t\right] \\
&= E^Q\left[e^{-\int_t^T r_s ds} X_T Z \middle| \mathcal{F}_t\right]
\end{aligned}
$$

where $X_T = S_T Y_T$ and $Z = I[S_T \geq K, Y_T < D]$, we rewrite the above expression as:

$$R_1 = E^Q \left[e^{-\int_t^T r_s ds} m_T R_T Z \middle| \mathcal{F}_t \right] \tag{19}$$

where $m_T = E^Q[X_T]$ and $R_T = \frac{X_T}{E^Q[X_T]}$.

We assume that $Y_T \geq 0$. Since S_T is the price of a traded stock, we have $S_T \geq 0$. Thus, we have $X_T \geq 0$ P-a.s. Also, we note that $E^Q[R_T] = 1$. These facts allow us to use R_T as a Radon-Nycodim derivative in a change of measure and define a measure $\hat{Q}$ by:

$$d\hat{Q} = R_T dQ \text{ on } \mathcal{F}_T \tag{20}$$

Using Bayes' Theorem, we can re-write (19) as:

$$R_1 = e^{-\int_t^T r_s ds} m_T E^Q \left[R_T | \mathcal{F}_t \right] E^{\hat{Q}} \left[Z | \mathcal{F}_t \right] \tag{21}$$

If we define the likelihood process L_t, $0 \leq t \leq T$, by:

$$d\hat{Q} = L_t dQ \text{ on } \mathcal{F}_t$$

we have by standard theory:

$$L_t = E^Q \left[L_T | \mathcal{F}_t \right] = E^Q \left[R_T | \mathcal{F}_t \right] \tag{22}$$

Note that even if $L_T = R_T$, we cannot draw the conclusion $L_t = R_t$ for $t < T$. This is a consequence of the fact that $S_T Y_T$ is not a traded asset. In order to proceed, we need to calculate the following:

- m_T,
- $E^Q \left[R_T | \mathcal{F}_t \right]$,
- the dynamics for L_t in order to indentify the Girsanov transformation $Q \to \hat{Q}$,
- $E^{\hat{Q}} \left[Z | \mathcal{F}_t \right]$

(a) In order to calculate m_T, we need to obtain the dynamics of $S_t Y_t$ under Q:

$$d(S_t Y_t) = S_t Y_t (2r + \gamma_t \sigma_t \rho) dt + S_t Y_t (\bar{\gamma}_t + \bar{\sigma}_t) dW_t \tag{23}$$

Hence, $m_T = S_0 Y_0 \exp \left\{ \int_0^T (2r + \gamma_t \sigma_t \rho) ds \right\}$.

(b) Using (23), we obtain $E^Q \left[X_T | \mathcal{F}_t \right] = S_t Y_t \exp \left\{ \int_t^T (2r + \gamma_t \sigma_t \rho) ds \right\}$, so

$$E^Q \left[R_T | \mathcal{F}_t \right] = \frac{S_t Y_t}{S_0 Y_0} \exp \left[- \int_0^t (2r + \gamma_t \sigma_t \rho) ds \right] \tag{24}$$

(c) Since L_t is a martingale under Q, we assume the dynamics of L_t are of form:

$$dL_t = L_t \varphi_t dW_t \tag{25}$$

From (22),(24) and (25), we obtain:

$$dL_t = L_t(\bar{\gamma}_t + \bar{\sigma}_t)dW_t \tag{26}$$

The Girsanov transformation $Q \to \hat{Q}$ is now identified and we can write

$$dW_t = (\bar{\gamma}_t + \bar{\sigma}_t)'dt + d\hat{W}_t$$

where $\hat{W}$ is $\hat{Q}$-Wiener.

(d) By applying this Girsanov transformation to S_t and Y_t, we obtain the following dynamics under $\hat{Q}$:

$$\begin{aligned} dS_t &= S_t[r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t']dt + S_t\bar{\gamma}_t dW_t \\ dY_t &= Y_t[r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t']dt + Y_t\bar{\sigma}_t dW_t \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} S_T &= S_0 \exp\left[\int_0^T (r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\gamma}_t\|^2)dt + \int_0^T \bar{\gamma}_t d\hat{W}_t\right] \\ Y_T &= Y_0 \exp\left[\int_0^T (r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2)dt + \int_0^T \bar{\sigma}_t d\hat{W}_t\right], \end{aligned}$$

which yields:

$$\begin{aligned} & S_T \geq K \\ & S_0 \exp\left[\int_0^T (r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\gamma}_t\|^2)dt + \int_0^T \bar{\gamma}_t d\hat{W}_t\right] \geq K \\ & \eta \geq \underbrace{\frac{\ln\frac{K}{S_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\gamma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}}_{a_3} \\ & Y_T < D \\ & Y_0 \exp\left[\int_0^T (r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2)dt + \int_0^T \bar{\sigma}_t d\hat{W}_t\right] < D \\ & \xi < \underbrace{\frac{\ln\frac{D}{Y_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}}_{b_3} \end{aligned}$$

Hence, by combining the previous results, we obtain:

$$\begin{aligned}
\Pi_t(R_1) &= e^{-r(T-t)} m_T E^Q\left[R_T|\,\mathcal{F}_t\right] E^{\hat{Q}}\left[Z|\,\mathcal{F}_t\right] \\
&= e^{-r(T-t)} S_0 Y_0 \exp\left[\int_0^T (2r+\gamma_t\sigma_t\rho)ds\right]\frac{S_tY_t}{S_0Y_0} \\
&\quad \exp\left[-\int_0^t (2r+\gamma_t\sigma_t\rho)ds\right]\hat{Q}[S_T \geq K, Y_T < D] \\
&= e^{-r(T-t)} S_t Y_t \exp\left[\int_t^T (2r+\gamma_t\sigma_t\rho)ds\right]\hat{Q}(\eta \geq a_3, \xi > b_3, \rho_1)
\end{aligned}$$

where η and ξ are standard normal, ρ_1 is the correlation coefficient between the two standard normal variables and

$$\begin{aligned}
a_3 &= \frac{\ln\frac{K}{S_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|_2]dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_3 &= \frac{\ln\frac{D}{Y_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}
\end{aligned}$$

We follow the same steps as before, and show by direct calculation that

$$\rho_1 = \frac{\rho\int_0^T \sigma_t\gamma_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}.$$

Using the properties of the bivariate normal distribution, we transform the probability in the above formula for $\Pi_t(R_1)$ in a CDF and have:

$$\Pi_t(R_1) = e^{-r(T-t)} S_t Y_t \exp\left[\int_t^T (2r+\gamma_t\sigma_t\rho)ds\right]\mathcal{N}(-a_3, b_3, \rho_1)$$

At this point, we have obtained all the necessary information in order to price the recovery payoff for the vulnerable option. We are going to collect the last calculations by presenting them in the following proposition:

Proposition 2.3 *Let assumptions 2.1 and 2.2 hold. Then, the fair price for the recovery payoff,Π_2, is:*

$$\begin{aligned}
\Pi_2 &= \frac{1-\beta}{D} e^{-rT} S_0 Y_0 \exp\left[\int_0^T (2r+\gamma\sigma\rho)ds\right]\mathcal{N}(-a_3, b_3, -\rho) \\
&\quad -\frac{1-\beta}{D} K Y_0 \mathcal{N}(-a_4, -b_4, \rho)
\end{aligned}$$

where

$$
\begin{aligned}
a_3 &= \frac{\ln\frac{K}{S_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\gamma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_3 &= \frac{\ln\frac{D}{Y_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\
a_4 &= \frac{\ln\frac{K}{S_0} + \int_0^T [r + \bar{\sigma}_t\bar{\gamma}_t']dt + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_4 &= \frac{\log\frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\
\rho_1 &= \frac{\rho\int_0^T \bar{\sigma}_t\bar{\gamma}_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}
\end{aligned}
$$

2.2.3 Collecting the results

So far, we were concerned with pricing vulnerable options in a complete market setup. In the previous subsections, we have obtained separate expressions for the payoff of a vulnerable option in the case of zero recovery and for the recovery payoff, respectively. For both expressions,we have started from the risk-neutral pricing formula and employed the change of numeraire. Now, we are going to collect the previous results into one formula.

