On Measuring the Degree of Market Efficiency

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Abstract

Measures of the degree of market efficiency are obtained via indices of acceptability for cash flows. Measures of the acceptability of cash flows are constructed using a sequence of convex cones containing the positive orthant and converging to it as the measure rises to infinity. We then focus on law invariant examples. Four new examples are provided by the measures $MINVAR$, $MAXVAR$, $MAXMINVAR$, and $MINMAXVAR$. The $MINVAR$ measure determines acceptability by evaluating the expectation of the minimum of a large number of draws from the distribution. The $MAXVAR$ measure writes the cash flow distribution as that of the maximum of a large number of draws from another distribution and evaluates the expectation of the latter distribution. The last two measures combine these approaches to constructing worst case scenarios. Computations of all measures are illustrated for simple strategies of writing options, holding them to maturity unhedged and paying out the required cash flows. The underliers used are the SPX and the $FTSE$.

We wish to measure the level of acceptability of trading strategies and thereby the efficiency of markets. Further we recognize that arbitrages are the best trades, but note also that an economy that merely excludes arbitrages but permits outcomes in the close proximity of arbitrages is not really very efficient. The mere exclusion of arbitrages may be viewed as a zero level of efficiency. A measure of the degree of efficiency is then axiomatized with concrete examples constructed in terms of measures of the distance of trades from arbitrage.

There is a large empirical literature testing the hypothesis that markets are efficient (see, for example, Fama and French (1992), Campbell, Lo, MacKinlay, and Whitelaw (1997)). This is done by recognizing that under efficient markets asset prices are martingales under a change of measure. One then tests the martingale property or the absence of abnormal returns for a specific measure change defined by an asset pricing theory. We then learn on conducting the test if particular trades provide evidence of inefficiency. What we have in fact is evidence of inefficiency relative to a selected asset pricing model. There is then a positive excess return from the perspective of a single pricing functional and from the view point of acceptability considered here, this is very weak support for a good trade or an acceptable cash flow. When judged by a single functional, the cone of acceptability is very wide and is in fact a half space.

From another perspective (Jacod and Shiryaev (1998)) we know that we have market efficiency just if zero is an interior outcome for the set of possible asset price moves. When zero is an interior point, one may construct a measure under which we have the martingale property. Describing this measure may be quite a complex task. In this sense we appreciate that mere market efficiency is quite a weak hypothesis and is probably satisfied in all liquid markets. The real question is not that of assessing market efficiency, but assuming market efficiency and the consequent exclusion of arbitrages, we need to understand better the degree of efficiency in markets. The exclusion of arbitrages is a zero level of efficiency. Markets that are more efficient than this zero level of efficiency, exclude trading opportunities that are at some distance from arbitrage. It is this distance we seek to understand and measure.

Much of the literature on market anomalies, and we cite Fama and French (1996) for an example, also evaluates potential trades from the limited view of acceptability as a high alpha strategy. These strategies may not be very good at all if all we know is that they provide us with a cash flow in a half space containing the positive orthant. What is of interest is the structure of the intersection of half spaces in which the strategy has detected a resulting cash flow. We note in this regard that the use of benchmarks (Kent, Grinblatt, Titman and Wermers (1997)) is a step in the direction of a conic view of acceptability provided they are used simultaneously as opposed to one at time. Anomalies could usefully be tested for their ability to attain high levels for the indices of acceptability that we introduce in this paper. As will become clear, high levels for such indices bring us closer to an arbitrage.

The first measure that one is accustomed to consider in connection with good trades is the Sharpe ratio (Sharpe (1964)). A high Sharpe ratio has traditionally been taken as an indicator of a good deal (see, for example, Hansen and Jagannathan (1991), Cochrane and Saa-Requejo (2000)). But as pointed out by Bernando and Ledoit (2000), this measure does not really respect arbitrage and can be zero for an arbitrage when we have a positive cash flow with a finite mean and an infinite variance. This led to the proposal by Bernando and Ledoit (2000) for the Gain-Loss ratio as a measure of performance. This ratio goes to infinity for arbitrages and its reciprocal may be a measure of distance from arbitrage.

A discomfort with the Gain-Loss ratio is that it treats small losses and large losses symmetrically. To rectify this situation we considered the tilt coefficient that essentially computed the highest level of risk aversion for exponential utility such that the exponentially tilted mean is still positive. Needless to say, the motivation for incorporating such an exaggeration of losses relative to a simultaneous deflation of gains is rooted in economic principles advocating weightings proportional to marginal utility with a similar effect.

