Spectral Properties of Trinomial Trees

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Background

- Binomial and trinomial trees are central to the theory and practice of mathematical finance (Cox, Ross & Rubinstein 1979).
- The prices obtained from binomial and trinomial models converge to the prices given by continuous-time and -space models (see e.g. (He 1990) and (Amin & Khanna 1994)).
- The *rate* of convergence has been examined in (Heston & Zhou 2000): for a call the rate is at least of the order $O(1/\sqrt{n})$, where n is the number of time-steps in the binomial model.
- If h is the lattice spacing, this is equivalent to $O(h)$ (we shall see later that $n = \frac{3\sigma^2 T}{h^2}$).

Background

(Heston & Zhou 2000) suggest smoothing the payoff to improve convergence.

Proposition 1 *Let the function g be in $C^2(\mathbb{R})$ and let $C_h(g)$ and $C(g)$ be the discrete and continuous-time prices of the contingent claim g at some expiry T. Then the following equality holds:*

$$C_h(g) = C(g) + O(h^2).$$

- Key assumption: payoff $g : \mathbb{R} \to \mathbb{R}$ must be in $C^2(\mathbb{R})$.
- This approach works well for vanilla payoffs because they are non-differentiable only at a single point.
- It works less well for European double digitals and butterfly spreads.

Background

- The probability mass function of a random walk on a trinomial tree can be viewed as the forward price of an Arrow-Debreu security.
- The probability density of a smooth process can be viewed as a forward price of a security whose payoff is the Dirac delta function.
- The singular nature of the latter payoff makes it difficult to find the correct convergence rates.

Aim

- We address the question of the convergence of the probability mass function of a random walk on a commonly used trinomial tree to the probability density of a Brownian motion with drift.
- The main result: the convergence is of the order $O(h^4)$, where h represents the distance between the nodes of the tree at any time-step.
- This convergence rate is optimal for the tree and is uniform in the state-space.
- Recall that the rate in proposition 1 for the C^2 payoffs was $O(h^2)$, which implies that the standard smoothing techniques cannot yield the correct convergence rate.

Definition of the tree

Task: approximate $Y_T := x + \mu T + \sigma W_T$, where $x \in \mathbb{R}$ is a starting point, $\mu, \sigma \in \mathbb{R}$, such that $\sigma > 0$, and T time horizon, by

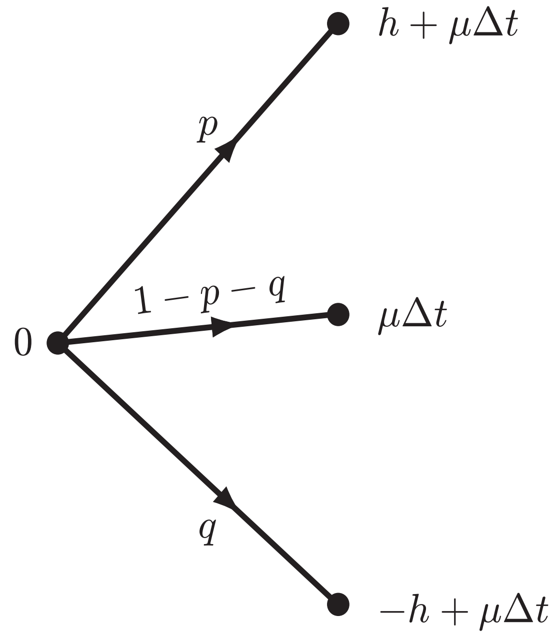
$$Y_T^h := x + \sum_{i=1}^{n} X_i,$$

where Δt is the size and $n := \frac{T}{\Delta t}$ is the number of time-steps.

The variables X_i are iid with a domain $\{-h + \mu\Delta t, \mu\Delta t, h + \mu\Delta t\}$, where h denotes the distance between the consecutive nodes in the tree.

Local geometry of the tree

The variable X_i is used to approximate the evolution of the process $\mu t + \sigma W_t$ in the short time interval of length Δt.



The local geometry (i.e. the lattice spacing h) and the probabilities p, q are chosen so that the moment-matching conditions hold:

$$\mathbb{E}_t\left[(Y_{t+\Delta t} - Y_t - \mu\Delta t)^k\right] = \mathbb{E}[(X_i - \mathbb{E}[X_i])^k],\, t = i\frac{\Delta t}{T},\, k \in \{1, \ldots, 5\}.$$

Local geometry of the tree

The moment-matching conditions yield the following identities:

$$h^2 = 3\sigma^2 \Delta t, \quad p = q = \frac{1}{6}.$$

The number of steps in the tree $n = \frac{T}{\Delta t}$ can now be seen to satisfy

$$h^2 = \frac{3\sigma^2 T}{n}.$$

For a fixed Δt we have just defined a discrete-time Markov chain Y_t^h where every time t is an integer multiple of Δt.

