Risk indifference option pricing in jump diffusion markets

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Pricing issue

Contingent claim with maturity T and terminal payoff G.

- Complete market: $\exists!$ arbitrage-free way to value an option: the value is defined as the cost of replicating it.
- Incomplete market (jumps, stochastic volatility or trading constraints):
  No unique equivalent martingale measure.

  Different ways to measure risk lead to different approaches to pricing and hedging.

Superreplication

Financial market: Bond with discounted price $= 1$ and Stock with discounted price S.
portfolio: π
wealth process: $X_x^{(\pi)}(t) = x + \int_0^t \pi(s) dS_s$
The upper and lower superreplication prices are defined via

$$p^{up}(G) = \inf\{x : \exists \pi,\ X_x^{(\pi)}(T) \geq G\}$$
$$p^{low}(G) = \sup\{x : \exists \pi,\ X_x^{(\pi)}(T) \geq -G\}$$

One can show that

$$p^{up}(G) = \sup_{\mathbb{Q} \in \mathcal{M}_1} E_Q[\exp(-\int_0^T r(u)du)G]$$
$$p^{low}(G) = \inf_{\mathbb{Q} \in \mathcal{M}_1} E_Q[\exp(-\int_0^T r(u)du)G]$$

where $\mathcal{M}_1$ denotes the set of EMM.
In most examples with jumps this approach is too conservative and these bounds are too wide.

Utility indifference pricing

Hodges and Neuberger (1989) introduced preferences via a utility fn: $U : [0, \infty) \to [-\infty, \infty)$ increasing, concave, $\mathcal{C}^1$.

- Utility maximization without option liability

$$V_0(x) = \max_{\pi \in \mathcal{P}} E[U(x + \int_0^T \pi_t dS_t)]$$

- Utility maximization in presence of liability

$$V_G(x) = \max_{\pi \in \mathcal{P}} E[U(x + \int_0^T \pi_t dS_t - G)]$$

Seller's indifference price p satisfies:

$$V_0(x) = V_G(x + p).$$

Risk indifference pricing

Similar approach based on *risk* rather than utility.
Let $\rho : \mathbb{F} \mapsto \mathbb{R}$ be a convex risk measure where $\mathbb{F}$ is the set of $\mathcal{F}_T$-measurable RV.

- Minimal risk without option liability

$$\Phi_0(x) = \inf_{\pi \in \mathcal{P}} \rho(X_x^{(\pi)}(T))$$

- Minimal risk in presence of liability

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{P}} \rho(X_{x+p}^{(\pi)}(T) - G)$$

risk indifference price p_{risk}: amount for which the investor is indifferent between issuing a claim or not wrt to the risk of his terminal position:

$$\Phi_0(x) = \Phi_G(x + p_{risk})$$

Convex risk measures

A convex risk measure: $\rho : \mathbb{F} \mapsto \mathbb{R}$ satisfies some axioms

- Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\forall \alpha \in [0,1], \forall X, Y$
- Positivity: $X \geq 0$ p.s. $\Rightarrow \rho(X) \leq \rho(0) = 0$
- Translation: $\rho(X + a) = \rho(X) - a, \forall a \in \mathbb{R}, \forall X$
- LSC: $\{X \in \mathbb{F}; \rho(X) \leq a\}$ closed in $\mathbb{F}$ for any $a \in \mathbb{R}$.

Delbaen, Fritelli, Gianin, Fölmer Schied

Representation of convex risk measures

A map $\rho : \mathbb{F} \rightarrow \mathbb{R}$ is a convex risk measure if and only if there exists a family $\mathcal{L}$ of measures Q on $\mathcal{F}_T$ such that

$$Q \ll P$$

and a convex "penalty function" $\zeta : \mathcal{L} \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\inf_{Q \in \mathcal{L}} \zeta(Q) = 0$ such that

$$\rho(F) = \sup_{Q \in \mathcal{L}} \{E_Q[-F] - \zeta(Q)\}; \qquad F \in \mathbb{F}.$$

Choosing a risk measure is equivalent of choosing the family $\mathcal{L}$ of measures and the penalty fn ζ. If $\zeta = 0$, then ρ is a *coherent* risk measure (Arzner, Delbaen, Eber, Heath).

