Good-deal Pricing of American Contingent Claims using a Stochastic Linear Programming Approach

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Abstract

We consider pricing of American contingent claims (ACC) and also different types of contingent claims as special cases of ACC's, in a multi-period, discrete time, discrete state space using the concept of good-deals. We analyze the problem to get lower and upper bounds for buyer's price and writer's price, respectively. In incomplete markets we have tighter bounds compared to no-arbitage pricing. We note that our results are useful in determining bid-ask spreads. Determining buyer's price for ACC's requires solving an integer program unlike European options for which solving a linear program is sufficient. However, we show that a relaxation of the integer programming problem which is a linear program, can be used to get the same lower bound for the price of the ACC. This result resolves the problem arising from the computational complexity of solving an integer program.

Keywords: American options, contingent claim, pricing, hedging, martingales, stochastic linear programming

1 Introduction

Since its introduction by Ross [8], arbitrage pricing theory (APT) has been widely studied in the pricing literature. Compared to capital asset pricing model, which was introduced before APT, it was more practical since it did not rely on utilities of individuals. Thus, while pricing financial instruments experts would not have to specify preferences of individuals. However, APT had its own weaknesses, e.g., it did not imply unique prices in incomplete markets. Since markets are almost never complete due to market imperfections as discussed in Carr *et al.* [2] APT did not seem to be a perfect solution for pricing. Then, a wide variety of studies tried to unify these two pricing theories. Some relatively recent examples are Cochrane and Saa-Requejo [3], Bernardo and Ledoit [1], Carr *et al.* [2] and Pınar and Salih [6].

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In this paper, we are interested in pricing of ACC's and also different types of contingent claims as special cases of ACC's, following closely the studies by [5, 6] both of which work under the mathematical setting initiated by King [4]. Pınar and Salih [6] analyze the problem of pricing and hedging using the concept of a sufficiently attractive expected gain opportunity (SAGE) to a European contingent claim's writer and buyer. Pennanen and King [5] characterizes the set of arbitrage-free prices for ACC's. We combine the setting of [6] with the approach of [5] in this paper. Our work is based on good-deals defined as investments with high expected gain-loss ratio in [1]. Our motivation is to derive tighter bounds for the price of an ACC in an incomplete market. We examine the problem in a multi-period, discrete time, discrete state space framework. Under this framework we define the stock price process as a non-recombinant tree. After defining buyer's problem similarly to the one in [5] we formulate an integer programming problem for the buyer's price. Then, we prove that the good-deal bound from the buyer's perspective can be computed by solving a linear program. This result extends Theorem 3 of [5] to the good-deal setting. It also provides a correct proof of Theorem 3 of [5]. We give our proof in section 3.

2 Preliminaries

2.1 The Stochastic Scenario Tree

We approximate the behavior of the stock market by assuming that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \ldots, T$. We further assume the market evolves as a discrete, non-recombinant scenario tree (hence, suitable for incomplete markets) in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. The σ-algebras are such that, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $0 \leq t \leq T-1$ and $\mathcal{F}_T = \mathcal{F}$. A stochastic process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if for each $t = 0, \ldots, T$, the outcome of the process only depends on which element of $\mathcal{F}_t$ has been realized at stage t. Similarly, a decision process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if for each $t = 0, \ldots, T$, the decision depends on which element of $\mathcal{F}_t$ has been realized at stage t. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathcal{N}$. $\mathcal{A}(n)$ denotes the ascendant nodes or path history of node n including itself and $\mathcal{D}(n)$ denotes the set of descendant nodes of n, again including itself. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{S}(n)} p_m, \;\; \forall\, n \in \mathcal{N}_t, \; t = T-1, \ldots, 0.$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω, [4]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathcal{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{S}(n)} \frac{p_m}{p_n} X_m.$$

The market consists of $J+1$ tradable securities indexed by $j = 0, 1, \dots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \dots, S_n^J)$. We assume as in [5, 6] that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset corresponds to the risk-free asset in the classical valuation framework.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. We refer to the collection of vectors $\theta_0, \theta_1, \dots, \theta_{|\mathcal{N}|}$ as Θ. The value of the portfolio at state n is

$$S_n \cdot \theta_n = \sum_{j=0}^{J} S_n^j \theta_n^j.$$

We need the following definition.

Definition 1 *If there exists a probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ such that*

$$S_t = \mathbb{E}^Q[S_{t+1}|\mathcal{N}_t] \ (t \leq T-1)$$

then the vector process $\{S_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process.