Generator of the random walk

The domain of the chain Y_t^h, at time t, is a subset of $\mu t + h\mathbb{Z}$ if and only if the starting point x lies in $h\mathbb{Z}$.

The generator of the chain Y_t^h can be viewed as a bounded operator $\mathbf{P}_{\Delta t}^h : l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})$ defined on the orthonormal basis $(\delta_y)_{y\in h\mathbb{Z}}$ of the Hilbert space $l^2(h\mathbb{Z})$ as follows:

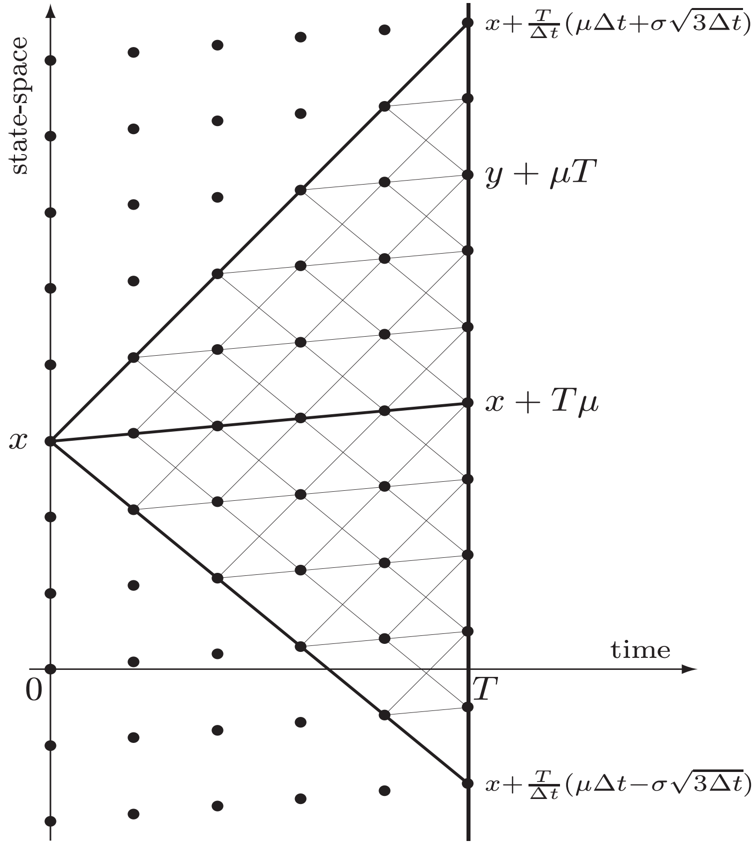
$$\mathbf{P}_{\Delta t}^h(\delta_y)(x) := \mathbb{P}(Y_{\Delta t}^h = \mu\Delta t + y | Y_0^h = x).$$

In coordinates $x, y \in h\mathbb{Z}$, the generator can be expressed as

$$\mathbf{P}_{\Delta t}^h(x,y) := \mathbf{P}_{\Delta t}^h(\delta_y)(x) = \frac{1}{6}\begin{cases} 1 & \text{if } |y-x| = h, \\ 4 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Global geometry of the tree

The probability mass function of the chain Y_t^h over the time interval $[0, T]$ is given by $p_T^h(x, y) := \mathbb{P}(Y_T^h = \mu T + y | Y_0^h = x), \quad x, y \in h\mathbb{Z}$.



Therefore $p_T^h(x, y) = (\mathbf{P}_{\Delta t}^h)^n(\delta_y)(x)$, where $n = \frac{3\sigma^2 T}{h^2}$.

Discrete Laplace operator

The bounded linear operator $\Delta_h : l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})$ defined by

$$\Delta_h f(x) := \frac{f(x+h) + f(x-h) - 2f(x)}{h^2},$$

where $x \in h\mathbb{Z}$, is known as the *discrete Laplace operator*.

