Good-deal Pricing of American Contingent Claims using a Stochastic Linear Programming Approach

Ahmet Camcı and Mustafa Ç. Pınar
Bilkent University
Industrial Engineering Department

Good-Deals

Expected Gain-Loss Ratio (Bernardo and Ledoit(2000)):

$$\frac{E^P[X^+]}{E^P[X^-]}$$

Pınar and Salih(2006) constructed optimization models to derive martingale expressions for the upper and lower pricing bounds of European contingent claims

American Contingent Claims (ACC)

Föllmer and Schied(2002) derive martingale expressions for the arbitrage-free prices of ACC's in general incomplete markets.

Pennanen and King(2006) constructed optimization problems to derive the pricing bounds first, then they derived martingale expressions.

Model (1)

Discrete time, discrete state stock price process that can be represented by a non-recombinant tree.
Notation:

- $\mathcal{N}_t$: The set of all nodes in time index t, $t = 0, \ldots, T$
- $\pi(n)$: The unique parent node of node n
- $A(n)$: The set of all ascendant nodes of node n
- $C(n)$: The set of child nodes of node n
- $D(n)$: The set of all descendant nodes of node n

Model (2)

- Probabilities for terminal nodes are assumed to be given: p_n
- Price of securities at node n: $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$
- $S_n^0 = 1$, $\forall n \in \mathcal{N}$
- Portfolio of the investor at node n: $\theta_n = (\theta_n^0, \theta_n^1, \ldots, \theta_n^J)$
- Payoff of American contingent claim at node n is F_n

Good-Deals

A portfolio strategy is said to yield a good-deal opportunity at level $\lambda > 1$ if,

$$S_0 \cdot \theta_0 = 0$$
$$S_0 \cdot (\theta_n - \theta_{\pi(n)}) = 0, \forall n \in \mathcal{N}_t,\ t \geq 1$$
$$E^P[X^+] > \lambda E^P[X^-]$$

Good-Deal Seeking Problem

$$
\begin{aligned}
\max \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\
\text{s.t.} \quad & S_0 \cdot \theta_0 \ = \ 0 \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) \ = \ 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& S_n \cdot \theta_n - x_n^+ + x_n^- \ = \ 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ \ \geq \ 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \ \geq \ 0,\ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

Lower Bound for the ACC Price

P1:

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall \, n \in \mathcal{N}_t, 1 \le t \le T \\
& S_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall \, n \in \mathcal{N}_T \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \ge 0 \\
& \sum_{m \in \mathcal{A}(n)} e_m \le 1, \ \forall \, n \in \mathcal{N}_T \\
& x_n^+, x_n^- \ge 0, \ \forall \, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall \, n \in \mathcal{N}.
\end{aligned}
$$

Lower Bound for the ACC Price (Relaxation)

P2:

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \le t \le T \\
& S_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \ge 0 \\
& \sum_{m \in \mathcal{A}(n)} e_m \le 1, \ \forall\, n \in \mathcal{N}_T \\
& x_n^+, x_n^- \ge 0, \ \forall\, n \in \mathcal{N}_T \\
& e_n \ge 0, \ \forall\, n \in \mathcal{N}.
\end{aligned}
$$

The Main Result (1)

Theorem 1 *There exists an optimal solution to P2 with* $e_n \in \{0, 1\}\,,\ \forall\, n \in \mathcal{N}$.

Proof: **Case 1:** Consider the case where e^* has a value not equal to 0 or 1 for the root, which is the starting node of the tree (i.e. $e_0^* \notin \{0, 1\}$).

The Main Result (2)

$$
\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T} z_n \\
\text{s.t.} \quad & y_0 = 1 \\
& \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N} \setminus \mathcal{N}_T \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \le 0, \ \forall\, n \in \mathcal{N} \\
& p_n u - y_n \le 0, \ \forall\, n \in \mathcal{N}_T \\
& y_n - \lambda p_n u \le 0, \ \forall\, n \in \mathcal{N}_T \\
& z_n \ge 0, \ \forall\, n \in \mathcal{N}_T \\
& u \ge 0.
\end{aligned}
$$

The Main Result (3)

- Optimal value becomes F_0 then
- $e_0 = 1$, $V = F_0$ and all the other variables as zeros constitutes a feasible solution to P2 with objective value F_0.

Case 2: Consider the case where e^* has a value not equal to 0 or 1 for some node but not the root. Select one of such nodes among the set of nodes those are closest to the root. Call it w.

The Main Result (4)

$$
\begin{aligned}
\max \quad & e_w \\
\text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n,\ \forall\, n \in \mathcal{D}(w) \backslash \{w\} \\
& S_n \cdot \theta_n - x_n^+ + x_n^- = 0,\ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x_n^- \geq 0 \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1,\ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& x_n^+, x_n^- \geq 0,\ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n \geq 0,\ \forall\, n \in \mathcal{D}(w),
\end{aligned}
$$

The Main Result (5)

$$
\begin{array}{lrcl}
\min & e_w & & \\
\text{s.t.} & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) & = & F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & F_n e_n, \ \forall\, n \in \mathcal{D}(w) \backslash \{w\} \\
& S_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x_n^- & \geq & 0 \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m & \leq & 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& x_n^+, x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n & \geq & 0, \ \forall\, n \in \mathcal{D}(w).
\end{array}
$$

Martingales

Theorem 2 *If there are no good-deals in the market price process, the buyer's price for American contingent claim F can be expressed as*

$$\max_{\tau \in \mathcal{T}} \min_{q \in Q(\lambda)} E^q[F_\tau] = \min_{q \in Q(\lambda)} \max_{\tau \in \mathcal{T}} E^q[F_\tau]$$

where,

$$Q(\lambda) = \{q \mid q_0 = 1;\ q_n S_n = \sum_{m \in \mathcal{C}(n)} q_m S_m,\ \forall n \in \mathcal{N} \backslash \mathcal{N}_T;$$
$$\exists u \geq 0 \ s.t.\ p_n u \leq q_n \leq \lambda p_n u,\ \forall n \in \mathcal{N}_T\}.$$

Stocks Paying Dividends

Corollary 1 *If each security $j = 1, \ldots, J$ pays dividend payments D_n^j in node n, under the assumption of no good-deals in the market price process, the buyer's price for American contingent claim F can be expressed as*

$$\max_{\tau \in \mathcal{T}} \min_{q \in Q'(\lambda)} E^q[F_\tau] = \min_{q \in Q'(\lambda)} \max_{\tau \in \mathcal{T}} E^q[F_\tau]$$

where

$$Q'(\lambda) = \{q \mid q_0 = 1,\ q_n S_n = \sum_{m \in \mathcal{C}(n)} q_m (S_m + D_m),\ \forall n \in \mathcal{N} \backslash \mathcal{N}_T;$$
$$\exists u \geq 0 \ s.t.\ p_n u \leq q_n \leq \lambda p_n u,\ \forall n \in \mathcal{N}_T\}.$$