2.2 Good-deals

In our context a good-deal opportunity is defined as follows. For $n \in \mathcal{N}_T$ let $S_n \cdot \theta_n = x_n^+ - x_n^-$ where x_n^+ and x_n^- are non-negative numbers, i.e., we express the final portfolio value at terminal state n as the difference of two non-negative numbers, x_n^+ and x_n^- representing gain and loss, respectively. Assume that there exist a set of vectors $\theta_0, \theta_1, \dots, \theta_{|\mathcal{N}|}$ such that

$$S_0 \cdot \theta_0 = 0$$

$$S_n \cdot (\theta_n - \theta_{\pi(n)}) = 0, \ \forall \, n \in \mathcal{N}_t, t \geq 1$$

and

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] > 0,$$

for $\lambda > 1$, where $X^+ = \{x_n^+\}_{n \in \mathcal{N}_T}$, and $X^- = \{x_n^-\}_{n \in \mathcal{N}_T}$. This sequence of portfolio holdings is said to yield a good-deal opportunity at level $\lambda > 1$. This formulation is similar to Bernardo and Ledoit [1] gain-loss ratio, and the Sharpe ratio restriction of Cochrane and Saa-Requejo [3]. Yet, it makes the problem easier to tackle within the framework of linear programming. Moreover, the parameter λ can be interpreted as the risk aversion parameter of the individual investor. As λ gets bigger, the individual will become more risk-averse, preferring near-arbitrage positions. As λ gets closer to 1, the individual weighs the gains and losses equally. In the limiting case of λ being equal to 1 the pricing operator (equivalent martingale measure) is unique if it exists.

Consider now the perspective of an investor who is content with the existence of a good-deal opportunity although an arbitrage opportunity may not exist. Such an investor is interested in the solution of the following stochastic linear programming problem:

$$\begin{array}{lrcl}
\max & \displaystyle\sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & & \\
\text{s.t.} & S_0 \cdot \theta_0 & = & 0 \\
 & S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
 & Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T, \\
 & x_n^+ & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\
 & x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T.
\end{array}$$

If there exists an optimal solution (i.e., a sequence of $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ vectors) to the above problem that yields a positive optimal value, the solution is said to give rise to a good-deal opportunity at level λ (the expected positive terminal wealth outweighing λ times the expected negative final wealth). If there exists a good-deal opportunity, then the problem is unbounded. We note that by the fundamental theorem of linear programming, when it is solvable, above problem has always a basic optimal solution in which no pair x_n^+, x_n^-, for all $n \in \mathcal{N}_t$, can be positive at the same time. This justifies our approach in defining x_n^+ and x_n^- as the terminal gain and loss.

The discrete state stochastic vector process $\{S_t\}$ is a good-deal free market price process at a fixed level λ if the optimal value of the above stochastic linear program is equal to zero. Clearly, if λ tends to infinity the problem turns out to be the stochastic programming problem that is used to seek an arbitrage in [4]. It is a well-accepted phenomenon that every rational investor is ready to lose if the benefits of the gains outweigh the costs of the losses. It is also reasonable to assume that the rational investor will try to limit losses. This type of behavior excluded by the no-arbitrage setting is easily modeled by the Expected Utility approach. This formulation allows investors to take reasonable risks without explicitly specifying a complicated utility function while it converges to the no-arbitrage setting in the limit. It is easy to see that an arbitrage opportunity is also a good-deal opportunity, and that absence of a good-deal opportunity (at any level λ) implies absence of arbitrage. Pınar and Salih [6] derives martingale expressions for no good-deal pricing of market stock price process and European contingent claims.

2.3 American Contingent Claims

An American contingent claim F generates payoff opportunities F_n, $(n \geq 0)$ to its holder depending on the states n of the market. Then, if the holder exercises the option at state n, he gets F_n and loses his chance to exercise the option at a later time. Using this definition, an American call option on a stock S with strike price K corresponds to $F = S - K$. American put is obtained by reversing the sign of F. We can define a European call option with maturity T by setting $F_t = 0$ for $t \neq T$. Bermudan call options having exercise date set $G \subset \{1, \ldots, T\}$ can be defined by setting $F_t = 0$ for $t \notin G$.