The identity

$$\mathbf{P}^h_{\Delta t} = I + \frac{\sigma^2 \Delta t}{2} \Delta_h$$

is key for the proof of the convergence result because

- the spectra of the generator $\mathbf{P}^h_{\Delta t}$ and of Δ_h are related by an affine transformation and
- a *spectral decomposition* of Δ_h yields a spectral decomposition for the generator $\mathbf{P}^h_{\Delta t}$.

Continuous-time Markov chain

A bounded operator $\mathcal{L}_h : l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})$, given by

$$\mathcal{L}_h := \frac{\sigma^2}{2}\Delta_h,$$

can be used to define a *continuous-time Markov chain* on $h\mathbb{Z}$.

The probability kernel $\mathbf{Q}_t^h$ of the chain is given by the backward Kolmogorov equation $\frac{\partial \mathbf{Q}_t^h}{\partial t} = \mathcal{L}_h \mathbf{Q}_t^h$, $\mathbf{Q}_0^h = I$. The explicit solution is given by

$$\mathbf{Q}_t^h = \exp(t\mathcal{L}_h).$$

Notice that $\mathbf{P}_{\Delta t}^h$ is the “first order” approximation of $\mathbf{Q}_{\Delta t}^h$.

Spectrum of the discrete Laplace operator

The *semidiscrete Fourier transform* is a unitary operator $\mathcal{F}_h : l^2(h\mathbb{Z}) \to L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$, defined by

$$\mathcal{F}_h(f)(p) := \sqrt{\frac{h}{2\pi}} \sum_{m\in\mathbb{Z}} f(hm)e^{-imhp}.$$

It maps any element of the basis $(\delta_y)_{y\in h\mathbb{Z}}$ to the function $p \mapsto \sqrt{\frac{h}{2\pi}} e^{-imhp}$ in $L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$, where $y = mh$.

Its inverse is a unitary transformation $\mathcal{F}_h^{-1} : L^2([-\frac{\pi}{h}, \frac{\pi}{h}]) \to l^2(h\mathbb{Z})$ given by the following formula for $\phi \in L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$:

$$\mathcal{F}_h^{-1}(\phi)(hm) := \sqrt{\frac{h}{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \phi(p)e^{imhp}dp.$$

Spectrum of the discrete Laplace operator

The following calculation in the Hilbert space $L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$ yields a spectral representation of the discrete Laplace operator Δ_h:

$$
\begin{aligned}
\mathcal{F}_h \Delta_h(f)(p) &= \sum_{m \in \mathbb{Z}} \frac{f(h(m+1)) + f(h(m-1)) - 2f(hm)}{h^2} \sqrt{\frac{h}{2\pi}} e^{-imhp} \\
&= \sqrt{\frac{h}{2\pi}} \sum_{m \in \mathbb{Z}} f(hm) e^{-imhp} \frac{e^{ihp} + e^{-ihp} - 2}{h^2} \\
&= \frac{2(\cos(hp) - 1)}{h^2} \mathcal{F}_h(f)(p), \qquad \text{for } p \in [-\frac{\pi}{h}, \frac{\pi}{h}].
\end{aligned}
$$

Therefore we get

$$
\mathcal{F}_h (\mathbf{P}^h_{\Delta t})^n \mathcal{F}_h^{-1}(\phi)(p) = \left(1 + \sigma^2 \Delta t \frac{\cos(hp) - 1}{h^2}\right)^n \phi(p).
$$

Spectral representation of the trinomial tree

Proposition 2 *The spectral representation of the probability kernel $p_T^h(x,y)$ is given by the integral*

$$p_T^h(x,y) = \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\frac{2}{3} + \frac{1}{3}\cos(hp)\right)^{\frac{3\sigma^2 T}{h^2}} e^{ip(x-y)} dp,$$

for any pair of elements x, y in $h\mathbb{Z}$.

Proof

$$\begin{aligned} p_T^h(x,y) &= (\mathbf{P}_{\Delta t}^h)^n(\delta_y)(x) \\ &= \mathcal{F}_h^{-1}\left(\left(1+\sigma^2\Delta t\frac{\cos(hp)-1}{h^2}\right)^n \sqrt{\frac{h}{2\pi}}e^{-ipy}\right)(x). \end{aligned}$$

□

Spectral representation of the Markov chain

We know the following:

- the probability kernel is given by $\mathbf{Q}_T^h = \exp(T\mathcal{L}_h)$ and
- $\mathcal{F}_h\mathcal{L}_h\mathcal{F}_h^{-1}(\phi)(p) = \sigma^2\frac{\cos(hp)-1}{h^2}\phi(p)$, for all $\phi \in L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$.