Proposition 2.4 *Let assumptions 2.1 and 2.2 hold. Then, the price for a vulnerable option at time 0,*Π*, is given by:*

$$
\begin{aligned}
\Pi_0 &= S_0\mathcal{N}[a_1, -b_1, -\rho_1] - Kp(0,T)\mathcal{N}[-a_2, -b_2, \rho_1] \\
&\quad + \frac{1-\beta}{D}e^{-rT}S_0Y_0\exp\left\{\int_0^T (2r + \gamma_s\sigma_s\rho)ds\right\}\mathcal{N}[-a_3, b_3, -\rho_1] \\
&\quad - \frac{1-\beta}{D}KY_0\mathcal{N}[-a_4, -b_4, \rho_1]
\end{aligned}
$$

with:

$$
a_1 = \frac{\log\frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}
$$

$$b_1 = \frac{\ln \frac{D}{Y_0} - \int_0^T (r + \bar{\gamma}_t \bar{\sigma}_t')dt + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$a_2 = \frac{\ln \frac{Kp(0,T)}{S_0} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

$$b_2 = \frac{\ln \frac{Dp(0,T)}{Y_0} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$a_3 = \frac{\ln \frac{K}{S_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t' - \frac{1}{2}\|\bar{\gamma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

$$b_3 = \frac{\ln \frac{D}{Y_0} - \int_0^T [r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t' - \frac{1}{2}\|\bar{\sigma}_t\|^2]dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$a_4 = \frac{\ln \frac{K}{S_0} + \int_0^T [r + \bar{\sigma}_t \bar{\gamma}_t']dt + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

$$b_4 = \frac{\log \frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 (t) dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$\rho_1 = \frac{\rho \int_0^T \bar{\sigma}_t \bar{\gamma}_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

2.3 Extensions to other products

2.3.1 Vulnerable exchange options

In the current subsection, I will show how one can easily extend the formula for a European call to price other options. The example used will be that of a vulnerable exchange option. An exchange option is a contract that gives the right, but not the obligation to exchange one stock for another. In this section, we will modify a bit the previous assumptions. We need to have a market consisting of two stock price processes. Also, the payoff function is being modified.
An exchange option has the payoff $\max[S_T^1 - S_T^2, 0]$. In its vulnerable form, the payoff of an exchange option becomes:

$$\mathcal{X} = \Phi(S_T^1, S_T^2, Y_T) = \max[S_T^1 - S_T^2, 0] I\{Y_T \geq D\} + \mathcal{R} I(Y_T < D)$$

where the recovery payoff, $\mathcal{R}$ is given by:

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T^1 - S_T^2, 0]$$

The assumptions needed in order to price a vulnerable exchange option are:

Assumption 2.3

1. *Let $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration given by the P-Wiener process $\tilde{W}$, which is defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dS_t^1 &= \alpha_1 S_t^1 dt + S_t^1 \bar{\gamma}_1 d\tilde{W}_t & (27)\\ dS_t^2 &= \alpha_2 S_t^2 dt + S_t^2 \bar{\gamma}_2 d\tilde{W}_t & (28)\\ dY_t &= \mu Y_t dt + Y_t \bar{\sigma} d\tilde{W}_t & (29)\\ dB_t &= B_t r dt & (30) \end{aligned}$$

   where Y_t is denoting the assets of the counterparty underwriting the option, S_t^1 and S_t^2 the price processes of the stocks on which the option is contracted and B_t the bank account.

3. *In the equations above, μ, α_1 and α_2 be scalars, and $\bar{\sigma}$, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are (1,3) row vectors specified as follows:*

$$\begin{aligned} \bar{\gamma}_1 &= (\,\gamma^1, \quad 0,, \quad 0\,) & (31)\\ \bar{\gamma}_2 &= (\,\gamma^2\rho_{12}, \quad \gamma^2\sqrt{1-\rho_{12}^2}, \quad 0\,) & (32)\\ \bar{\sigma} &= \left(\sigma\rho_{13}, \quad \sigma\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad \sigma\sqrt{1-\rho_{13}^2-\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}\right]^2}\right) & (33) \end{aligned}$$

4. *$\tilde{W}$ is a three dimensional P-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2, \\ \tilde{W}^3 \end{pmatrix} \qquad (34)$$

   with $\tilde{W}^1$, $\tilde{W}^2$ and $\tilde{W}^3$ being independent scalar P-Wiener processes.

5. *Assume that both the assets of the counterparty underwriting the option and the stock are traded on the market.*

Remark 2.1 *Note that, in this section, the model parameters μ, α_1, α_2, σ, γ_1, γ_2 are constants. This is done for notational convenience. In the case of time varying but deterministic coefficients, the calculations are easily extended, but become very messy.*

Since derivations are very similar to the ones in the previous section, we are going to summarize the results in the following proposition. A proof for the results follows.

Proposition 2.5 *Let Assumptions 2.3 hold. Then, the price for a vulnerable option at time zero,*$\Pi(0;\Phi)$*, is given by:*

$$\begin{aligned}\Pi(0;\Phi) &= S_0^1\mathcal{N}(a_1,-b_1,\rho)-S_0^2\mathcal{N}(-a_2,-b_2,\rho)\\ &\quad+\frac{1-\beta}{D}Y_0S_0^1\exp\{T(r+\gamma_1\sigma\rho_{13})ds\}\mathcal{N}(a_3,b_3,\rho)\\ &\quad-\frac{1-\beta}{D}Y_0S_0^2\exp\{T(r+\gamma_2\sigma\rho_{23})\}\mathcal{N}(a_4,b_4,-\rho).\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\log\left(\frac{S_0^1}{S_0^2}\right)+\frac{1}{2}T\|(\bar{\gamma}_2-\bar{\gamma}_1)\|^2}{\sqrt{T\|(\bar{\gamma}_2-\bar{\gamma}_1)\|^2}}\\
b_1 &= \frac{\log(D/Y_0)-T(r+\sigma\gamma_1\rho_{13})+\frac{1}{2}T\|\bar{\sigma}\|^2}{\sqrt{T\|\bar{\sigma}\|^2}}\\
a_2 &= \frac{\log\left(\frac{S_0^2}{S_0^1}\right)+\frac{1}{2}T\|(\bar{\gamma}_1-\bar{\gamma}_2)\|^2}{\sqrt{\frac{1}{2}T\|(\bar{\gamma}_1-\bar{\gamma}_2)\|^2}}\\
b_2 &= \frac{\log(D/Y_0)-T(r+\sigma\gamma_2\rho_{23})+\frac{1}{2}T\|\bar{\sigma}\|^2}{\sqrt{\frac{1}{2}T\|\bar{\sigma}\|^2}}\\
a_3 &= \frac{\log\left(\frac{S_0^1}{S_0^2}\right)-T[(\bar{\gamma}_2-\bar{\gamma}_1)(\bar{\gamma}_2+\bar{\gamma}_1+\bar{\sigma})'-\frac{1}{2}\|\bar{\gamma}_2-\bar{\gamma}_1\|^2]}{\sqrt{T\|\bar{\gamma}_2-\bar{\gamma}_1\|^2}}\\
b_3 &= \frac{\log\left(\frac{D}{Y_0}\right)-T[r+(\bar{\gamma}_1+\bar{\sigma})\bar{\sigma}'-\frac{1}{2}\|\bar{\sigma}\|^2]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
a_4 &= \frac{\log\left(\frac{S_0^2}{S_0^1}\right)-T[(\bar{\gamma}_2-\bar{\gamma}_1)(\bar{\gamma}_2+\bar{\gamma}_1+\bar{\sigma})'-\frac{1}{2}\|\bar{\gamma}_1-\bar{\gamma}_2\|^2]}{\sqrt{T(\bar{\gamma}_1-\bar{\gamma}_2)}}\\
b_4 &= \frac{\log\left(\frac{D}{Y_0}\right)-T[r+(\bar{\gamma}_2+\bar{\sigma})\bar{\sigma}'-\frac{1}{2}\|\bar{\sigma}\|^2]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
\rho &= -\frac{\gamma_2\rho_{23}-\gamma_1\rho_{13}}{\sqrt{(\gamma_1)^2+(\gamma_2)^2-2\gamma_2\gamma_1\rho_{12}}}
\end{aligned}$$

Proof. By a change of numeraire from the risk neutral measure Q to Q^2, the EMM with S_t^2 as numeraire, we obtain

$$\Pi(0;\Phi) = S_0^2 E^2 \left[\max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{Y_T \geq D\right\} + \frac{(1-\beta)Y_T}{D} \max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{Y_T < D\right\}\right]$$

We denote $\frac{S_t^1}{S_t^2}$ by Z_t and fromstandard theory we know that Z is a Q^2-martingale, i.e. it has a zero rate of return.
Our goal is now to apply the formula for the simple European call writen on Z_t, with strike $K = 1$ and local rate of return 0.
However, we cannot apply the previous result directly. In the formula for a vulnerable option, both the underlying stock and the assets of the counterparty have the same rate of return r. Under Q^2, the underlying stock has zero rate of return, but the assets of the counterparty do not. From the standard Girsanov transformation, we obtain the following dynamics for Y_t:

$$dY_t = (r + \gamma_2 \sigma \rho_{23}) Y_t dt + \bar{\sigma} dW_t^2 \tag{35}$$

We denote $r + \gamma_2 \sigma \rho_{23} = c$, and we can re-write Y_t as:

$$Y_t = e^{ct} \tilde{Y}_t$$

where $\tilde{Y}_t$ is defined below.

Definition 2.2 *We define $\tilde{Y}_t$ as the stochastic process given by*

$$\tilde{Y}_t = Y_t \exp\left\{-(r + \gamma_2 \sigma \rho_{23}) T\right\}.$$

The point of this is that $\tilde{Y}_t$ is a Q^2-martingale. Thus the price of the vulnerable exchange option can be written as:

$$\begin{aligned}
\Pi(0;\Phi) &= S_0^2 E^2 \left[\max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{\tilde{Y}_T \geq e^{-cT} D\right\}\right] \\
&+ S_0^2 e^{cT} E^2 \left[\frac{(1-\beta)\tilde{Y}_T}{D} \max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{\tilde{Y}_T < e^{-cT} D\right\}\right]
\end{aligned}$$

Since both Z and $\tilde{Y}$ are Q^2-martingales, we obtain the price of a vulnerable exchange option simply by tranfering the result for the price vulnerable European call option, written on Z_t, with strike 1, local rate of return 0; the assets of the counterparty are given by $\tilde{Y}_t$ and the default barrier becomes $e^{-cT} D$.