Risk indifference price p of a contingent claim with terminal payoff G amounts then to solve 2 stochastic differential games.

$$\Phi_0(x) = \inf_{\pi \in \mathcal{P}} \left(\inf_{Q \in \mathcal{L}} \{ E_Q[-X_x^{(\pi)}(T)] - \zeta(Q) \} \right)$$

and

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{P}} \sup_{Q \in \mathcal{L}} \{ E_Q[-X_{x+p}^{(\pi)}(T) + G] - \zeta(Q) \})$$

for a given family of measures $\mathcal{L}$ and a given penalty fn ζ.

Jump model setup

Financial market:

- riskless asset with discounted price constant at 1
- a risky asset with discounted price S:

$$dS(t) = S(t^-)\Big[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)\Big]; \; S(0) > 0;$$

where $B(t)$ is a Brownian motion and $\tilde{N}(dt,dz)$ is a compensated Poisson random measure on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, P)$ with intensity Lévy measure ν. We assume

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$$

$\beta(t)$ and $\gamma(t,z)$ are given $\mathbb{F}_t$-predictable processes, such that

$$\int_0^T \Big\{|\alpha(s)| + \beta^2(t) + \int_{\mathbb{R}_0} |\ln(1+\gamma(t,z))^2| \nu(dz)\Big\} dt < \infty \quad \text{a.s.}$$

$$\gamma(s,z) \geq -1 \qquad \text{for a.a. } s,z \in [0,T]\times\mathbb{R}_0 \;\; (\mathbb{R}_0 = \mathbb{R}\backslash\{0\})$$

Self-financing trading strategy

Portfolio $\pi(t)$: number of shares of risky asset held at time t.
Wealth $X(t) = X_0 + \int_0^t \pi(t) dS(t)$.

The strategy π is admissible if $\pi(t)$ is predictable, S-integrable and $X^\pi(t)$ is lower bounded for $t \in [0, T]$ a.s.

The set of such portfolios is denoted by $\mathcal{P}$.

Set of measures

For given $\mathcal{F}_t$-predictable processes $\theta_0(t)$ and $\theta_1(t,z)); t \geq 0, z \in \mathbb{R}_0$ s.t.

$$\int_0^T \{\theta_0^2(s) + \int_{\mathbb{R}} (\ln(1+\theta_1(s,z)))^2 \nu(dz\} ds < \infty a.s., \qquad (1)$$

define the process $K_\theta(t)$ as the solution of the SDE

$$dK_\theta(t) = K_\theta(t^-)[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{N}(dt,dz)]; \quad 0 \leq t \leq T$$

$$K_\theta(0) > 0$$

Then define the measures $Q = Q_\theta$ by

$$dQ_\theta(\omega) = K_\theta(T)dP(\omega) \quad \text{on } \mathcal{F}_T$$

We define our controlled state process
$Y(t) = Y^{\theta,\pi}(t) = (K_\theta(t), S(t), X^{(\pi)}(t)) \in \mathbb{R}^3$

$$\mathbf{dY(t)} = \begin{bmatrix} 0 \\ S(t)\alpha(t) \\ S(t)\alpha(t)\pi(t) \end{bmatrix} \mathbf{dt} + \begin{bmatrix} \theta_0(t)K_\theta(t) \\ S(t)\beta(t) \\ S(t)\pi(t)\beta(t) \end{bmatrix} \mathbf{dB(t)}$$

$$+ \begin{bmatrix} \int_{\mathbb{R}} K_\theta(t^-)\theta_1(t,z) \\ \int_{\mathbb{R}} S(t^-)\gamma(t,z) \\ \int_{\mathbb{R}} S(t^-)\pi(t)\gamma(t,z) \end{bmatrix} \tilde{N}(dt,dz);$$

$$Y(0) = y = (K_\theta(0), S(0), X^{(\pi)}(0)) = (k,s,x) \in \mathbb{R}^3.$$

Put $\tilde{Y}^\theta(t) = (K_\theta(t), S(t)) \in \mathbb{R}^2$.
We assume the coefficients are $\tilde{Y}$- Markovian:
$\alpha(t) = \alpha(t, \tilde{Y}(t))$, $\beta(t) = \beta(t, \tilde{Y}(t))$, $\gamma(t,z) = \gamma(t, \tilde{Y}(t), z)$
and that the claim $G = g(S(T))$.