3 Buyer's Price

Under the assumption of no good-deal opportunities for the stock price process, the price of the ACC that provides no good-deals to the buyer must be greater than or equal to the optimal value of the following optimization problem.

$$
\begin{array}{lrcl}
\max & V & & \\
\text{s.t.} & S_0 \cdot \theta_0 & = & F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\
& S_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T \\
& \displaystyle\sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & \geq & 0 \\
& \displaystyle\sum_{m \in \mathcal{A}(n)} e_m & \leq & 1, \ \forall\, n \in \mathcal{N}_T \\
& x_n^+, x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T \\
& e_n & \in & \{0, 1\}, \ \forall\, n \in \mathcal{N}.
\end{array}
$$

This problem is a mixed integer programming (MIP) problem, and let us call it as P1. It is well known that MIP problem is NP-hard, i.e., solving MIP is much more difficult than solving a linear programming problem. Hence, in the worst case the time required to solve this problem increases exponentially as the number of periods or nodes increases. One natural relaxation of the above problem is the following linear programming relaxation.

$$
\begin{array}{lrcl}
\max & V & & \\
\text{s.t.} & S_0 \cdot \theta_0 & = & F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) & = & F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\
& S_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T \\
& \displaystyle\sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & \geq & 0 \\
& \displaystyle\sum_{m \in \mathcal{A}(n)} e_m & \leq & 1, \ \forall\, n \in \mathcal{N}_T \\
& x_n^+, x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T \\
& e_n & \geq & 0, \ \forall\, n \in \mathcal{N}.
\end{array}
$$

We will refer to the above problem as P2. In P2 the option looks different from an ACC. Because the investor can exercise the option partially along the path history under the model of P2. In other words, the formulation allows the buyer to exercise not the whole but only some part of the option at a time while reserving his right to exercise the rest of the option at a later time. Note that P2 is a linear programming problem. Our next result which is the main result of this paper, shows that P1 and P2 have the same optimal value. Before showing this result, let us note that θ, e, V, x^+ and x^- are the decision variables in P2 as in P1. Besides, setting all these variables to zero gives a feasible solution for both problems. Hence, the feasible regions of these problems are nonempty and they always have an optimal solution. Finiteness of this optimal solution follows from no good-deals assumption for the stock price process and finite payoffs of the ACC.

Theorem 1 *There exists an optimal solution to P2 with* $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$.

Proof: Assume that P2 has an optimal solution V^*, e^*, θ^*, x^{+*} and x^{-*} such that $e_n^* \notin \{0,1\}$ for some $n \in \mathcal{N}$.

Case 1: We will first consider the case where e^* has a value not equal to 0 or 1 for the root, which is the starting node of the tree (i.e. $e_0^* \notin \{0,1\}$). In order to deal with this case, we will form the Lagrangian function for P2. That is

$$\begin{aligned} L(V,e,\theta,v,u,y,z) &= V - y_0[S_0 \cdot \theta_0 - F_0 e_0 + V] - \sum_{n\in\mathcal{N}\setminus\{0\}} y_n[S_n \cdot (\theta_n - \theta_{\pi(n)}) - F_n e_n] \\ &\quad + \sum_{n\in\mathcal{N}_T} v_n[S_n \cdot \theta_n - x_n^+ + x_n^-] + u[\sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^-] \\ &\quad - \sum_{n\in\mathcal{N}_T} z_n[\sum_{m\in\mathcal{A}(n)} e_m - 1]. \end{aligned}$$

After rearranging the above function we have

$$\begin{aligned} L(V,e,\theta,v,u,y,z) &= (1-y_0)V + \sum_{n\in\mathcal{N}_T} (v_n - y_n) S_n \cdot \theta_n + \sum_{n\in\mathcal{N}\setminus\mathcal{N}_T} \theta_n \cdot [\sum_{m\in\mathcal{C}(n)} y_m S_m - y_n S_n] \\ &\quad + \sum_{n\in\mathcal{N}} [y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m] e_n + \sum_{n\in\mathcal{N}_T} [p_n u - v_n] x_n^+ \\ &\quad + \sum_{n\in\mathcal{N}_T} [v_n - \lambda p_n u] x_n^- + \sum_{n\in\mathcal{N}_T} z_n. \end{aligned}$$

Then the dual problem of P2 can be formulated as

$$\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T} z_n \\
\text{s.t.} \quad & y_0 = 1 \\
& [v_n - y_n] S_n = 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N} \backslash \mathcal{N}_T \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{N} \\
& p_n u - v_n \leq 0, \ \forall\, n \in \mathcal{N}_T \\
& v_n - \lambda p_n u \leq 0, \ \forall\, n \in \mathcal{N}_T \\
& z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& u \geq 0.
\end{aligned}$$