The only thing we will need to calculate the correlation coefficient $\tilde{\rho}$ between Z_t and Y_t. A reasoning similar to the one in the proof of lemma (2.1) gives us:

$$\tilde{\rho} = Corr[(\bar{\gamma}_2 - \bar{\gamma}_1) W_T^1, \bar{\sigma} W_T^1]$$

By applying the definition of the correlation coefficient, one obtains:

$$\tilde{\rho} = \frac{\gamma_2 \rho_{23} - \gamma_1 \rho_{13}}{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - 2\gamma_2 \gamma_1 \rho_{12}}} \tag{36}$$

Thus, by transfering results from Proposition 2.4, we obtain the price for the vulnerable exchange option given in Proposition 3.5.

□

2.3.2 Linearly Homogeneous Payoffs

Let Assumption 2.3 hold. In this section, we extend results from section 2.3.1 to a more general class of contracts. More specifically, we now consider a T-claim $X = \Phi(S_T^1, S_T^2)$. In order to do so, we need a homogeneity assumption.

Assumption 2.4 *We assume $\Phi(x, y)$ is a linearly homogenous function, i.e.*

$$\Phi(\lambda x, \lambda y) = \lambda \Phi(x, y), \forall \lambda \geq 0$$

Furthermore, we define the contract function ψ by

$$\psi(z) = \Phi(z, 1) \tag{37}$$

A well known result in mathematical finance relates the non-vulnerable pricing problem of Φ to the simpler problem of pricing ψ. We would like to see if it is possible to find such a relation between vulnerable versions of the contracts defined above.

We denote the vulnerable version of the contract function $\Phi(S_t^1, S_t^2)$ by $\Phi^V(S_t^1, S_t^2, Y_t)$ and the vulnerable version of the contract function $\psi(S_t)$ by $\psi^V(S_t, Y_t)$. In general, the vulnerable version of a contract function $F(x)$, denoted by $F^V(x, y)$ is given by:

$$F^V(x, y) = F(x) I\{y \geq D\} + \frac{(1-\beta)y}{D} F(x) I\{y < D\} \tag{38}$$

By applying the risk neutral valuation formula to the claim $X^V = \Phi^V(S_t^1, S_t^2, Y_t)$, we obtain the following expression for the price of the claim, $\Pi(0, X^V)$:

$$\begin{aligned} \Pi(0, X^V) &= e^{(-rT)} E^Q \left[\Phi^V(S_t^1, S_t^2, Y_t) \right] = S_t^2 E^2 \left[\Phi\left(\frac{S_T^1}{S_T^2}, 1 \right) I\{Y_T \geq D\} \right] \\ &+ S_t^2 E^2 \left[\frac{(1-\beta) Y_T}{D} \Phi\left(\frac{S_T^1}{S_T^2}, 1 \right) I\{Y_T < D\} \right] \end{aligned}$$

where $E^2[\bullet]$ is the expectation operator taken under the equivalent martingale measure Q^2 where S^2 is numeraire.
The present argument follows the same lines as the reasoning outlined in the previous section. We denote $\frac{S_t^1}{S_t^2}$ by Z_t. Under Q^2, Z is a martingale, and has a zero rate of return. In order to obtain a similar calculation formula to the one used in the non-vulnerable claims case, we need Y_t also to be a Q^2-martingale. Since this is not the case, we rewrite Y_t as:

$$Y_t = \tilde{Y}_t e^{ct} \tag{39}$$

where $c = r + \gamma_2\sigma\rho_{23}$ and the process $\tilde{Y}_t$ is defined in Definition 2.2. We remember that $\tilde{Y}_t$ is a martingale under Q^2.
Thus, we can write the price of the claim X^V, as

$$\begin{aligned}\Pi(0, X^V) &= S_t^2 E^2\left[\Phi(Z_T, 1) I\left\{\tilde{Y}_T \geq De^{-cT}\right\}\right] \\ &+ S_t^2 E^2\left[\frac{(1-\beta)\tilde{Y}_T}{De^{-cT}}\Phi(Z_T, 1) I\left\{\tilde{Y}_T < De^{-cT}\right\}\right] \\ &= S_t^2 E^2\left[\psi(Z_T, \tilde{Y}_T)\right]\end{aligned}$$

where the default barrier for the vulnerable claim $\mathcal{X}^V = \psi(Z_T, \tilde{Y}_T)$ is De^{-cT}. The result is summarized in the following proposition. Again, we reduce the problem of pricing a contract written on two assets S^1 and S^2 to the pricing problem of a contract written for a single asset, Z.

Proposition 2.6 *Let Assumptions 2.3 and 2.4 hold. Then, we have the following equivalence between two pricing problems:*

$$\Pi[0, \Phi^V(S_T^1, S_T^2, Y_T)] = S_t^2 \Pi[0, \psi^V(Z_T, \tilde{Y}_T)] \tag{40}$$

where Z_t and Y_t are defined as above. The claim ψ^V is prices in a world of zero local return and the default barrier for the vulnerable claim $\mathcal{X}^V = \psi(Z_T, \tilde{Y}_T)$ is De^{-cT}.

3 Incomplete Markets and Good Deal Bounds

One of the main limitations of the previous approach is the assumption that the assets of the counterparty, or the default "trigger", are liquidly traded on the market. It is a stong assumption, which allows us to obtain a unique price for the vulnerable option. If both the stock and the assets of the counterparty are traded on the market, we have a complete market model and, hence, a unique price.
However, if the assets of the counterparty are not liquidly traded, we are not in a complete market setup, and hence, we are not entitled to use the formula derived in the previous section. One of the consequences of having an incomplete market setup is the fact that we no longer have a unique EMM, and consequently not a unique price. One could simply calculate the bounds of the prices, generated by the interval of all possible risk-neutral measures. These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.
Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterium, chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature dealing with this approach see [12], [6], [1] However, there is no clear cut way of choosing between different criteria and some of

them are somewhat ad-hoc, in the sense that they do not have a clear economic interpretation.
In contrast to this, Cochrane -Saa-Requejo proposed in [4], the methodof good deal bounds. The good deal approach aims at obtaining an interval of "reasonable" prices in incomplete markets, rather than concentrating at obtaining a unique price. Since the no-arbitrage bounds are too large to be used, [4] propose to rule out not only arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered "too good to be true", hence the name of "good deal bounds". One possible measure for the "goodness" of a deal is its Sharpe Ratio, and thus, trades/portfolios which have a Sharpe Ratio (SR) above a certain threshold are eliminated. The SR is chosen as a measure for the "goodness of the deal" because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Thus, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.
The next step in developing a theory for "good deal bounds" was done by [3]. They proposed a new frame for solving the optimization problem defined by [4]while at the same time allowing for more complex dynamics for the underlying assets, such as jumpdiffusion processes, to be taken into account.

3.1 Setup

First, we will consider the classical structural model, dropping only the market completeness assumption. The model is identical to the one presented in the previous section, except for one feature. The assumption that the assets of the counterparty are traded on the market is dropped. We make the following assumptions:

Assumption 3.1

1. *Let* $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ *be given.*

2. *The market model is given by the following dynamics under the objective probability measure* P.

$$
\begin{aligned}
dY_t &= \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t, && (41)\\
dS_t &= \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t, && (42)\\
dB_t &= B_t r dt. && (43)
\end{aligned}
$$

Here Y_t *is denoting the assets of the counterparty underwriting the option,* S_t *the price of the stock on which the option is contracted and* B_t *the bank account.*

3. *μ_t and α_t are scalar deterministic functions of time, $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are positive deterministic functions of time specified as follows:*

$$\bar{\gamma}_t = (\, \gamma_t, \quad 0 \,) \tag{44}$$
$$\bar{\sigma}_t = (\, \sigma_t \rho, \quad \sigma_t \sqrt{1-\rho^2} \,) \tag{45}$$

4. *$\tilde{W}$ is a two dimensional P-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2 \end{pmatrix} \tag{46}$$

with $\tilde{W}^1$ and $\tilde{W}^2$ being independent scalar P-Wiener processes.

5. *We assume that the assets of the counterparty underwriting the option are not traded on the market and that the stock is traded.*

6. *The payoff of a vulnerable European call option, $X = \Phi(S_T, Y_T)$, is given by*

$$X = \Phi(S_T, Y_T) = \max(S_T - K, 0) I(Y_T \geq D) + \mathcal{R} I(Y_T < D)$$

where D is the total value of the claims against the counter-party.

7. *Recovery payoff is given by:*

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T - K, 0]$$

Notice that the above assumptions are identical to assumptions 2.1 and 2.2, with the exception of point 5, which leads to market incompleteness.

3.1.1 Q-dynamics:

Since we are in an incomplete market set-up, we do not have a unique equivalent martingale measure (EMM), but a whole class of EMM. For any potential EMM $Q \sim P$ we define the corresponding likelihood process L by:

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \tag{47}$$

Since $\mathcal{F}_t = \mathcal{F}_t^W$, L_t must have dynamics of the form:

$$dL_t = L_t \varphi_t' d\tilde{W}_t \tag{48}$$
$$L_0 = 1 \tag{49}$$

where $\varphi_t = (\, \varphi_t^1, \quad \varphi_t^2 \,)'$ is adapted to $\underline{\mathcal{F}}$. From Girsanov's theorem, it follows that:

$$d\tilde{W}_t = \varphi_t dt + dW_t$$

where W_t is a Q-Wienner process.
Thus, the dynamics of the two assets under the potential martingale measure Q are:

$$\begin{aligned} dY_t &= (\mu_t + \bar{\sigma}_t \varphi_t) Y_t dt + Y_t \sigma_t dW_t \\ dS_t &= (\alpha_t + \bar{\gamma}_t \varphi_t) S_t dt + S_t \gamma_t dW_t \\ dB_t &= B_t r dt \end{aligned}$$

Since S_t is a traded asset, its drift must equal the risk free interest rate under an equivalent martingale measure. Thus, in order for Q to be a martingale measure, φ has to satisfy the **martingale condition**:

$$r = \alpha_t + \bar{\gamma}_t \varphi_t \tag{50}$$

i.e

$$r = \alpha_t + \gamma_t \varphi_t^1 \tag{51}$$

The martingale condition does not determine a unique Girsanov kernel φ_t, but only the first term of the φ_t. Thus we do not have a unique equivalent martingale measure, but we obtain a class of martingale measures. They are defined as the class of measures obtained by (47)- (49) and satisfying the martingale condition (50).