Set of controls and set of measures

- Let Π be the set of $Y(t)$-Markovian investment strategies π of $\mathcal{P}$, i.e. $\pi(t) = \pi(Y(t))$.
- Let $\mathbb{L}$ be the set of $\tilde{Y}$-Markovian controls $\theta(t) = (\theta_0(t, \tilde{Y}(t)), \theta_1(t, \tilde{Y}(t), z))$; satisfying (1) and such that

$$E[K_\theta(T)] = K_\theta(0)$$

  (this implies that $K_\theta(t)$ is a martingale).
- $\mathbb{M} = \{\theta \in \mathbb{L}; M\theta = 0\}$, with

$$M\theta(t) = \alpha(t) + \theta_0(t)\beta(t) + \int_{\mathbb{R}_0} \theta_1(t)\gamma(t)\nu(dz);$$

We identify the processes π, θ with the feedback functions $\theta(\tilde{y})$, $\pi(y)$.
We define the two sets $\mathcal{L}, \mathcal{M}$ of measures

$$\mathcal{L} = \{Q_\theta; \theta \in \mathbb{L}\}; \quad \mathcal{M} = \{Q_\theta; \theta \in \mathbb{M}\},$$

By Girsanov, the measures $Q_\theta \in \mathcal{M}$ with $K_\theta(0) = 1$ are EMM.

Penalty function

We assume that the penalty function ζ has the form

$$\zeta(Q_\theta) = E\Big[\int_0^T \int_{\mathbb{R}_0} \lambda(\theta_0(t), \theta_1(t,z), K(t), S(t), z)\nu(dz)dt + h(K(T), S(T))\Big]$$

for some convex functions $\lambda \in C^1(\mathbb{R}^2 \times \mathbb{R}_0)$, $h \in C^1(\mathbb{R})$ s.t.

$$E\Big[\int_0^T \int_{\mathbb{R}_0} \big|\lambda(\theta_0(t), \theta_1(t,z), K(t), S(t), z)\big|\nu(dz)dt + \big|h(K(T), S(T))\big|\Big] < \infty$$

for all $(\theta, \pi) \in \mathbb{L} \times \Pi$.
This includes the relative entropy $\zeta(Q) = E[\frac{dQ}{dP} \ln \frac{dQ}{dP}]$ as a special case (take $\lambda = 0$ and $h(k,s) = k \ln k$) .

Game problem

Our game problem

$$\Phi_G(x) = \inf_{\pi \in \mathcal{P}} \left(\sup_{Q \in \mathcal{L}} \{ E_Q[-X_x^{(\pi)}(T) + G] - \zeta(Q) \} \right)$$

can be rewritten as:
Problem 1: Find $\Phi_G(t,y)$ and $(\theta^*, \pi^*) \in \mathbb{L} \times \Pi$ such that

$$\Phi_G(t,y) := \inf_{\pi \in \Pi} \left(\sup_{\theta \in \mathbb{L}} J^{\theta,\pi}(t,y) \right) = J^{\theta^*,\pi^*}(t,y),$$

where $J^{\theta,\pi}(t,y) =$

$$E^{t,y}\Big[-\int_t^T \Lambda\theta(t)dt - h(K(T),S(T)) + K(T)g(S(T)) - K(T)X^{(\pi)}(T) \Big],$$

and

$$\Lambda\theta(t) = \int_{\mathbb{R}} \lambda(\theta_0(t), \theta_1(t,z), K(t), S(t), z)\nu(dz).$$

Auxiliary stochastic control problem

Problem 2:

$$\Psi_G = \sup_{Q \in \mathcal{M}} \{E_Q[G] - \zeta(Q)\}$$

Putting this into a Markovian context, the problem gets the form:
Find $\Psi_G(t,s,k)$ and $\check{\theta} \in \mathbb{M}$ such that

$$\Psi_G(t,s,k) = \sup_{\theta \in \mathbb{M}} J_0^{\theta}(t,s,k) = J_0^{\check{\theta}}(t,s,k), \tag{2}$$

where

$$J_0^{\theta}(t,s,k) = E^{t,s,k}\Big[-\int_t^T \Lambda(\theta(t))dt - h(K(T),S(T)) + K(T)g(S(T))\Big];$$