Since $S_n \neq 0$, second constraint implies that $v_n = y_n$, $\forall\, n \in \mathcal{N}_T$. Thus the dual problem can be rearranged as

$$\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T} z_n \\
\text{s.t.} \quad & y_0 = 1 \\
& \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N} \backslash \mathcal{N}_T \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{N} \\
& p_n u - y_n \leq 0, \ \forall\, n \in \mathcal{N}_T \\
& y_n - \lambda p_n u \leq 0, \ \forall\, n \in \mathcal{N}_T \\
& z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& u \geq 0.
\end{aligned}$$

We have an optimal solution to P2 with $e_0^* \notin \{0,1\}$. Then complementary slackness implies that third constraint of the above program should be satisfied as an equality for the corresponding optimal solution of the dual problem (i.e., $y_0 F_0 - \sum_{m \in \mathcal{N}_T} z_m = 0$). Since $y_0 = 1$, we have $F_0 = \sum_{m \in \mathcal{N}_T} z_m$. Thus, the optimal solution to the dual problem is found to be F_0. Then, by strong duality we know that F_0 is the optimal value of P2. One can easily check that $e_0 = 1$, $V = F_0$ and all the other variables as zeros (θ_n's as zero vectors) constitute a feasible solution to P2 with objective value F_0. This is an optimal solution with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$, thus the proof for the first case is complete.

Case 2: Now assume that optimal solution e^* is such that $e_0^* = 0$ and $e_n^* \notin \{0,1\}$ for some $n \in \mathcal{N}$. Let $I = \{i | e_i^* \notin \{0,1\}, i \in \mathcal{N}\}$. Let $G = \{g | g \in I, \mathcal{A}(g) \cap I = \{g\}\}$. Let w be the element with the smallest time index (that is closest to the root) in G. Note that $e_n^* = 0$, $\forall\, n \in \mathcal{A}(w) \backslash \{w\}$ in this case. Also, let k denote the time index for node w.

Claim: One can always find an optimal solution to P2 with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w) \backslash \{w\}$.

To prove the claim we will consider the two following linear programs to which we will refer as R1 and R2 respectively:

$$
\begin{aligned}
\max \quad & e_w \\
\text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& S_n \cdot \theta_n - x^+_n + x^-_n = 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x^+_n - \lambda \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x^-_n \geq 0 \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& x^+_n, x^-_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n \geq 0, \ \forall\, n \in \mathcal{D}(w),
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & e_w \\
\text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& S_n \cdot \theta_n - x^+_n + x^-_n = 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x^+_n - \lambda \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} p_n x^-_n \geq 0 \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& x^+_n, x^-_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n \geq 0, \ \forall\, n \in \mathcal{D}(w).
\end{aligned}
$$

Let us denote the optimal solution of R1 as $\bar{\theta}_{\mathcal{D}(w)}$, $\bar{e}_{\mathcal{D}(w)}$, $\bar{x}^+_{\mathcal{D}(w)}$, $\bar{x}^-_{\mathcal{D}(w)}$ and to R2 as $\tilde{\theta}_{\mathcal{D}(w)}, \tilde{e}_{\mathcal{D}(w)}$, $\tilde{x}^+_{\mathcal{D}(w)}$, $\tilde{x}^-_{\mathcal{D}(w)}$. If the optimal value of R1 is 1, then we see that V^*, $(\bar{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\bar{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\bar{x}^+_{\mathcal{D}(w)\cap\mathcal{N}_T}, x^{+*}_{\mathcal{N}_T\backslash\mathcal{D}(w)})$, $(\bar{x}^-_{\mathcal{D}(w)\cap\mathcal{N}_T}, x^{-*}_{\mathcal{N}_T\backslash\mathcal{D}(w)})$ form another optimal solution of P2 with $e_w = 1$. For this optimal solution we have $e_w = 1$ and $e_i = 0$, $\forall\ i \in \mathcal{A}(w)\backslash\{w\}$ (we have also $e_i = 0$, for all $i \in \mathcal{D}(w)\backslash\{w\}$ for this solution). Similarly, if the optimal value of R2 is 0, then V^*, $(\tilde{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\tilde{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\tilde{x}^+_{\mathcal{D}(w)\cap\mathcal{N}_T}, x^{+*}_{\mathcal{N}_T\backslash\mathcal{D}(w)})$, $(\tilde{x}^-_{\mathcal{D}(w)\cap\mathcal{N}_T}, x^{-*}_{\mathcal{N}_T\backslash\mathcal{D}(w)})$ form another optimal solution of P2 with $e_w = 0$. Then, for this optimal solution we have $e_i = 0$, for all $i \in \mathcal{A}(w)$. So, our claim will be proved if we can show that R2 having an optimal value greater than 0 implies that optimal value of R1 is 1. To show that we will consider the dual problems of