3.2 Optimization Problem

As mentioned before, the "good deal bound" valuation framework rests on the idea of placing constraints on the Sharpe ratio of the claim to be priced. The problem becomes that of finding the highest and the lowest arbitrage free price processes, subject to a constraint on the maximum Sharpe Ratio (SR). However, if we want to be consistent, we should look for a framework which allows us to place an upper bound on the SR not only of the derivative unde rconsideration, but also of all the **portfolios** that can be formed on the market consisting of the underlying assets, the derivative claim and the money account. It then turns out that binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds.
An extended version of the Hansen Jaganathan bounds is derived and proven in [3]. This inequality provides the bounds for the Sharpe ratio of the assets on the market, as well as for all derivatives and self financing portfolios formed on the market, and reads as follows:

$$|SR_t|^2 \leq \|\lambda_t\|^2. \tag{52}$$

Here we denote by λ_t the market price of risk and by SR_t the Sharpe ratio on a particular asset derivative or self financing portfolio on the market; $\| \bullet \|$ stands for the Euclidian norm. As we can see, the Sharpe ratio is bounded by the norm of the price of risk on the market. Standard theory gives us the relationship between the Girsanov kernel, φ_t, and the market price of risk:

$$\varphi_t = -\lambda_t.$$

Thus, our pricing problem can be reformulated as follows: we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk or equivalently, a bound on the Girsanov kernel φ_t for every t. Dealing with the market price of risk translates to dealing with the Girsanov kernel of the equivalent martingale measures. Following the above reasoning, we can now define the good deal bounds.

Definition 3.1 *The **upper good deal bound** price process for a vulnerable option is defined the optimal value process for the following optimal control problem:*

$$\max_{\varphi} \quad E^Q[e^{-r(T-t)}(\max[S_T - K, 0]I\{Y_T \geq D\} + \mathcal{R}I\{Y_T \leq D\})] \quad (53)$$

$$dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t \quad (54)$$

$$dS_t = rS_t dt + S_t\bar{\gamma}_t dW_t \quad (55)$$

$$\alpha_t + \bar{\gamma}_t\varphi_t = r \quad (56)$$

$$\|\varphi_t\|^2 \leq C^2 \quad (57)$$

*The **lower good deal bound** process is the optimal value process for a similar optimal control problem, with the only difference that we minimize the expression, subject to the same constraints.*
*We denote the optimal value process by $V(t, S_t, Y_t)$, where V is the **optimal value function**.*

Before proceeding, let us comment on the structure of the optimization problem. The objective function is the arbitrage-free price for the payoff function, where the expectation is computed under the risk neutral measure generated by φ. Since we have to select this measure from a continuum of eligible EMM, we maximize with respect to the Girsanov kernel φ.
The optimization is subject to the dynamics of the assets on the market, under the appropiate probability measure.
The constraints:

$$dS_t = rS_t dt + S_t\bar{\gamma}_t dW_t$$
$$\alpha_t + \bar{\gamma}_t\varphi_t = r$$

are the usual constraints on the drift of the traded assets on the market that establish the probability measure as a risk neutral measure.
If all the elements of φ could be identified from these constraints, we would be in a complete market setup and would be able to find a unique price. Since the

number of traded assets is smaller than the number of risk sources, we cannot price all the risk factors and need the last inequality in order to tighten the no arbitrage price bounds. We will refer to this inequality:

$$\|\varphi_t\|^2 \leq C^2, \quad 0 \leq t \leq T$$

as the good deal bounds condition.

3.3 The Hamilton Jacobi Bellman equation

The optimization problem stated above is a standard stochastic optimal control problem and we will solve it with the aid of the Hamilton Jacobi Bellman equation. We restrict ourselves to the case when the market price of risk depends only on the stock and the assets of the counterparty; thus, we have $\varphi_t = \varphi(t, S_t, Y_t)$ According to the general theory of dynamic programing, the optimal value function satisfies the following PDE, also known as the Hamilton Jacobi Bellman equation, where $\mathcal{A}$ denotes the infinitesimal operator for (S, Y).

$$\frac{\partial V}{\partial t}(t, s, y) + \sup_{\varphi} \mathcal{A}V(t, s, y) - rV(t, s, y) = 0 \tag{58}$$

$$V(T, s, y) = \Phi(s, y). \tag{59}$$

Here

$$\Phi(s, y) = \max(s - K, 0)I(y \geq D) + \mathcal{R}(s, y)I(y \leq D) \tag{60}$$

and

$$\mathcal{R}(s, y) = (1 - \beta)\frac{y}{D} \max[s - K, 0] \tag{61}$$

The infinitesimal operator is given by:

$$\begin{aligned}\mathcal{A}V \quad = \quad & \frac{\partial V}{\partial s} rs + \frac{\partial V}{\partial y}(\mu_t + \bar{\sigma}_t\varphi_t)y \\ & + \frac{1}{2}\frac{\partial^2 V}{\partial s^2} s^2 \bar{\gamma}_t \bar{\gamma}_t\prime + \frac{1}{2}\frac{\partial^2 V}{\partial y^2} y^2 \bar{\sigma}_t \bar{\sigma}_t\prime + \frac{\partial^2 V}{\partial s \partial y} sy\bar{\gamma}_t\bar{\sigma}_t\prime\end{aligned}$$

The first step in solving the PDE is to solve for each t, s, y the embedded static maximization problem, corresponding to $sup\mathcal{A}V$ subject to constraints. In our case, for fixed t, s, y, the static problem takes the form:

$$\max_{\varphi} \qquad \frac{\partial V}{\partial y}\sigma\varphi y \tag{62}$$

$$\alpha + \bar{\gamma}\varphi = r \tag{63}$$

$$\|\varphi\| \leq C^2 \tag{64}$$

We notice that the above problem is in fact a linear optimization problem and therefore, the solution will be a boundary solution. Thus, both constraints are binding. Since the Girsanov kernel φ is a (2,1) column vector, by solving the system of equations:

$$\alpha + \bar{\gamma}\varphi = r$$
$$\|\varphi\| = C^2$$

we obtain:

$$\hat{\varphi}(t,s,y)' = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \pm\sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2} \right) \tag{65}$$

Thus, we have two candidates for the optimal φ and it remains to determine which is the optimal one. Since our objective function is linear in φ:

$$\frac{\partial V}{\partial y}\sigma\varphi y$$

and σ and y are positive by assumption, we need to investigate the sign of $\frac{\partial V}{\partial y}$ in order to decide which of the possible Girsanov kernels we choose.

Lemma 3.1 *Under assumptions 3.2 and if φ does not depend on s and y, we have*

$$\frac{\partial V}{\partial y} \geq 0 \tag{66}$$

Proof. We are going to prove that $\frac{\partial V}{\partial y} \geq 0$, or, equivalently, that the value function is increasing in y. We do this by first showing that the payoff function is increasing in y. Then we prove that this implies that the associated pricing function is increasing in y, and hence, the optimal value function is too.

To see that the payoff function $\Phi(s,y)$ is non-decreasing in y, we note that for $y \geq D$,

$$\Phi(s,y) = \max(s-K,0),$$

which does not depend on the value of y, hence, it is non-decreasing in y. For $y < D$, the payoff function is

$$\Phi(s,y) = \mathcal{R}(s,y) < \max(s-K,0)$$

and thus, $\Phi(s,y)$ is non-decreasing as $y = D_-$. Also, the recovery payoff is a linear function of the assets of the counterparty,

$$\mathcal{R}(s,y) = (1-\beta)\frac{y}{D}\max[s-K,0],$$

with the coefficient of y positive. Hence, $\Phi(s,y)$ is non-decreasing in y.
Let $\Pi^Q(t,s,y)$ be a pricing function, i.e.

$$\Pi^Q(t,s,y) = E^Q[e^{-r(T-t)}\Phi[S_T,Y_T]|S_t = s, Y_t = y] \tag{67}$$

where Q is some admisible EMM.
We now want to prove that if the payoff function $\Phi(s,y)$ is increasing in y and the Girsanov kernel is a deterministic function of time

$$\varphi(t,s,y) = \varphi(t),$$

then also the pricing function $\Pi^Q(t,s,y)$ is increasing in the variable y.
We solve the SDE giving the dynamics of Y_t under Q:

$$dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t$$

and obtain the following formula for Y_T, given $Y_t = y$:

$$Y_T = y\exp\left(\int_t^T \left[\mu_t + \bar{\sigma}_t\varphi_t - \frac{1}{2}\|\bar{\sigma}_t\|^2\right] dt + \int_t^T \bar{\sigma}_t dW_t\right)$$