Note that $(y=(s,k,x))$

$$J^{\theta,\pi}(t,s,k,x) = J_0^{\theta}(t,s,k) - E^{t,y}[K(T)X^{(\pi)}(T)]$$

Infinitesimal generators

The process $Y^{\theta,\pi}(t) = (K_\theta(t), S(t), X^\pi(t))$ is Markovian with generator

$$A^{\theta,\pi}\varphi(y) = A^{\theta,\pi}\varphi(s,k,x) = \alpha s \frac{\partial \varphi}{\partial s} + s\alpha\pi \frac{\partial \varphi}{\partial x} + \tfrac{1}{2}\theta_0^2 k^2 \frac{\partial^2 \varphi}{\partial k^2} + \tfrac{1}{2}\beta^2 s^2 \frac{\partial^2 \varphi}{\partial s^2}$$
$$+ \tfrac{1}{2}\beta^2\pi^2 s^2 \frac{\partial^2 \varphi}{\partial x^2} + \theta_0 \beta k s \frac{\partial^2 \varphi}{\partial k \partial s} + \theta_0 \pi \beta k s \frac{\partial^2 \varphi}{\partial k \partial x} + \pi \beta^2 s^2 \frac{\partial^2 \varphi}{\partial s \partial x}$$
$$+ \int_{\mathbb{R}_0} \Big\{ \varphi(k + k\theta_1, s + s\gamma, x + s\pi\gamma) - \varphi(k,s,x)$$
$$- k\theta_1 \frac{\partial \varphi}{\partial k} - s\gamma \frac{\partial \varphi}{\partial s} - s\pi\gamma \frac{\partial \varphi}{\partial x} \Big\} \nu(dz)$$

The process $\tilde{Y}^\theta(t) = (K_\theta(t), S(t))$ is Markovian with generator

$$A^\theta \psi(k,s) = \alpha s \frac{\partial \psi}{\partial s} + \tfrac{1}{2}\theta_0^2 k^2 \frac{\partial^2 \psi}{\partial k^2} + \tfrac{1}{2}\beta^2 s^2 \frac{\partial^2 \psi}{\partial s^2} + \theta_0 \beta k s \frac{\partial^2 \psi}{\partial k \partial s}$$
$$+ \int_{\mathbb{R}_0} \Big\{ \psi(k + k\theta_1, s + s\gamma) - \psi(k,s)$$
$$- k\theta_1 \frac{\partial \psi}{\partial k} - s\gamma \frac{\partial \psi}{\partial s} \Big\} \nu(dz)$$

A useful lemma

Lemma 1: Let $\psi \in C^2(\mathbb{R}_+^2)$ and define

$$\varphi(k,s,x) := \psi(k,s) - kx.$$

Then,

$$A^{\theta,\pi}\varphi(k,s,x) = A^{\theta}\psi(k,s) - ks\pi\Big[\alpha + \theta_0\beta + \int_{\mathbb{R}_0} \theta_1\gamma\nu(dz)\Big]$$

Proof: We have $A^{\theta,\pi}\psi(k,s,x) = A^{\theta}\psi(k,s)$, so it remains to compute

$$\begin{aligned} A^{\theta,\pi}(kx) &= \alpha s\pi k + \theta_0\pi\beta ks + \int_{\mathbb{R}_0} \{(k + k\theta_1)(x + s\pi\gamma) \\ &\quad - kx - k\theta_1 x - \pi s\gamma k\}\nu(dz) \\ &= ks\pi\Big[\alpha + \theta_0\beta + \int_{\mathbb{R}_0} \theta_1\gamma\nu(dz)\Big]. \end{aligned}$$

Theorem 1: Put $\mathcal{S} = [0, T) \times \mathbb{R}^3$. Suppose $\varphi \in C^{1,2}(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and $(\hat{\theta}, \hat{\pi}) \in \mathbb{L} \times \Pi$ satisfy the following conditions:

(i) $\frac{\partial \varphi}{\partial t} + A^{\theta,\hat{\pi}}\varphi(t,y) - \Lambda(\theta) \leq 0$ for all $\theta \in \mathbb{R}^2$, $(t,y) \in \mathcal{S}$