The spectral representation for the continuous-time Markov chain therefore equals

$$\mathbf{Q}_T^h(x,y) \quad = \quad \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{T\sigma^2\frac{\cos(hp)-1}{h^2}} e^{ip(x-y)}dp.$$

Probability density for Brownian motion with drift

The transition density $p_T(x, y)$ of the process $Y_t = x + \mu t + \sigma W_t$ has a spectral representation of the form:

$$p_T(x, y) := \mathbb{P}(Y_T = \mu T + y | Y_0 = x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{p^2}{2}\sigma^2 T} e^{ip(x-y)} dp.$$

This is a consequence of the spectral decomposition of the Laplace operator given by Fourier transforms.

Technical point

The limiting procedure $h \to 0$ involves a passage from the discrete state-space $h\mathbb{Z}$ to the continuum of $\mathbb{R}$.

Fix a sequence of positive real numbers $(h_n)_{n\in\mathbb{N}}$ with the following two properties:

$$\lim_{n\to\infty} h_n = 0 \quad \text{and} \quad \frac{h_i}{h_j} \in \mathbb{N} \text{ for all } j \geq i.$$

The second property ensures that $h_j\mathbb{Z} \supseteq h_i\mathbb{Z}$, when $j \geq i$.

For example choose $h_n := (\frac{1}{2})^n$.

Main result

Theorem 3 (M) *For any $x, y \in h\mathbb{Z}$ the following holds*

$$p_T(x, y) = \frac{1}{h} p_T^h(x, y) + O(h^4)$$

and the error term $O(h^4)$ is independent of x and y. Equivalently there exist positive constants C and δ such that the inequality $|p_T(x, y) - \frac{1}{h} p_T^h(x, y)| \leq Ch^4$ holds for all $h < \delta$ and all $x, y \in h\mathbb{Z}$. Furthermore the probability kernel $\frac{1}{h} p_T^h(x, y)$ does NOT converge to the density $p_T(x, y)$ at the rate $O(h^4 f(h))$, for any non-constant function f such that $\lim_{h \to 0} f(h) = 0$.

Continuous-time theorem

Theorem 4 (Albanese, M) *For any $x, y \in h\mathbb{Z}$ the following holds*

$$p_t(x,y) = \frac{1}{h}\mathbf{Q}_t^h(x,y) + O(h^2),$$

where $\mathbf{Q}_t^h$ is the probability kernel of the continuous-time Markov chain with the infinitesimal generator $\mathcal{L}_h = \frac{\sigma^2}{2}\Delta_h$. The error term $O(h^2)$ is independent of x and y and the convergence rate is optimal in the same sense as in theorem 3.

Idea of the proof of theorems 3 and 4

Define $f_h(p) := \frac{3}{h^2}\left(\log(2+\cos(hp)) - \log(3)\right)$ and compare the probability kernels

$$p_T^h(x,y) = \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{f_h(p)\sigma^2 T} e^{ip(x-y)}\,dp,$$