R1 and R2. The dual problems DR1 and DR2 of R1 and R2 are

$$\begin{aligned}
\min \quad & \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n \geq 1 \\
& y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& p_n u - y_n \leq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& y_n - \lambda p_n u \leq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& u \geq 0,
\end{aligned}$$

$$\begin{aligned}
\max \quad & -\sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n - y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n \geq -1 \\
& y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& p_n u - y_n \leq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& y_n - \lambda p_n u \leq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& u \geq 0.
\end{aligned}$$

We will denote the optimal value of R2 by α, which is equal to the optimal value of DR2. We know that $\alpha \leq 1$. Assume that $\alpha > 0$. Then by complementary slackness we know that the second constraint of DR2 must be satisfied as an equality at the corresponding optimal solution, since $e_w \neq 0$ at the optimal solution of R2. Then at the optimal solution of DR2, we have

$$0 > \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n = -1 \tag{1}$$

Moreover, we must have $y_w \geq 0$ for any feasible solution of DR1 and DR2. This follows from the following fact. If we look at the fourth constraints of DR1 and DR2 we see that for any feasible solution to DR1 and DR2 we must have $y_n \geq 0$, $\forall n \in \mathcal{N}_T \cap \mathcal{D}(w)$. Since the variable u and the probabilities are greater than or equal to 0 we must have positive y_n's for terminal nodes in order to satisfy these constraints. Then, since $S_n^0 > 0$ for all n, we have $y_n \geq 0$, $\forall n \in \mathcal{N}_{T-1} \cap \mathcal{D}(w)$ by the first constraints of DR1 and DR2. Same argument follows successively for $(T-2), (T-3), \ldots, k$. So, we have $y_w \geq 0$. Then, using the second inequality of (1) we have

$$\sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n$$

$$y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w$$

$$S_w \cdot \theta^*_{\pi(w)} \geq -F_w$$

where last step follows from $y_w \geq 0$. Then, for DR1 at any feasible solution we have

$$1 \leq -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n \leq y_w S_w \cdot \theta^*_{\pi(w)} + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n$$

whence we see that optimal solution of DR1 cannot be less than 1. It is easy to see by R1 that optimal value of DR1 cannot be greater than 1 either. Hence, we conclude that the optimal value of DR1. and therefore that of R1, is 1. This completes the proof of our claim.
Using the claim we see that there always exists an optimal solution to P2 with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w)$. So, one can eliminate all the nodes having time index k in I by applying the above procedure. Then, by proceeding successively with the nodes in $(k+1)^{st}, (k+2)^{nd} \ldots (T)^{th}$ time indices one can find an optimal solution for P2 with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$. We note that at each step the size of I might increase, but no nodes with a time index less than or equal to that of the node eliminated at that particular step can show up in I at the next step. This completes the proof of the theorem. ▮

The above theorem and its proof can also be re-iterated for the case of arbitrage-free pricing bound of Theorem 3 of [5]. We give this theorem and its proof in the appendix. Our result shows that one can always find a feasible solution to P1 which gives the optimal value of the relaxed problem P2. Then, since the optimal value of a problem cannot be better than the optimal value of its relaxation we say that optimal value of P1 can be found by solving P2. One major implication of this result is since P2 is a linear program the problem arising from the computational complexity of P1 (NP-hardness of P1) is eliminated. Besides, one can use duality to get expressions for the buyer's price in terms of martingale measures and stopping times. In order to get these expressions we need the result of [6] regarding buyer's price for a European contingent claims. For simplicity, we assume that $S^0_n = 1, \forall n = 1, \ldots, T$. Under this assumption buyer's price for a European contingent can be expressed as

$$\min_{q \in Q(\lambda)} \sum_{n>0} q_n F_n \tag{2}$$

where

$$Q(\lambda) = \{q \mid q_0 = 1,\, q_n S_n = \sum_{m \in \mathcal{C}(n)} q_m S_m,\, \forall n \in \mathcal{N} \backslash \mathcal{N}_T;\, p_n \leq q_n \leq \lambda p_n,\, \forall n \in \mathcal{N}_T\}.$$

The constraint $q_0 = 1$ is required if the option has a payoff in the root node, which is the case in our formulation. Then, by the existence of this constraint $Q(\lambda)$ forms a set of martingale measures for the process $(S_n)^T_{n=0}$. Thus, we can use expectations instead of summations. Let us define the sets

$$E = \{e \mid e \text{ is } (\mathcal{F}_t)^T_{t=0}\text{-adapted}, \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \in \{0,1\}\ P\text{-a.s.}\},$$

$$\tilde{E} = \{e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted}, \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \geq 0 \ P\text{-a.s.}\}.$$

One common way to describe exercise strategies of ACC's is by stopping times. These are functions $\tau : \Omega \rightarrow \{0, \ldots, T\} \cup \{+\infty\}$ such that $\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{F}_t$, for each $t = 0, \ldots, T$. The relation $e_t = 1 \Leftrightarrow \tau = t$ defines a one-to-one correspondence between stopping times and decision processes $e \in E$. The set of stopping times will be denoted by $\mathcal{T}$.