(ii) $\frac{\partial \varphi}{\partial t} + A^{\hat{\theta},\pi}\varphi(t,y) - \Lambda(\hat{\theta}) \geq 0$ for all $\pi \in \mathbb{R}$, $(t,y) \in \mathcal{S}$

(iii) $\frac{\partial \varphi}{\partial t} + A^{\hat{\theta},\hat{\pi}}\varphi(t,y) - \Lambda(\hat{\theta}) = 0$; $(t,y) \in \mathcal{S}$

(iv) $\varphi(T,k,s,x) = kg(s) - h(k,s) - kx$; $(k,s) \in \mathbb{R}_+^2$

(v) the family $\{\varphi(\tau, Y^{\theta,\pi}(\tau))\}_{\tau \in \mathcal{T}}$ is uniformly integrable for all $(\theta,\pi) \in \mathbb{L} \times \Pi$, $y \in \mathcal{S}$, where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times $\tau \leq T$.

Then

$$\varphi(t,y) = \Phi_G(t,y) = \inf_{\pi \in \Pi}\left(\sup_{\theta \in \mathbb{L}} J^{\theta,\pi}(t,y)\right) = \sup_{\theta \in \mathbb{L}}\left(\inf_{\pi \in \Pi} J^{\theta,\pi}(t,y)\right)$$

$$= \sup_{\theta \in \mathbb{L}} J^{\theta,\hat{\pi}}(t,y) = \inf_{\pi \in \Pi} J^{\hat{\theta},\pi}(t,y) = J^{\hat{\theta},\hat{\pi}}(t,y); \quad y \in \mathcal{S}.$$

Theorem 2: Suppose the value function $\Psi_G(t,k,s)$ for Problem 2 belongs to $C^{1,2}([0,T]\times\mathbb{R}_+^2)\cap C([0,T]\times\mathbb{R}_+^2)$ and that an optimal $\check{\theta}\in\mathbb{M}$ exists. Moreover, suppose that the map

$$\theta\to A^{\theta}\Psi_G(t,k,s);\qquad \theta\in\mathbb{R}^2$$

is strictly concave. Then the value function for Problem 1 is

$$\Phi_G(t,y)=\Psi_G(t,k,s)-kx \tag{3}$$

and for all $\pi\in\Pi$ the pair

$$(\theta^*,\pi^*)=(\check{\theta},\pi)$$

is optimal for Problem 1 (game problem).

Proof: By the HJB equation for Problem 2 we know that

$$\frac{\partial \Psi_G}{\partial t} + \sup_{\theta \in \mathbb{M}} \{A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta)\} = \frac{\partial \Psi}{\partial t} + A^{\check{\theta}}\Psi_G - \Lambda(\check{\theta}) = 0 \quad t < T$$

$$\Psi_G(T,k,s) = -h(k,s) + kg(s)$$

Define

$$\varphi(t,k,s,x) = \Psi_G(t,k,s) - kx$$

By Lemma 1 we have

$$A^{\theta,\pi}\varphi(y) = A^{\theta}\Psi_G(t,k,s) - ks\pi M(\theta)$$

where $M(\theta) = \alpha + \theta_0\beta + \int_{\mathbb{R}_0} \theta_1 \gamma \nu(dz)$.

Conditions (i)–(iii) of Theorem 1 get the form

(i)' $\frac{\partial \Psi_G}{\partial t} + A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta) - ks\hat{\pi}M(\theta) \leq 0$ for all $\theta \in \mathbb{R}^2$

(ii)' $\frac{\partial \Psi_G}{\partial t} + A^{\hat{\theta}}\Psi_G(t,k,s) - \Lambda(\hat{\theta}) - ks\pi M(\hat{\theta}) \geq 0$ for all $\pi \in \mathbb{R}$

(iii)' $\frac{\partial \Psi_G}{\partial t} + A^{\hat{\theta}}\Psi_G(t,k,s) - \Lambda(\hat{\theta}) - ks\hat{\pi}M(\hat{\theta}) = 0.$

Optimality conditions

The first order conditions for the maximum point $\hat{\theta} = \hat{\theta}(\pi)$ of the map

$$\theta \to A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta) - ks\pi M(\theta); \qquad \theta \in \mathbb{R}^2 \tag{4}$$