The more recently developed theory of acceptable risks (Artzner, Delbaen, Eber, and Heath (1999), Carr, Geman, and Madan (2001)) on the other hand conceptualizes acceptable risks in terms of a convex cone or set containing the positive orthant. We show that the tilt coefficient as described above does not yield a convex set of acceptable risks. We recognize that the tilt coefficient is a preliminary approach to using risk aversion as a measure of acceptability, and this is here refined to levels of an index of acceptability. Our indices are related to coherent utilities that are supported by a set of measures defining a cone of acceptability. The extreme measures of this set form the counterpart of state-price densities in classical expected utility theory. We choose between indices on the basis of the properties of densities associated with these extreme measures.

The indices of acceptability introduced here may also be differentiated from more classical performance measures that are typically constructed as ratios of reward to risk (see for example Biglova, Ortobelli, Rachev and Stoyanov (2004), Rockafellar and Uryasev (2002)). In contrast, our indices may be viewed as coherent analogs of the tilt coefficient and thereby are more closely related to the intuitions embedded in classical economics.

We formulate a set of axioms for indices of acceptability and characterize the entire class of such indices. A particularly important subclass of indices is given by the characterization of law invariant cones of acceptability by Kusuoka (2001). We provide a number of explicit and tractable examples in this subclass. To develop an appreciation of the values attainable in the economy we study these measures for a variety of simple trades in options markets and report the results.

Four new law invariant measures are introduced, and they are termed $MINVAR$, $MAXVAR$, $MAXMINVAR$, and $MINMAXVAR$. $MINVAR$ constructs a worst case scenario by forming the expectation of the minimum of numerous draws from the cash flow distribution. $MAXVAR$ constructs a distribution from which one draws numerous times and takes the maximum to get the cash flow distribution being evaluated. The last two measures $MINMAXVAR$ and $MAXMINVAR$ combine these approaches to constructing worst case scenarios. For each of these measures we describe the determining set of measures and construct the extreme measures that constitute the equivalent of the state-price densities. The last two measures share the property with classical state-price densities of tending to infinity and zero as losses tend to negative infinity or gains tend to infinity.

The outline of the paper is as follows. Section 1 develops our axiomatic structure for the class of performance measures. The characterization theorem is presented in Section 2. Section 3 considers examples of popular measures as well as new measures motivated by our axiomatic structure and describes the properties satisfied by these measures. Section 4 describes the data set used to evaluate the performance measures of simple trading strategies available in the economy. The results of evaluating these measures for our simple trades are also presented in Section 4. Section 5 concludes.

1 Axiomatizing Performance Measures

We consider here the axioms to be satisfied by any measure of trading performance. The trading opportunities are modeled by the access to a space of random variables defined on a standard probability space $(\Omega, \mathcal{F}, P)$. In this regard we follow Artzner, Delbaen, Eber, and Heath (1999) in restricting the domain of application of our measures to an economically relevant set of alternatives. In particular we exclude arbitrary lotteries that combine random variables into hypothetical gambles that are not available as outcomes of realistic trading strategies.

To avoid technicalities associated with the finiteness of moments, we restrict attention to the class of bounded random variables given by $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$. Our measure of performance, that we refer to here as an acceptability index, is a real valued map α from L^∞ to the extended positive reals $[0, \infty]$. For a random variable $X \in L^\infty$ meaning the cash flow of a transaction, the acceptability index of X is then $\alpha(X)$. We consider eight properties that such an acceptability index should satisfy and discuss these in turn in short subsections of this section. The first four properties define what we term to be a coherent acceptability index. The remaining properties are additional properties that enable us to provide a number of tractable examples of coherent acceptability indices.

1.1 Convexity

The set of trades acceptable at level x is naturally defined as

$$\mathcal{A}_x = \{X : \alpha(X) \geq x\}.$$

We require this set to be a convex set, and we shall in fact end up requiring it to be a convex cone. Acceptable opportunities have already been characterized in Artzner, Delbaen, Eber, and Heath (1999) and Carr, Geman, and Madan (2001) as convex cones or sets containing the positive orthant. We now introduce the idea of levels of acceptability indexed by the real number x, but keep the basic property of convexity for all the levels.