Theorem 2 *If there are no good-deals in the market price process, the buyer's price for American contingent claim F can be expressed as*

$$\max_{\tau \in \mathcal{T}} \min_{q \in Q(\lambda)} \mathbb{E}^q[F_\tau] = \min_{q \in Q(\lambda)} \max_{\tau \in \mathcal{T}} \mathbb{E}^q[F_\tau]$$

Proof: If we set e fixed in P1 and maximize with respect to θ, we have a European contingent claim with payoffs $F_t e_t$ for $t = 1, \ldots, T$. Then, by 2 for the buyer's price of this ECC, we have

$$\min_{q \in Q(\lambda)} \mathbb{E}^q[\sum_{t=0}^{T} F_t e_t].$$

Then, maximizing with respect to e, for the buyer's price of the ACC we have

$$\max_{e \in E} \min_{q \in Q(\lambda)} \mathbb{E}^q[\sum_{t=0}^{T} F_t e_t].$$

The correspondence between stopping times and the process $e \in E$ implies the first expression in the theorem. By Theorem 1, instead of last expression we can use

$$\max_{e \in \tilde{E}} \min_{q \in Q(\lambda)} \mathbb{E}^q[\sum_{t=0}^{T} F_t e_t].$$

Since $\tilde{E}$ and $Q(\lambda)$ are bounded convex sets, by Corollary 37.6.1 of [7] we can change the order of max and min without changing the value. Then, for each fixed $q \in Q(\lambda)$, the objective is linear in e. So the maximum over $\tilde{E}$ is attained at an extreme point of $\tilde{E}$. We know that extreme points of $\tilde{E}$ are the elements of the sets E. Thus, we get the second expression in the theorem. ∎

We can extend our results for stocks that pay dividends or interest payments. We assume that there is no dividend associated with S^0.

Corollary 1 *If each security $j = 1, \ldots, J$ pays dividend payments D_n^j in node n, under the assumption of no good-deals in the market price process, the buyer's price for American contingent claim F can be expressed as*

$$\max_{\tau \in \mathcal{T}} \min_{q \in Q'(\lambda)} \mathbb{E}^q[F_\tau] = \min_{q \in Q'(\lambda)} \max_{\tau \in \mathcal{T}} \mathbb{E}^q[F_\tau]$$

where

$$Q'(\lambda) = \{q \mid q_0 = 1,\ q_n S_n = \sum_{m \in \mathcal{C}(n)} q_m(S_m + D_m),\ \forall n \in \mathcal{N} \backslash \mathcal{N}_T;\ p_n \leq q_n \leq \lambda p_n,\ \forall n \in \mathcal{N}_T\}.$$

Proof: If dividends exist, self-financing constraints of P1 becomes

$$S_n \cdot (\theta_n - \theta_{\pi(n)}) - D_n \cdot \theta_{\pi(n)} = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \le t \le T$$

The rest of the argument, including the proof of Theorem 1 follows as it is in the case of stocks without dividends. ∎

Appendix

We will give a proof of Theorem 3 of [5]. An arbitrage seeking buyer's problem can be formulated as the following problem that we will refer as AP1.

$$\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \le t \le T \\
& S_n \cdot \theta_n \ge 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n)} e_m \le 1, \ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0,1\}, \ \forall\, n \in \mathcal{N}.
\end{aligned}$$

A linear programming relaxation of AP1 is the following problem AP2:

$$\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \le t \le T \\
& S_n \cdot \theta_n \ge 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n)} e_m \le 1, \ \forall\, n \in \mathcal{N}_T \\
& e_n \ge 0, \ \forall\, n \in \mathcal{N}.
\end{aligned}$$

Theorem 3 *There exists an optimal solution to AP2 with* $e_n \in \{0,1\}, \ \forall\, n \in \mathcal{N}$.

Proof: Assume that AP2 has an optimal solution V^*, e^* and θ^* such that $e_n^* \notin \{0,1\}$ for some $n \in \mathcal{N}$.