(for fixed y, π) are

$$\nabla_{\theta}(A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta) - ks\pi M(\theta))_{\theta=\hat{\theta}} = 0. \tag{5}$$

Since $\theta \to A^{\theta}\Psi_G(y)$ is strictly concave, this equation is solved by the maximum point $\hat{\theta}(\pi)$ of (4) only.
Since $\theta = \check{\theta}$ is a maximum point for the function

$$\theta \to A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta); \qquad \theta \in \mathbb{R}^2$$

with the constraint $M(\theta) = 0$, there exists a constant (with respect to θ) $C = C(k,s)$ such that

$$\nabla_{\theta}(A^{\theta}\Psi_G(t,k,s) - \Lambda(\theta) - C(k,s)M(\theta))\big|_{\theta=\check{\theta}} = 0 \tag{6}$$

Therefore, if we choose

$$\hat{\pi} = \hat{\pi}(y) = (ks)^{-1}C(k,s), \tag{7}$$

then by comparing (5) and (6) we see that

$$\check{\theta} = \hat{\theta}(\hat{\pi}). \tag{8}$$

We verify that with this choice of $\hat\theta = \hat\theta(\hat\pi)$ and $\hat\pi$, (i)'–(iii)' hold:

$$\text{(i)'}\quad \sup_{\theta\in\mathbb{L}}\{A^{\theta}\Psi_G - \Lambda(\theta) - ks\hat\pi M(\theta)\} = A^{\hat\theta(\hat\pi)}\Psi_G - \Lambda(\hat\theta(\hat\pi)) - ks\hat\pi M(\hat\theta(\hat\pi))$$

$$= A^{\check\theta}\Psi_G - \Lambda(\check\theta) - ks\hat\pi M(\check\theta) = A^{\check\theta}\Psi_G - \Lambda(\check\theta) = \sup_{\theta\in\mathbb{M}}\{A^{\theta}\Psi_G - \Lambda(\theta)\} = -\frac{\partial\Psi}{\partial}$$

by HJB equation for value function Ψ_G of Problem 2. Hence (i)' holds. Similarly, for all $\pi \in \Pi$ we have

$$A^{\hat\theta}\Psi_G - \Lambda(\hat\theta) - ks\pi M(\hat\theta) = A^{\hat\theta}\Psi_G - \Lambda(\hat\theta) = A^{\check\theta}\Psi_G - \Lambda(\check\theta) = -\frac{\partial\Psi_G}{\partial t},$$

again by the HJB equation for Ψ_G. Therefore (ii)' and (iii)' hold also. Finally we check that (iv) holds:

$$\varphi(T,k,s,x) = \Psi_G(T,k,s) - kx = kg(s) - h(k,s) - ks.$$

So φ and $(\hat\theta(\hat\pi),\hat\pi)$ satisfy all the requirements of Theorem 1 and

$$\varphi(t,y) = \Phi_G(t,y) = \Psi_G(t,k,s) - kx.$$

Moreover, $\theta^* := \hat\theta(\hat\pi)$ and $\pi^* := \hat\pi$ defined by (7) and (8) constitute an optimal pair.

Now let $\pi \in \Pi$ be arbitrary. Note that

$$E^{t,y}[K_{\theta^*}(T)X^{(\pi)}(T)] = E^{t,y}[K_{\check{\theta}}(T)X^{(\pi)}(T)] = kE^{t,y}_{\frac{1}{k}Q_{\check{\theta}}}\left[X^{(\pi)}(T)\right] = kx$$

because $\frac{1}{k}Q_{\check{\theta}}$ is an equivalent martingale measure.
Let $Y^* = Y^{\theta^*,\pi^*}$, $Y = Y^{\check{\theta},\pi}$.

$$\Phi_G(t,y) = \inf_{\pi\in\Pi}\left(\sup_{\theta\in\mathbb{L}} J^{\theta,\pi}(y)\right) = J^{\hat{\theta}(\hat{\pi}),\hat{\pi}}(y)$$

$$= E^{t,y}\Big[-\int_t^T \Lambda(\hat{\theta}(K_{\theta^*}(t),S(t))dt + K_{\theta^*}(T)g(S(T))$$

$$- h(K_{\theta^*}(T),S(T)) - K_{\theta^*}(T)X^{(\pi^*)}(T)\Big]$$

$$= E^{t,y}\Big[-\int_t^T \Lambda(\check{\theta}(K_{\check{\theta}}(t),S(t))dt + K_{\check{\theta}}(T)g(S(T) - h(K_{\check{\theta}}(T),S(T))\Big] - kx$$

$$= J^{\hat{\theta}(\hat{\pi}),\pi}(t,y).$$

We conclude that for all $\pi \in \Pi$, the pair $(\theta^*,\pi) = (\check{\theta},\pi) \in \mathbb{M}\times\Pi$ is optimal for Problem 2.