This condition then requires the function α to satisfy the condition that

$$\text{if } \alpha(X) \geq x \text{ and } \alpha(Y) \geq x, \text{ then } \alpha(\lambda X + (1-\lambda)Y) \geq x \text{ for any } \lambda \in [0,1].$$

1.2 Monotonicity

A basic property of acceptability is monotonicity. This is the condition that if X is acceptable and Y dominates X as a random variable, then Y is acceptable. We require this condition for all the acceptability levels, and so we must have the property

$$\text{if } X \leq Y, \text{ then } \alpha(X) \leq \alpha(Y).$$

1.3 Scale Invariance

There is much debate on whether sets of acceptability should be convex sets or convex cones. The axiomatization in Artzner, Delbaen, Eber, and Heath (1999) opted for a conic structure, while Carr, Geman, and Madan (2001), Föllmer and Schied (2002), and Frittelli and Rosazza Gianin (2002) argued for convex sets. When designing or evaluating trades, there are two considerations involved, the direction of the trade and its size or scale. A number of issues impact the scale of trades and include the level of personal risk aversion, the depth of the market, and the resulting impact of the trade on the terms of trade, or the wealth or borrowing ability of the individual trading. Some of these considerations are market based, while others are personalized. The direction of trade is presumably a relatively more objective consideration with the size then left to more personal matters. Additionally many of the examples of early measures of performance like the Sharpe ratio or the Gain-Loss ratio are scale invariant.

We consider here measures of performance that are scale invariant by requiring our sets of acceptability at all levels to be convex cones. Equivalently we require that

$$\alpha(\lambda X) = \alpha(X) \text{ for } \lambda > 0.$$

1.4 Fatou Property

This is a continuity or closure property and is a relatively technical condition. For any countable collection of random variables X_n with $|X_n| \leq 1$ such that $\alpha(X_n) \geq x$, we require that if X_n converges to X in probability, then $\alpha(X) \geq x$.

These are the four basic properties we require of all coherent acceptability indices. In addition we consider four other properties useful in constructing practical examples of such indices. The first additional property is that of law invariance. Under this condition we may invoke the characterization of law invariant risk measures by Kusuoka (2001) to construct our examples. Furthermore, we note that early performance measures like the Sharpe ratio and the Gain-Loss ratio are law invariant in that the measure depends only on the probability law of the random variable.

1.5 Law Invariance

This property requires that

$$\text{if } X \overset{law}{=} Y, \text{ then } \alpha(X) = \alpha(Y),$$

where $X \overset{law}{=} Y$ means that X and Y have the same probability distribution.

1.6 Second Order Monotonicity

Of interest in defining acceptability indices is the issue of consistency with expected utility theory. Recall that Y dominates X in the second order $\left(X \preceq^2 Y\right)$ if $E\left[f(X)\right] \leq E\left[f(Y)\right]$ for any increasing concave function f. For the index to be consistent with expected utility, we must have that

$$\text{if } X \preceq^2 Y, \text{ then } \alpha(X) \leq \alpha(Y).$$

The last two properties we consider are related to the extreme values of the index.

1.7 Arbitrage Consistency

Here we require that the index is infinity for arbitrages or equivalently that

$$X \geq 0 \text{ if and only if } \alpha(X) = \infty.$$

1.8 Expectation Consistency

Here we relate the positivity of the index to the positivity of the mean and require that

$$\begin{aligned} &\text{if } E[X] < 0, \text{ then } \alpha(X) = 0; \\ &\text{if } E[X] > 0, \text{ then } \alpha(X) > 0. \end{aligned}$$

We shall consider a variety of candidate measures of performance and study their properties in Section 3. We comment here on the relationship between measures of performance as we have defined them and individual objective functions like expected utility or monetary utility. The latter seek to determine a preference level of an accessed cash flow. They measure how good a trade is from the perspective of a particular market participant. Expected utility is clearly not scale invariant, and so it is different from our acceptability indices. Monetary utility has the scale invariance property, but its form is completely different from that for indices:

$$u(\lambda X) = \lambda u(X) \text{ for } \lambda \geq 0.$$

In addition monetary utilities are superadditive and require that

$$u(\lambda X + (1-\lambda)Y) \geq \lambda u(X) + (1-\lambda)u(Y) \text{ for } \lambda \in [0,1].$$

This property ensures that convex portfolios have a preference level strictly above the minimum of two unequal preference levels. Our acceptability index as a performance measure is not a preference assessment for any market participant, and so we do not require such a superadditivity property. We only require that convex portfolios have an acceptability index no smaller than the minimum of two unequal index levels. Hence, acceptability indices are not monetary utilities.