Thus, for a given φ which does not depend on s and y, we can write $Y_T = yZ$, where Z is a lognormal variable that does not depend on y.
One can easily see that if $\Phi(s,y)$ is increasing in the second variable, than also the pricing function $\Pi^Q(t,s,y)$is increasing in the variable y.
In our case, we know that

$$V = \Pi^Q$$

when Q is generated by $\hat{\varphi}$. Since we see from (65) does not depends on s and y, we conclude that $\Pi^Q(t,s,y)$ and thus V is nondecreasing in y. □

In conclusion, the optimal Girsanov kernel is:

$$\hat{\varphi}_t = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right)' \tag{68}$$

Proposition 3.2 *Under assumptions 3.2, the Girsanov kernel corresponding to the upper good deal bound EMM is*

$$\hat{\varphi}'_{max} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right) \tag{69}$$

The Girsanov kernel for the lower good deal bound EMM is given by

$$\hat{\varphi}'_{min} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{B^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right) \tag{70}$$

3.4 A closed form solution

Introducing the result above in the HJB equation, we obtain:

$$\begin{aligned} V_t + V_s rs + V_y(\mu_t + \bar{\sigma}_t\hat{\varphi}_t)y + \frac{1}{2}V_{ss}s^2\bar{\gamma}_t\prime\bar{\gamma}_t & \\ +\frac{1}{2}V_{yy}y^2\bar{\sigma}_t\bar{\sigma}_t + +V_{sy}sy\bar{\gamma}_t\prime\bar{\sigma}_t - rV &= 0 \\ V(T,s,y) &= \Phi(T,s,y) \end{aligned}$$

where $\hat{\varphi}_t \in \{\hat{\varphi}_{min}, \hat{\varphi}_{max}\}$.

By applying Feynman Kac to the above equation, we have obtained the following formula:

$$\Pi(t, s, y) = V(t, s, y) = E^{\hat{Q}}\left[e^{-r(T-t)}\Phi(s, y)\middle| \mathcal{F}_t\right] \tag{71}$$

where is $\hat{Q}$ is defined by the Radon Nykodim derivative:

$$L_t = \frac{d\hat{Q}}{dP} \tag{72}$$

$$dL_t = L_t\hat{\varphi}_t' dW_t \tag{73}$$

where $\hat{\varphi}_t \in \{\hat{\varphi}_{min}, \hat{\varphi}_{max}\}$.

By using the change of numeraire and similar techniques to the ones presented in the section dealing with good-deal bounds, one can obtain a closed form solution for the price of vulnerable options in incomplete markets. We present the result below, followed by the proof.

Proposition 3.3 (Incomplete markets) *Let assumptions 3.2 hold. The upper good deal bound price of a vulnerable option is given by:*

$$\begin{aligned}
\Pi(t) &= S_t\mathcal{N}[-a_1, -b_1, \rho_2] - e^{-r(T-t)}K\mathcal{N}[-a_2, -b_2, \rho_2] \\
&+ \frac{1-\beta}{D}S_tY_t \exp\left\{\int_t^T [\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \bar{\sigma}_s\bar{\gamma}_s']\, ds\right\}\mathcal{N}[-a_3; b_3; -\rho_2] \\
&- e^{-r(T-t)}\frac{K(1-\beta)}{D}Y_t \exp\left\{\int_t^T (\mu_s + \bar{\sigma}_s\hat{\varphi}_s) ds\right\}\mathcal{N}(-a_4, -b_4, \rho_2)
\end{aligned}$$

where

$$a_1 = \frac{\log\frac{K}{S_t} - \int_t^T \left\{r(T-t) + \bar{\gamma}\bar{\gamma}' - \frac{1}{2}\|\bar{\gamma}\|^2\right\} ds}{\sqrt{\int_t^T \|\bar{\gamma}\|^2(s)dst}}$$

$$b_1 = \frac{\log\frac{D}{Y_t} - \int_t^T \left[\mu + \bar{\sigma}\hat{\varphi} + \bar{\sigma}\bar{\gamma}' - \frac{1}{2}\|\bar{\sigma}\|^2(s)\right] ds}{\sqrt{\int_t^T \|\bar{\sigma}\|^2(s)ds}}$$

$$a_2 = \frac{\log\frac{K}{S_t} - r(T-t) + \frac{1}{2}\int_t^T \|\bar{\gamma}\|^2(t)dt}{\sqrt{\int_t^T \|\bar{\gamma}\|^2(s)ds}}$$

$$b_2 = \frac{\log\frac{D}{Y_t} - \int_t^T \left[\mu + \hat{\varphi}\bar{\sigma} - \frac{1}{2}\|\bar{\sigma}\|^2(s)\right] ds}{\sqrt{\int_t^T \|\bar{\sigma}\|^2(s)ds}}$$

$$
\begin{aligned}
a_3 &= \frac{\log\frac{K}{S_t} - \int_t^T \left\{r + (\bar\gamma+\bar\sigma)\bar\gamma' - \frac{1}{2}\|\bar\gamma\|^2\right\} ds}{\sqrt{\int_t^T \|\bar\gamma\|^2 ds}} \\
b_3 &= \frac{\log\frac{D}{Y_t} - \int_t^T \left\{\mu + \hat\varphi\bar\sigma + (\bar\gamma+\bar\sigma)\bar\sigma' - \frac{1}{2}\|\bar\sigma\|^2\right\} ds}{\sqrt{\int_t^T \|\bar\sigma\|^2 ds}} \\
a_4 &= \frac{\log\frac{K}{S_t} - \int_t^T \left[r + \bar\gamma\bar\sigma - \frac{1}{2}\|\bar\gamma\|^2\right] ds}{\sqrt{\bar\gamma\bar\gamma'(T-t)}} \\
b_4 &= \frac{\log\frac{D}{Y_t} - \int_t^T \left[\mu + \bar\sigma\hat\varphi + \bar\sigma\bar\sigma' - \frac{1}{2}\|\bar\sigma\|^2\right] ds}{\sqrt{\bar\sigma\bar\sigma'(T-t)}} \\
\rho_2 &= \frac{\rho\int_0^T \sigma_t\gamma_t\prime dt}{\sqrt{\int_0^T \|\bar\sigma_t\|^2 dt}\sqrt{\int_0^T \|\bar\gamma_t\|^2 dt}} \\
\hat\varphi_t &= \left(-\frac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\right)\prime.
\end{aligned}
$$

The lower good deal bound price is given by a similar pricing formula, with the only exception that

$$
\hat\varphi_t = \left(-\frac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{B^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\right)\prime.
$$

Proof. After a few easy transformations on (71), we obtain the following expression:

$$
\begin{aligned}
\Pi(t,s,y) &= E^Q\left[e^{-r(T-t)}S_T I\{S_T > K, Y_T > D\}\Big|\mathcal{F}_t\right] - e^{-r(T-t)}KQ_t[S_T > K, Y_T > D] \\
&+ \frac{1-\beta}{D}E^Q\left[e^{-r(T-t)}S_T Y_T I\{S_T > K, Y_T < D\}\Big|\mathcal{F}_t\right] \\
&- e^{-r(T-t)}\frac{K(1-\beta)}{D}E^Q\left[Y_T I\{S_T > K, Y_T < D\}\right|\mathcal{F}_t]
\end{aligned}
$$

As before, we need to calculate several expectations and we will start calculating the easiest and moving towards the more complicated ones.

- We start with the expression $Q[S_T > K, Y_T > D]$. From the dynamics of S_t and Y_t under Q, we obtain:

$$
S_T = S_t \exp\left(r(T-t) - \frac{1}{2}\int_t^T \|\bar\gamma_s\|^2 ds + \int_t^T \bar\gamma_s dW_s\right) \quad (74)
$$

$$
Y_T = Y_t \exp\left(\int_t^T \left[\mu + \hat\varphi\bar\sigma_s - \frac{1}{2}\|\bar\sigma_t\|^2\right] dt + \int_t^T \bar\sigma_s dW_s\right) \quad (75)
$$

Through a chain of inequalities similar to the ones performed in the section on complete markets, one obtains:

$$S_T > K \Leftrightarrow \eta > \underbrace{\frac{\log \frac{K}{S_t} - r(T-t) + \frac{1}{2}\int_t^T \|\bar{\gamma}_s\|^2 ds}{\sqrt{\int_t^T \|\bar{\gamma}_s\|^2 ds}}}_{a_2} \tag{76}$$

$$Y_T > D \Leftrightarrow \xi > \underbrace{\frac{\log \frac{D}{Y_t} - \int_t^T \left[\mu_s + \hat{\varphi}_s \bar{\sigma}_s - \frac{1}{2}\|\bar{\sigma}_s\|^2\right] ds}{\sqrt{\int_t^T \|\bar{\sigma}_s\|^2 ds}}}_{b_2} \tag{77}$$

where η and ξ are standard normal with correlation coefficient ρ_2. Summarizing the last computations, we can say:

$$Q_t[S_T > K, Y_T > D] = \mathcal{N}[-a_2, -b_2, \rho_2] \tag{78}$$

with a_2 and b_2 as above. We obtain ρ_2 through similar computations to the ones in the previous section and obtain:

$$\rho_2 = \frac{\rho \int_0^T \sigma_t \gamma_t \prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \tag{79}$$

- The next expectation we are going to calculate is $E^Q\left[e^{-r(T-t)} S_T I\left\{S_T > K, Y_T > D\right\}\middle| \mathcal{F}_t\right]$. We can rewrite the expectation as $S_0 E^{\tilde{Q}}\left[I\left\{S_T > K, Y_T > D\right\}\middle| \mathcal{F}_t\right]$, where $\tilde{Q}$ is defined by:

$$d\tilde{Q} = L_t dQ$$
$$dL_t = \bar{\gamma} dW_t$$

Through a chain of inequalities similar to the ones before, one obtains:

$$S_T > K \Leftrightarrow \eta > \underbrace{\frac{\log \frac{K}{S_t} - \int_t^T \left\{r(T-t) + \bar{\gamma}_s \bar{\gamma}_s' - \frac{1}{2}\|\bar{\gamma}_s\|^2\right\} ds}{\sqrt{\int_t^T \|\bar{\gamma}_s\|^2 ds}}}_{a_1} \tag{80}$$

$$Y_T > D \Leftrightarrow \xi > \underbrace{\frac{\log \frac{D}{Y_t} - \int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \bar{\sigma}_s \bar{\gamma}_s' - \frac{1}{2}\|\bar{\sigma}_s\|^2\right] ds}{\sqrt{\int_t^T \|\bar{\sigma}_s\|^2 ds}}}_{b_1} \tag{81}$$

and

$$E^Q\left[e^{-r(T-t)} S_T I\left\{S_T > K, Y_T > D\right\}\middle| \mathcal{F}_t\right] = S_0 \mathcal{N}[-a_1, -b_1, \rho_2] \tag{82}$$

with a_1 and b_1 as above and ρ_2 the coefficient of correlation between η and ξ, given by the equation (79).