Viscosity solution approach

General Elliptic Integro-Differential equations:

$$F(x, u, \nabla u, D^2 u, \mathcal{I}[x, u]) = 0 \tag{9}$$

where F is a continuous function satifying degenerate ellipticity conditions and $\mathcal{I}[x, u]$ is a nonlocal term.

$$F(x, u, p, M, l_1) \leq F(x, u, p, M, l_2) \quad \text{if } M \geq N, l_1 \geq l_2$$

Definition (Viscosity sub and supersolutions): An usc function is a viscosity subsolution of (9) if for any test function $\phi \in \mathcal{C}^2$, if x is a global maximum point of $u - \phi$, then

$$F(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}[x, \phi]) \leq 0$$

An lsc function is a viscosity subsolution of (9) if for any test function $\phi \in \mathcal{C}^2$, if x is a global minimum point of $u - \phi$, then

$$F(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}[x, \phi]) \geq 0$$

Pham, Karlsen Jacobsen, Arizawa, Barles Imbert

Lemma (*in progress*): Suppose u is a viscosity subsolution (*resp. super*) for the HJB equation of Pb 2:

$$\frac{\partial \psi}{\partial t} + \sup_{\theta \in \mathbb{M}} \{A^{\theta}\psi(t,k,s) - \Lambda(\theta)\} = 0 \quad t < T$$
$$\psi(T,k,s) = -h(k,s) + kg(s)$$

Then

$$w := u - kx$$

is a viscosity subsolution (*resp. super*) for the HJBI equation of Pb 1:

$$\frac{\partial \varphi}{\partial t} + \inf_{\pi \in \mathbb{R}} (\sup_{\theta \in \mathbb{L}} A^{\theta,\pi}\varphi(t,y) - \Lambda(\theta)) = 0; \quad t < T$$
$$\varphi(T,k,s,x) = kg(s) - h(k,s) - kx$$

Theorem 3 (*in progress*) : Suppose the HJBI equation for Pb 1 has a unique viscosity solution (which then is the value function Φ_G). Then

$$\Phi_G(t, y) = \Psi_G(t, y) - kx$$

where Ψ_G is the value function of Problem 2.

Remark 1: sufficient condition for uniqueness : the jumps are bounded away from zero.

Remark 2 : drawback of viscosity solution: do not give information on optimal control.

Risk indifference pricing

We apply Theorem 2 to find the risk indifference price solution of

$$\Phi_G(t,k,s,x+p)=\Phi_0(t,k,s,x)$$

where Φ_G is the solution of Pb 1. By Th. 2 this equation becomes

$$\Psi_G(t,k,s)-k(x+p)=\Psi_0(t,k,s)-kx$$

which has the solution

$$p=p_{risk}=k^{-1}(\Psi_G(t,k,s)-\Psi_0(t,k,s)).$$

In particular, choosing $k=1$ (i.e. all measures $Q\in\mathcal{L}$ are probability measures), we get

$$p_{risk}(G)=\sup_{Q\in\mathcal{M}}\{E_Q[G]-\zeta(Q)\}-\sup_{Q\in\mathcal{M}}\{-\zeta(Q)\},$$

where $\mathcal{M}$ is the set of equivalent martingale measures defined on slide 13.

Remark:

$$p_{risk}(G)\leq\sup_{Q\in\mathcal{M}}E_Q[G]+\sup_{Q\in\mathcal{M}}\{-\zeta(Q)\}-\sup_{Q\in\mathcal{M}}\{-\zeta(Q)\}\leq\sup_{Q\in\mathcal{M}_1}E_Q[G]$$

$$=p_{up}(G),\quad \text{the upper hedging price of } G\,,$$