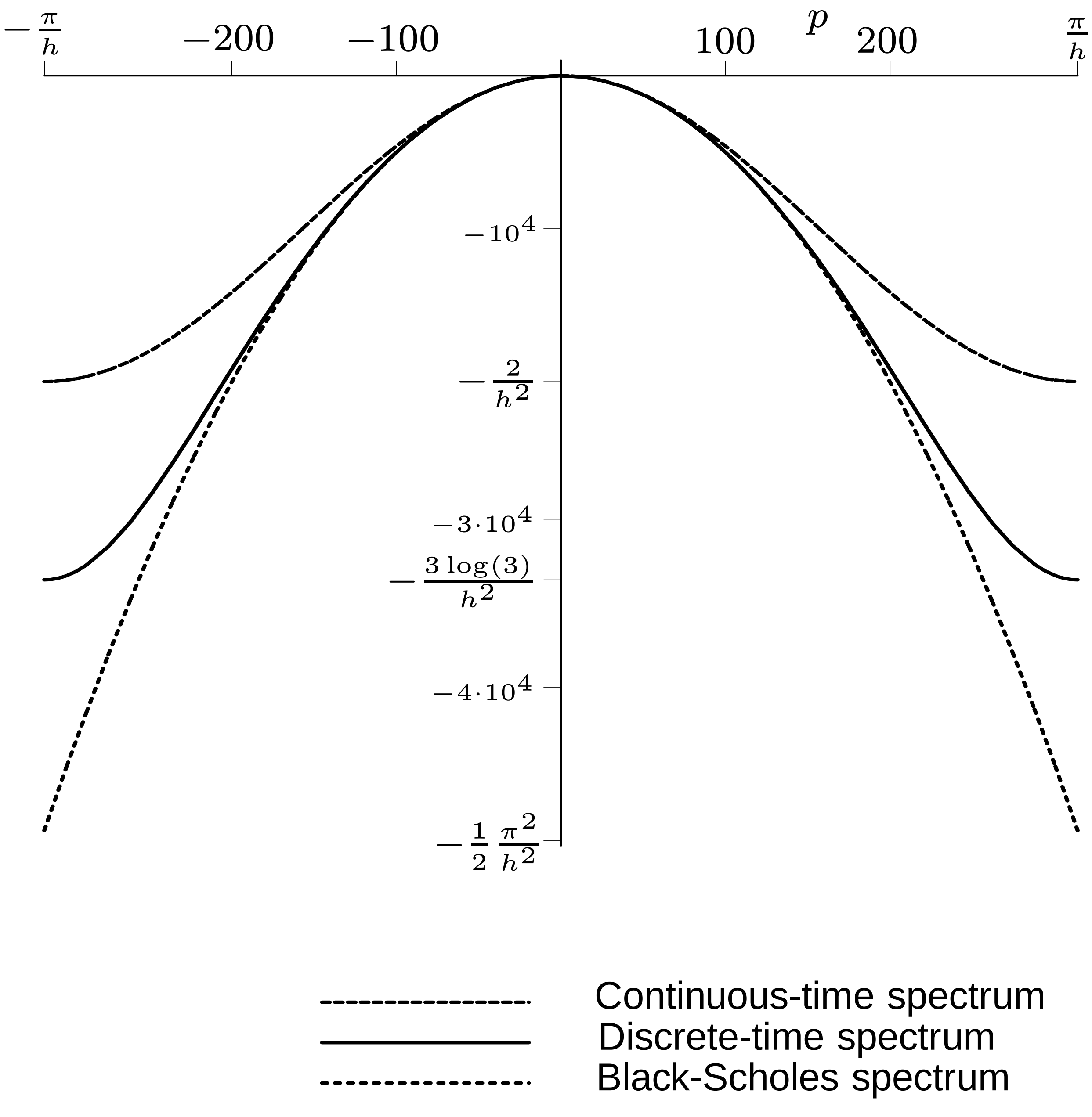
$$\mathbf{Q}_T^h(x,y) = \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{T\sigma^2 \frac{\cos(hp)-1}{h^2}} e^{ip(x-y)}\,dp.$$

with the probability density of the continuous state-space models

$$p_T(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{p^2}{2}\sigma^2 T} e^{ip(x-y)}\,dp.$$

The problem reduces to the comparison of the integrals of $p \mapsto f_h(p)$, $p \mapsto \frac{\cos(hp)-1}{h^2}$ and $p \mapsto -\frac{p^2}{2}$.

The spectra



$-\frac{\pi}{h}$
-200
-100
100
p
200
$\frac{\pi}{h}$
-10^4
$-\frac{2}{h^2}$
$-3\cdot 10^4$
$-\frac{3\log(3)}{h^2}$
$-4\cdot 10^4$
$-\frac{1}{2}\frac{\pi^2}{h^2}$
Continuous-time spectrum
Discrete-time spectrum
Black-Scholes spectrum

Uniform comparison on $\mathbb{R}$ is not feasible.

We can decompose the integration domains in the following way: $\mathbb{R} = [-K(h), K(h)] \cup (\mathbb{R} - [-K(h), K(h)])$, where

$$K(h) := \frac{1}{\sigma\sqrt{T}}\sqrt{10\log(1/h)}.$$

In the figure above $h = 100^{-1}$ and $K(h) < 7$.

- On the interval $[-K(h), K(h)]$ the functions $f_h(p)$, $-\frac{p^2}{2}$ are very close to each other and
- on its complement there is exponential damping because the spectra are negative.

Future research

- Can this approach be generalized to n dimensions?
- How do discretizations of Lévy process converge to the continuous state-space limit?
- Can this be done for diffusions with non-constant coefficients?
- ...

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Also available at `http://qfrmc.imaa.ic.ac.uk/~amijatov/`