- Now we are going to turn to $E^Q\left[Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\mathcal{F}_t\right]$. In order to calculate this expectation, we are going to use a variant of the change of numeraire technique as follows:

$$E^Q\left[Y_T \underbrace{I\left\{S_T > K, Y_T < D\right\}}_{Z}\middle|\mathcal{F}_t\right] = E^Q\left[\underbrace{E^Q[Y_T]}_{m_T}\underbrace{\frac{Y_T}{E^Q[Y_T]}}_{R_T} Z\middle|\mathcal{F}_t\right]$$

$$= m_T E^Q\left[R_T\middle|\mathcal{F}_t\right] E^{\tilde{Q}}\left[Z\middle|\mathcal{F}_t\right]$$

where $\tilde{Q}$ is the equivalent martingale measure defined by:

$$d\tilde{Q} = L_T dQ \quad (83)$$

$$L_T = R_T \quad (84)$$

$$L_t = E^Q\left[R_T\middle|\mathcal{F}_t\right] \quad (85)$$

By calculating each of the parts of the formula separately, we obtain:

  - $m_T = Y_0 \exp\left\{\int_0^T (\mu_s + \bar{\sigma}_s\hat{\varphi}_s)ds\right\}$
  - $E^Q\left[R_T\middle|\mathcal{F}_t\right] = \frac{Y_t}{Y_0}\exp\left\{-\int_0^t (\mu_s + \bar{\sigma}_s\hat{\varphi}_s)ds\right\}$
  - In order to calculate the last expectation, we need the dynamics of L_t, given by $dL_t = L_t\bar{\sigma}_t dW_t$ from (85).
  - Before obtaining a formula for $E^{\tilde{Q}}\left[Z\middle|\mathcal{F}_t\right]$, we must have the dynamics of S_t and Y_t under $\tilde{Q}$. By applying Girsanov's transformation, we have:

$$dS_t = (r + \bar{\gamma}_t\bar{\sigma}_t')S_t dt + S_t\bar{\gamma}_t dW_t \quad (86)$$

$$dY_t = (\mu_t + \bar{\sigma}_t\hat{\varphi}_t + \bar{\sigma}_t\bar{\sigma}_t')Y_t dt + Y_t\bar{\sigma}_t dW_t \quad (87)$$

    and from the formula for the geometric brownian motion and a chain of inequalities similar to the ones in the previous section, we obtain:

$$S_T \geq K \Leftrightarrow \eta \geq \underbrace{\frac{\log\frac{K}{S_t} - \int_t^T\left[r + \bar{\gamma}_s\bar{\sigma}_s' - \frac{1}{2}\|\bar{\gamma}_s\|^2\right]ds}{\sqrt{\int_t^T\|\bar{\gamma}_s\|^2 ds}}}_{a_4} \quad (88)$$

$$Y_T \leq D \Leftrightarrow \xi \geq \underbrace{\frac{\log\frac{D}{Y_t} - \int_t^T\left[\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \bar{\sigma}_s\bar{\sigma}_s' - \frac{1}{2}\|\bar{\sigma}_s\|^2\right]ds}{\sqrt{\int_t^T\|\bar{\sigma}_s\|^2 ds}}}_{b_4} \quad (89)$$

By summing up the last calculations, we obtain the following equality:

$$E^Q\left[Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\mathcal{F}_t\right] = Y_t \exp\left\{(\mu + \sigma\hat{\varphi})(T-t)\right\}\mathcal{N}(-a_4, -b_4, \rho_2) \quad (90)$$

where a_4 and b_4 are defined as above and ρ_2 is the correlation coefficient between ξ and η, given by equation (79).

- The last expectation to be calculated is

$$E^Q\left[e^{-r(T-t)}\underbrace{S_TY_T}_{X_T}\underbrace{I\left\{S_T>K, Y_T<D\right\}}_{Z}\Bigg|\mathcal{F}_t\right]$$

  and we will apply the same technique as above. We denote $E^Q[X_T]$ by m_T and $\frac{X_T}{E^Q[X_T]}$ by R_T and rewrite the expectation to be calculated as $e^{-r(T-t)}m_TE^Q\left[R_T|\,\mathcal{F}_t\right]E^{\bar{Q}}\left[Z|\,\mathcal{F}_t\right]$, where $\bar{Q}$ is defined by

$$d\bar{Q} = L_TdQ \quad (91)$$
$$L_T = R_T \quad (92)$$
$$L_t = E^Q\left[R_T|\,\mathcal{F}_t\right] \quad (93)$$

  Before proceeding, we need to calculate the dynamics of S_tY_t under Q. An easy application of Ito's lemma yields:

$$d(S_tY_t) = S_tY_t\left[r+\mu_t+\bar{\sigma}_t\hat{\varphi}_t+\bar{\sigma}_t\bar{\gamma}_t'\right]dt+S_tY_t(\bar{\sigma}_t+\bar{\gamma}_t)dW_t \quad (94)$$

  From (94), we obtain:

  - $m_T = S_0Y_0\exp\left\{\int_0^T\left[r+\mu_s+\bar{\sigma}_s\hat{\varphi}_s+\bar{\sigma}_s\bar{\gamma}_s'\right]ds\right\}$
  - $E^Q\left[R_T|\,\mathcal{F}_t\right] = \frac{S_tY_t}{S_0Y_0}\exp\left\{-\int_0^t\left[r+\mu_s+\bar{\sigma}_s\hat{\varphi}_s+\bar{\sigma}_s\bar{\gamma}_s'\right]ds\right\}$
  - the Girsanov kernel corresponding to the dynamics of L_t, given by $\tilde{\varphi}_t = (\bar{\sigma}_t+\bar{\gamma}_t)'$
  - $E^{\bar{Q}}\left[Z|\,\mathcal{F}_t\right]$, from the dynamics of S_t and Y_t under $\bar{Q}$. According to the definition of $\bar{Q}$ and Girsanov's transformation, the dynamics of S_t and Y_t are given by:

$$dS_t = S_t[r+(\bar{\gamma}_t+\bar{\sigma}_t)\bar{\gamma}_t']dt+S_t\bar{\gamma}_tdW_t \quad (95)$$
$$dY_t = Y_t[\mu_t+\hat{\varphi}_t\bar{\sigma}_t+(\bar{\gamma}_t+\bar{\sigma}_t)\bar{\sigma}_t']dt+Y_t\bar{\sigma}_tdW_t \quad (96)$$

    By solving the above stochastic differential equations for S_T and, respectively, for Y_T, we obtain the following inequalities:

$$S_T\geq K \Leftrightarrow \eta \geq \underbrace{\frac{\log\frac{K}{S_t}-\int_t^T\left\{r+(\bar{\gamma}_s+\bar{\sigma}_s)\bar{\gamma}_s'-\frac{1}{2}\|\bar{\gamma}_s\|^2\right\}ds}{\sqrt{\int_t^T\|\bar{\gamma}_s\|^2ds}}}_{a_3} \quad (97)$$

$$Y_T< D \Leftrightarrow \xi < \underbrace{\frac{\log\frac{D}{Y_t}-\int_t^T\left\{\mu_s+\hat{\varphi}_s\bar{\sigma}_s+(\bar{\gamma}_s+\bar{\sigma}_s)\bar{\sigma}_s'-\frac{1}{2}\|\bar{\sigma}_s\|^2\right\}ds}{\sqrt{\int_t^T\|\bar{\sigma}_s\|^2ds}}}_{b_3} \quad (98)$$

where η and ξ are standard normal. The correlation coefficient between the two standard variables is denoted by ρ_2 and it is given by (79).
We can write the initial expectation

$$A = E^Q\left[e^{-r(T-t)}S_T Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\mathcal{F}_t\right]$$

as:

$$A = S_t Y_t \exp\left\{\int_t^T \left[\mu_s + \bar{\sigma}_s\varphi_s + \bar{\sigma}_s\bar{\gamma}_s'\right]dt\right\}\mathcal{N}[-a_3; b_3; -\rho_2] \quad (99)$$

By summing up the calculations from (78), (82), (90) and (99), we obtain the closed form solution from proposition (3.3) □

3.5 The case of a non-traded stock

The above technique can be easily applied to the case of a vulnerable option when the underlying is not traded. Assumptions 3.2 still hold, except for 5. Now, we have:

Assumption 3.2 *We assume that the assets of the counterparty underwriting the option and the underlying stock are not traded on the market.*

The **upper good deal bound** price process for a vulnerable option becomes the optimal value process for the following optimal control problem:

$$\max_{\varphi} \quad E^Q[e^{-r(T-t)}(\max[S_T - K, 0]I\{Y_T \geq D\} + \mathcal{R}I\{Y_T \leq D\})] \quad (100)$$

$$dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t \quad (101)$$

$$dS_t = (\alpha_t + \bar{\gamma}_t\varphi_t)S_t dt + S_t\bar{\gamma}_t dW_t \quad (102)$$

$$\|\varphi_t\|^2 \leq C^2 \quad (103)$$

The difference between this incomplete market problem and the previous one being summed up by the constraint (102) - we no longer have a martingale condition to determine the stock dynamics under the risk neutral measure.
The HJB equation becomes:

$$\frac{\partial V}{\partial t}(t, s, y) + \sup_{\varphi}\mathcal{A}V(t, s, y) - rV(t, s, y) = 0 \quad (104)$$

$$V(T, s, y) = \Phi(s, y). \quad (105)$$

Here

$$\Phi(s, y) = \max(s - K, 0)I(y \geq D) + \mathcal{R}(s, y)I(y \leq D) \quad (106)$$

and

$$\mathcal{R}(s, y) = (1 - \beta)\frac{y}{D}\max[s - K, 0] \quad (107)$$

The infinitesimal operator is given by:

$$\begin{aligned}\mathcal{A}V &= \frac{\partial V}{\partial s}(\alpha_t + \bar{\gamma}_t\varphi_t)s + \frac{\partial V}{\partial y}(\mu_t + \bar{\sigma}_t\varphi_t)y \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2}s^2\bar{\gamma}_t\bar{\gamma}_t\prime + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}y^2\bar{\sigma}_t\bar{\sigma}_t\prime + \frac{\partial^2 V}{\partial s\partial y}sy\bar{\gamma}_t\bar{\sigma}_t\prime\end{aligned}$$

For each t, s, y, the embedded static maximization problem becomes:

$$\max_{\varphi} \quad \left[\frac{\partial V}{\partial y}y\bar{\sigma} + \frac{\partial V}{\partial s}s\bar{\gamma}\right]\varphi \tag{108}$$

$$\|\varphi\| \leq C^2. \tag{109}$$

The solution to the above maximization problem is given by:

$$\hat{\varphi} = \frac{\frac{\partial V}{\partial y}y\bar{\sigma} + \frac{\partial V}{\partial s}s\bar{\gamma}}{\|\frac{\partial V}{\partial y}y\bar{\sigma} + \frac{\partial V}{\partial s}s\bar{\gamma}\|}C \tag{110}$$

Since the optimal Girsanov kernel depends on V and on the norm of the derivatives of V, we are no longer in the position of obtaining a closed form solution for the HJB equation. However, we can solve the PDE numerically.

3.6 Extension to other products

One of the advantages of the GDB is that one can transfer results for the European calls to other simple vanilla products on a manner similar to the one used in the complete market case. Below, we exemplify how to do this for the case of an exchange option. The assumptions needed are similar to the ones stated for the complete market case, except that the assets of the counter-party underwriting the option are not traded.

Assumption 3.3

1. *Let $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration given by the P-Wiener process $\tilde{W}$, which is defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$dS_t^1 = \alpha_1 S_t^1 dt + S_t^1\bar{\gamma}_1 d\tilde{W}_t \tag{111}$$

$$dS_t^2 = \alpha_2 S_t^2 dt + S_t^2\bar{\gamma}_2 d\tilde{W}_t \tag{112}$$

$$dY_t = \mu Y_t dt + Y_t\bar{\sigma} d\tilde{W}_t \tag{113}$$

$$dB_t = B_t r dt \tag{114}$$

where Y_t is denoting the assets of the counterparty underwriting the option, S_t^1 and S_t^2 the price processes of the stocks on which the option is contracted and B_t the bank account.

3. *Let μ, α_1 and α_2 be scalars, $\bar{\sigma}$, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be (1,3) row vectors specified as follows:*

$$\bar{\gamma}_1 = (\,\gamma_1, \quad 0,, \quad 0\,) \tag{115}$$

$$\bar{\gamma}_2 = (\,\gamma_2\rho_{12}, \quad \gamma_2\sqrt{1-\rho_{12}^2}, \quad 0\,) \tag{116}$$

$$\bar{\sigma} = \left(\sigma\rho_{13}, \quad \sigma\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad \sigma\sqrt{1-\rho_{13}^2-\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}\right]^2}\right) \tag{117}$$

4. *Let $\tilde{W}$ be a three dimensional P-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2, \\ \tilde{W}^3 \end{pmatrix} \tag{118}$$

with $\tilde{W}^1$, $\tilde{W}^2$ and $\tilde{W}^3$ being independent scalar P-Wiener processes.

5. *Assume that the two stocks are traded on the market, but the assets of the counterparty underwriting the option are not liquidly traded.*

Remark 3.1 *Note that, in this section, the model parameters μ, α_1, α_2, σ, γ_1, γ_2 are constants. This is done for notational convenience. In the case of time varying coefficients, calculations are easily extended, but become very messy.*

Before we continue, we remember that an exchange option has the payoff $\max[S_T^1 - S_T^2, 0]$. In its vulnerable form, the payoff of an exchange option becomes:

$$\mathcal{X} = \Phi(S_T^1, S_T^2, Y_T, T) = \max[S_T^1 - S_T^2, 0] I\{Y_T \geq D\} + \mathcal{R} I(Y_T < D)$$

where the recovery payoff, $\mathcal{R}$ is given by:

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T^1 - S_T^2, 0]$$

Definition 3.2 *The* ***upper good deal bound*** *price process for a vulnerable exchange option is defined as the optimal value process for the following optimal control problem:*

$$\max_{\varphi} \quad E^Q[e^{-r(T-t)}\mathcal{X}] \tag{119}$$

$$dY_t = (\mu + \bar{\sigma}\varphi_t)Y_t dt + Y_t\bar{\sigma}dW_t \tag{120}$$

$$dS_t^1 = rS_t^1 dt + S_t^1\bar{\gamma}_1 dW_t \tag{121}$$

$$dS_t^2 = rS_t^2 dt + S_t^2\bar{\gamma}_2 dW_t \tag{122}$$

$$\alpha_1 + \bar{\gamma}_1\varphi_t = r \tag{123}$$

$$\alpha_2 + \bar{\gamma}_2\varphi_t = r \tag{124}$$

$$\|\varphi_t\|^2 \leq C^2. \tag{125}$$

The ***lower good deal bound*** *is the optimal value process for a similar optimal control problem, except that we minimize instead of maximizing subject to the same constraints.*

Our aim is to obtain an equivalent good deal bounds problem expressed under Q^2, the measure where S^2 is numeraire. We are going to show how to obtain this equivalent good deal bounds problem which allows a direct transfer from the pricing problem of a vulnerable exchange option to the pricing problem of a simple vulnerable European call, which is more simple.
We will do this by obtaining equivalent expressions to the objective function and the constraints, under the new measure Q^2 and involving Girsanov kernel corresponding to the change of measure $P \to Q^2$, denoted by ψ.

We are going to present how we have obtained the equivalent problem:

- We apply a standard change of measure to the **objective function** of the upper good deal bound problem and we obtain:

$$E^Q[e^{-rT}\mathcal{X}] = S_0^2 E^2[\mathcal{Z}] \tag{126}$$

  where

$$\mathcal{Z} = \max\left[Z_T - 1, 0\right] I\left\{Y_T \geq D\right\} + \mathcal{R}(Z_T) I\left\{Y_T < D\right\}$$

  and $Z_T = \frac{S_T^1}{S_T^2}$. We have

$$\mathcal{R}(Z_T) = (1-\beta)\frac{Y_T}{D}\max[Z_T - 1, 0]$$

  and $E^2(\bullet)$ denotes the expectations operator under Q_2.

  We denote by ψ the Girsanov kernel corresponding to the change of measure $P \to Q^2$.

- Since our objective function is under Q^2, we would like to have also the **dynamics** of the assets under the same measure. From the dynamics of S^1 and S^2 given by equations (121) and (122) , we can derive the following dynamics under Q^2. We obtain the the dynamics of Y_t from the standard Girsanov transformation. The dynamics for $\frac{S_t^1}{S_t^2}$ is a Q^2-martingale, according to the definition of Q^2.

$$dY_t = (\mu + \bar{\sigma}\psi_t)Y_t dt + Y_t \bar{\sigma} dW_t^2 \tag{127}$$

$$d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2) dW_t^2 \tag{128}$$

  where W_t^2 is Q^2-Wiener.

- The next step is deriving the **martingale conditions** corresponding to Q^2. They are obtained in the following way: we calculate the dynamics of

$\frac{S_t^1}{S_t^2}$ and $\frac{B_t}{S_t^2}$ under the P-measure; we perform a Girsanov transformation $P \to Q^2$. We know that both $\frac{S_t^1}{S_t^2}$ and $\frac{B_t}{S_t^2}$ are martingales under Q^2 and hence we impose the drift of the two processes to be zero. We obtain:

$$r - \alpha_2 = \bar{\gamma}_2 \psi_t - \bar{\gamma}_2 \bar{\gamma}_2'$$
$$\alpha_1 - \alpha_2 = \bar{\gamma}_1 \bar{\gamma}_2' - \bar{\gamma}_2 \bar{\gamma}_2' - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t$$

- The next step in our equivalence problem is to take the good deal bound condition for the transformation $P \to Q$:

$$\|\varphi_t\|^2 \leq C^2$$

and find an equivalent good deal bound condition for the transformation $P \to Q^2$.