<u>**Case 1**</u>**:** We will first consider the case where e^* has a value not equal to 0 or 1 for the root, which is the starting node of the tree (i.e. $e_0^* \notin \{0,1\}$). In order to deal with this case, we will form the Lagrangian function for AP2. That is

$$\begin{aligned}
L(V,e,\theta,x,y,z) \quad = \quad & V - y_0[S_0 \cdot \theta_0 - F_0 e_0 + V] - \sum_{n \in \mathcal{N} \setminus \{0\}} y_n [S_n \cdot (\theta_n - \theta_{\pi(n)}) - F_n e_n] \\
& + \sum_{n \in \mathcal{N}_T} x_n S_n \cdot \theta_n - \sum_{n \in \mathcal{N}_T} z_n [\sum_{m \in \mathcal{A}(n)} e_m - 1]
\end{aligned}$$

After rearranging the above function we have

$$\begin{aligned} L(V,e,\theta,x,y,z) &= (1-y_0)V + \sum_{n\in\mathcal{N}_T}(x_n-y_n)S_n\cdot\theta_n + \sum_{n\in\mathcal{N}\backslash\mathcal{N}_T}\theta_n\cdot[\sum_{m\in\mathcal{C}(n)}y_mS_m - y_nS_n] \\ &\quad + \sum_{n\in\mathcal{N}}[y_nF_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T}z_m]e_n + \sum_{n\in\mathcal{N}_T}z_n \end{aligned}$$

Then the dual problem of AP2 can be formulated as

$$\begin{aligned} \min \quad & \sum_{n\in\mathcal{N}_T} z_n \\ \text{s.t.} \quad & y_0 = 1 \\ & [x_n-y_n]S_n = 0,\ \forall\, n\in\mathcal{N}_T \\ & \sum_{m\in\mathcal{C}(n)} y_mS_m = y_nS_n,\ \forall\, n\in\mathcal{N}\backslash\mathcal{N}_T \\ & y_nF_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \le 0,\ \forall\, n\in\mathcal{N} \\ & x_n, z_n \ge 0,\ \forall\, n\in\mathcal{N}_T \end{aligned}$$

Since $S_n \neq 0$, second constraint implies that $x_n = y_n$, $\forall\, n\in\mathcal{N}_T$. Thus the dual problem can be rearranged as

$$\begin{aligned} \min \quad & \sum_{n\in\mathcal{N}_T} z_n \\ \text{s.t.} \quad & y_0 = 1 \\ & \sum_{m\in\mathcal{C}(n)} y_mS_m = y_nS_n,\ \forall\, n\in\mathcal{N}\backslash\mathcal{N}_T \\ & y_nF_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \le 0,\ \forall\, n\in\mathcal{N} \\ & y_n, z_n \ge 0,\ \forall\, n\in\mathcal{N}_T. \end{aligned}$$

We have an optimal solution to AP2 with $e_0^* \notin \{0,1\}$. Then complementary slackness implies that third constraint of the above program should be satisfied as an equality for the corresponding optimal solution of the dual problem (i.e., $y_0F_0 - \sum_{m\in\mathcal{N}_T} z_m = 0$). Since $y_0 = 1$, we have $F_0 = \sum_{m\in\mathcal{N}_T} z_m$. Thus, the optimal solution to the dual problem is found to be F_0. Then, by strong duality we know that F_0 is the optimal value of AP2. One can easily check that $e_0 = 1$, $V = F_0$ and all the other variables as zeros (θ_n's as zero vectors) constitute a feasible solution to AP2 with objective value F_0. This is an optimal solution with $e_n \in \{0,1\}$, $\forall\, n\in\mathcal{N}$, thus the proof for the first case is complete.

Case 2: Now assume that optimal solution e^* is such that $e_0^* = 0$ and $e_n^* \notin \{0,1\}$ for some $n\in\mathcal{N}$. Let $I = \{i|e_i^* \notin \{0,1\}, i\in\mathcal{N}\}$. Let $G = \{g|g\in I, \mathcal{A}(g)\cap I = \{g\}\}$. Let w be the element with the smallest time index (that is closest to the root) in G. Note that $e_n^* = 0$, $\forall\, n\in\mathcal{A}(w)\backslash\{w\}$in this case. Also, let k denote the time index for node w.

Claim: One can always find an optimal solution to AP2 with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i\in\mathcal{A}(w)\backslash\{w\}$.