We define the following transformations:

– $P \to Q$, defined by the following relationship:

$$\begin{aligned} L &= \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \\ dL &= L\varphi' d\tilde{W} \end{aligned}$$

– $P \to Q^2$, defined by the following relationship:

$$\begin{aligned} L^2 &= \frac{dQ^2}{dP} \quad \text{on } \mathcal{F}_T \\ dL^2 &= L^2\psi' d\tilde{W} \end{aligned}$$

– $Q \to Q^2$, defined by the following relationship:

$$\begin{aligned} L^{1,2} &= \frac{dQ^2}{dP} \quad \text{on } \mathcal{F}_T \\ dL^{1,2} &= L^{1,2}\bar{\gamma}_2 dW \end{aligned}$$

We notice that

$$\frac{\frac{dQ^2}{dP}}{\frac{dQ}{dP}} = \frac{dQ^2}{dQ}.$$

The above equation together with the dynamics of the three Radon-Nikodym derivatives yield the following relation between φ and ψ:

$$\varphi = \psi - \bar{\gamma}_2' \tag{129}$$

Thus, the good deal bounds contraint becomes:

$$\|\psi - \bar{\gamma}_2'\|^2 \leq C^2 \tag{130}$$

Hence, the problem below is equivalent to the original upper good deal bound problem:

$$\max_{\psi} \quad S_t^2 E^2[\mathcal{Z}] \tag{131}$$

$$dY_t = (\mu + \bar{\sigma}\psi)Y_t dt + Y_t \bar{\sigma} dW_t^2 \tag{132}$$

$$d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2)dW_t^2 \tag{133}$$

$$r - \alpha_2 = \bar{\gamma}_2\psi_t - \bar{\gamma}_2\bar{\gamma}_2' \tag{134}$$

$$\alpha_1 - \alpha_2 = \bar{\gamma}_1\bar{\gamma}_2' - \bar{\gamma}_2\bar{\gamma}_2' - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t \tag{135}$$

$$\|\psi - \bar{\gamma}_2'\|^2 \leq C^2 \tag{136}$$

Thus, we have reduced the problem of pricing a vulnerable claim written on 2 assets to the problem of pricing a vulnerable claim written on one asset.

Proposition 3.4 *The upper good deal bound price process defined in 3.2 is also the optimal value process for the optimal control problem given below:*

$$\max_{\psi} \quad S_t^2 E^2[\mathcal{Z}] \tag{137}$$

$$dY_t = (\mu + 2\bar{\sigma}\psi - \bar{\sigma}\bar{\gamma}_2)Y_t dt + Y_t \bar{\sigma} dW_t^2 \tag{138}$$

$$d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2)dW_t^2 \tag{139}$$

$$r - \alpha_2 = \bar{\gamma}_2\psi_t - \bar{\gamma}_2\bar{\gamma}_2' \tag{140}$$

$$\alpha_1 - \alpha_2 = \bar{\gamma}_1\bar{\gamma}_2' - \bar{\gamma}_2\bar{\gamma}_2' - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t \tag{141}$$

$$\|\psi - \bar{\gamma}_2'\|^2 \leq C^2 \tag{142}$$

The lower good deal bound is the optimal value process for a similar optimal control problem, where we minimize subject to the same constraints as above.

By a reasoning very similar to the one in the previous section, we calculate the upper good deal bound Girsanov kernel, $\psi_u = (\psi_u^1, \psi_u^2, \psi_u^3)'$, as given below:

$$\psi_u = \begin{pmatrix} \frac{r-\alpha_1}{\gamma_1} + \gamma_2\rho_{12}, \\ \frac{1}{\gamma_2\sqrt{1-\rho_{12}}}\left[r - \alpha_2 - \gamma_2\rho_{12}\frac{r-\alpha_1}{\gamma_1}\right] + \gamma_2\sqrt{1-\rho_{12}}, \\ \sqrt{C^2 - (\psi^1)^2 - (\psi^2)2^{+}(\gamma_2)^2} \end{pmatrix} \tag{143}$$

The lower good deal bound Girsanov kernel is given by:

$$\psi_l = \begin{pmatrix} \frac{r-\alpha_1}{\gamma_1} + \gamma_2\rho_{12}, \\ \frac{1}{\gamma_2\sqrt{1-\rho_{12}}}\left[r - \alpha_2 - \gamma_2\rho_{12}\frac{r-\alpha_1}{\gamma_1}\right] + \gamma_2\sqrt{1-\rho_{12}}, \\ -\sqrt{C^2 - (\psi^1)^2 - (\psi^2)2^{+}(\gamma_2)^2} \end{pmatrix} \tag{144}$$

Thus, we can deduce the following proposition concerning the upper and lower GDB prices. This is done by applying the formula for a European call written on $Z_t = \frac{S_t^1}{S_t^2}$ with strike 1 and local rate of return 0 for the process Z_t.

Proposition 3.5 *Let assumptions 3.3 hold. Then, the upper good deal bound price for a vulnerable option at time zero,* $\Pi(S_1, S_2, Y)$*, is given by:*

$$\begin{aligned}\Pi(S_1, S_2, Y) &= S_0^1\mathcal{N}(a_1, -b_1, \rho) - S_0^2\mathcal{N}(-a_2, -b_2, \rho)\\ &+\frac{1-\beta}{D}Y_0S_0^1\exp\left\{T\left[\mu+\bar{\sigma}\hat{\psi}_u+\bar{\sigma}\bar{\gamma}_{1\prime}\right]\right\}\mathcal{N}(a_3, b_3, \rho)\\ &-\frac{1-\beta}{D}Y_0S_0^2\exp\left\{T(\mu+\bar{\sigma}\hat{\psi}_u+\bar{\sigma}\bar{\gamma}_{2\prime})\right\}\mathcal{N}(a_4, b_4, -\rho).\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\log\left(\frac{S_0^1}{S_0^2}\right)+\frac{1}{2}T\|(\bar{\gamma}_2-\bar{\gamma}_1)\|^2}{\sqrt{T\|(\bar{\gamma}_2-\bar{\gamma}_1)\|^2}}\\
b_1 &= \frac{\log\frac{D}{Y_0}-T\left[\mu+\bar{\sigma}\hat{\psi}_u+\bar{\sigma}(\bar{\gamma}_1-\bar{\gamma}_2)'-\frac{1}{2}\|\bar{\sigma}\|^2\right]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
a_2 &= \frac{\log\left(\frac{S_0^2}{S_0^1}\right)+\frac{1}{2}T\|(\bar{\gamma}_1-\bar{\gamma}_2)\|^2}{\sqrt{\frac{1}{2}T\|(\bar{\gamma}_1-\bar{\gamma}_2)\|^2}}\\
b_2 &= \frac{\log\frac{D}{Y_0}-T\left[\mu+\hat{\psi}_u\bar{\sigma}-\frac{1}{2}\|\bar{\sigma}\|^2\right]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
a_3 &= \frac{\log\left(\frac{S_0^1}{S_0^2}\right)-T[(\bar{\gamma}_2-\bar{\gamma}_1)(\bar{\gamma}_2+\bar{\gamma}_1+\bar{\sigma})'-\frac{1}{2}\|\bar{\gamma}_2-\bar{\gamma}_1\|^2]}{\sqrt{T\|\bar{\gamma}_2-\bar{\gamma}_1\|^2}}\\
b_3 &= \frac{\log\left(\frac{D}{Y_0}\right)-T[\mu+\hat{\psi}\bar{\sigma}+(\bar{\gamma}_1-\bar{\gamma}_2+\bar{\sigma})\bar{\sigma}'-\frac{1}{2}\|\bar{\sigma}\|^2]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
a_4 &= \frac{\log\left(\frac{S_0^2}{S_0^1}\right)-T[(\bar{\gamma}_2-\bar{\gamma}_1)(\bar{\gamma}_2+\bar{\gamma}_1+\bar{\sigma})'-\frac{1}{2}\|\bar{\gamma}_1-\bar{\gamma}_2\|^2]}{\sqrt{T\|\bar{\gamma}_1-\bar{\gamma}_2\|}}\\
b_4 &= \frac{\log\left(\frac{D}{Y_0}\right)-T\left[\mu+\bar{\sigma}\hat{\psi}_u+\bar{\sigma}\bar{\sigma}'-\frac{1}{2}\|\bar{\sigma}\|^2\right]}{\sqrt{T\|\bar{\sigma}\|^2}}\\
\rho &= -\frac{\gamma_2\rho_{23}-\gamma_1\rho_{13}}{\sqrt{(\gamma_1)^2+(\gamma_2)^2-2\gamma_2\gamma_1\rho_{12}}}
\end{aligned}$$

In the above formula $\hat{\psi}_u$ is given by (143) The lower good deal bound price is given by the same general formula, but replacing $\hat{\psi}_u$ by the lower bound Girsanov kernel $\hat{\psi}_l$ given by (144)

4 Conclusion

In the current paper, we apply good deal bounds in order to price European vulnerable options. The default is modeled in a structural set up. We extend closed form results for European options to options with linearly homogeneous payoffs. As topic for research in progress we mention the pricing of vulnerable options in an intensity based setup.

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