To prove the claim we will consider the two following linear programs to which we will refer as AR1 and AR2 respectively:

$$\begin{aligned}
\max \quad & e_w \\
\text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n \geq 0, \ \forall\, n \in \mathcal{D}(w),
\end{aligned}$$

$$\begin{aligned}
\min \quad & e_w \\
\text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\
& e_n \geq 0, \ \forall\, n \in \mathcal{D}(w).
\end{aligned}$$

Let us denote the optimal solution of AR1 as $\bar{\theta}_{\mathcal{D}(w)}$, $\bar{e}_{\mathcal{D}(w)}$ and to AR2 as $\tilde{\theta}_{\mathcal{D}(w)}, \tilde{e}_{\mathcal{D}(w)}$. If the optimal value of AR1 is 1, then we see that $(\bar{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\bar{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\backslash\mathcal{D}(w)})$ form another optimal solution of AP2 with $e_w = 1$. For this optimal solution we have $e_w = 1$ and $e_i = 0, \ \forall\, i \in \mathcal{A}(w)\backslash\{w\}$ (we have also $e_i = 0$, for all $i \in \mathcal{D}(w)\backslash\{w\}$ for this solution). Similarly, if the optimal value of AR2 is 0, then $(\tilde{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\tilde{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\backslash\mathcal{D}(w)})$ form another optimal solution of AP2 with $e_w = 0$. Then, for this optimal solution we have $e_i = 0$, for all $i \in \mathcal{A}(w)$. So, our claim will be proved if we can show that AR2 having an optimal value greater than 0 implies that optimal value of AR1 is 1. To show that we will consider the dual problems of AR1 and AR2. The dual problems DAR1 and DAR2 of AR1 and AR2 are

$$\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n \geq 1 \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& y_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w),
\end{aligned}$$

$$
\begin{aligned}
\max \quad & -\sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n - y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m\in\mathcal{C}(n)} y_m S_m \;=\; y_n S_n,\ \forall\, n\in\mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n \;\geq\; -1 \\
& y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \;\leq\; 0,\ \forall\, n\in\mathcal{D}(w)\backslash\{w\} \\
& y_n, z_n \;\geq\; 0,\ \forall\, n\in\mathcal{N}_T\cap\mathcal{D}(w).
\end{aligned}
$$

We will denote the optimal value of AR2 by α, which is equal to the optimal value of DAR2. We know that $\alpha \leq 1$. Assume that $\alpha > 0$. Then by complementary slackness we know that the second constraint of DAR2 must be satisfied as an equality at the corresponding optimal solution, since $e_w \neq 0$ at the optimal solution of AR2. Then at the optimal solution of DAR2, we have

$$
0 > \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n = -1 \tag{3}
$$

Moreover, we must have $y_w \geq 0$ for any feasible solution of DAR1 and DAR2. This follows from the following fact. We have $y_n \geq 0$, $\forall n \in \mathcal{N}_T \cap \mathcal{D}(w)$. Then, since $S^0_n > 0$ for all n, we have $y_n \geq 0$, $\forall n \in \mathcal{N}_{T-1} \cap \mathcal{D}(w)$ by the first constraints of DAR1 and DAR2. Similarly, we can show the same successively for $(T-2), (T-3), \ldots, k$. So, we have $y_w \geq 0$. Then, using the second inequality of (3) we have

$$
\begin{aligned}
\sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} &\geq -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n \\
y_w S_w \cdot \theta^*_{\pi(w)} &\geq -y_w F_w \\
S_w \cdot \theta^*_{\pi(w)} &\geq -F_w
\end{aligned}
$$

where last step follows from $y_w \geq 0$. Then, for DAR1 at any feasible solution we have

$$
1 \leq -y_w F_w + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n \leq y_w S_w \cdot \theta^*_{\pi(w)} + \sum_{n\in\mathcal{N}_T\cap\mathcal{D}(w)} z_n
$$

whence we see that optimal solution of DAR1 cannot be less than 1. It is easy to see by AR1 that optimal value of DAR1 cannot be greater than 1 either. Hence, we conclude that the optimal value of DAR1. and therefore that of AR1, is 1. This completes the proof of our claim.
Using the claim we see that there always exists an optimal solution to AP2 with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w)$. So, one can eliminate all the nodes having time index k in I by applying the above procedure. Then, proceeding successively with the nodes in $(k+1)^{st}, (k+2)^{nd} \ldots (T)^{th}$ time indices one can find an optimal solution for AP2 with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$. We note that, at each step the size of I might increase, but no nodes with a time index less than or equal to that of the node eliminated at that particular step can show up in I at the next step. This completes the proof of the theorem. ∎

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