Apart from these formal distinctions we note the differences that flow from the different objectives being pursued. Both expected utility and monetary utility measure, as we have noted, a preference level. Hence they both rank in preference two non-negative cash flows. An acceptability index is designed to measure the degree of efficiency in an economy and culminates at plus infinity for any nonnegative cash flow attained at zero cost. We measure zero cost cash flows with respect to a family of cones that all contain the positive orthant, while the better than sets of expected utility or monetary utility need not contain the positive orthant. Our interest is in studying the highest cone attainable in efficient economies at zero cost and not in studying the choice problem facing market participants under uncertainty.

2 Characterization and State-Price Densities

2.1 Coherent Utility Functions

The theory of acceptability indices we propose here builds on the theory of acceptability sets studied in Artzner, Delbaen, Eber, and Heath (1999) and Carr, Geman, and Madan (2001). Let us recall that a coherent utility function is a map u from L^∞ to the real line satisfying the conditions

(Superadditivity) $u(X+Y) \geq u(X) + u(Y)$;

(Monotonicity) if $X \leq Y$, then $u(X) \leq u(Y)$;

(Scale Invariance) $u(\lambda X) = \lambda u(X)$ for $\lambda \geq 0$;

(Translation Invariance) $u(X+m) = u(X) + m$ for $m \in \mathbb{R}$;

(Fatou Property) if $|X_n| \leq 1$ converge to X in probability and $u(X_n) \geq x$, then $u(X) \geq x$.

The corresponding coherent risk measure is then $\rho(X) = -u(X)$. According to the basic representation theorem proved by Artzner, Delbaen, Eber, and Heath (1999) for a finite Ω and by Delbaen (2002) in the general case, u is a coherent utility function if and only if it can be represented as

$$u(X) = \inf_{Q \in \mathcal{D}} E^Q[X], \tag{1}$$

with some set $\mathcal{D}$ of probability measures absolutely continuous with respect to P. The measures from $\mathcal{D}$ are called generalized scenarios in Artzner, Delbaen, Eber, and Heath (1999) and are test measures in the terminology of Carr, Geman, and Madan (2001).

A set $\mathcal{D}$ defining a coherent utility function through (1) is not unique, but there exists the largest such set given by

$$\mathcal{D} = \left\{Q \in \mathcal{P} : E^Q[X] \geq u(X)\ \forall X \in L^\infty\right\}, \tag{2}$$

where $\mathcal{P}$ denotes the set of probability measures absolutely continuous with respect to P. The set $\mathcal{D}$ might be called the determining set of u, and it is of primary importance in applications of coherent risks to the problems of pricing. For example, the fundamental theorem of asset pricing in the form provided by Carr, Geman, and Madan (2001) relates the absence of strictly acceptable opportunities in the market to the existence of a risk-neutral measure in this set. For the usefulness of determining sets in applications to equilibrium, we refer to Cherny (2005).

An important object associated with a coherent utility u is the extreme measure. The extreme measure corresponding to X is defined as the element $Q^*(X)$ of the determining set $\mathcal{D}$ at which the infimum of expectations $E^Q[X]$ is attained. In typical situations (see the examples in Section 4) such a measure exists and is unique. The role of this measure becomes clear from the following identity (see, for example, Cherny (2005)):

$$\lim_{\varepsilon \downarrow 0} [u(X + \varepsilon Y) - u(X)] = E^{Q^*(X)}[Y]. \tag{3}$$

In other words, a cash flow Y is profitable for an agent whose portfolio produces a cash flow X if and only if $E^{Q^*(X)}[Y] > 0$. Thus $Q^*(X)$ serves as the coherent analog of the state-price density based on the classical utility, while $E^{Q^*(X)}[Y]$ is the analog of the reservation price for a contract paying out an amount Y.

2.2 Coherent Acceptability Indices

As shown above, there exists a deep relationship between coherent utilities and classical expected utility functions. We will define a coherent acceptability index essentially as a coherent counterpart of a family of expected utilities with different degrees of risk aversion. Before stating the result we briefly discuss the nature of a coherent acceptability index. For each positive real x we essentially define a classical acceptable set $\mathcal{A}_x$ of random variables associated with test measures $\mathcal{D}_x \subset \mathcal{P}$ such that $X \in \mathcal{A}_x$ if and only if for all $Q \in \mathcal{D}_x$ we have that $E^Q[X] \geq 0$. We make the cones of acceptability $\mathcal{A}_x$ smaller and converging onto the positive orthant as we raise the level x by increasing the sizes of the test measures $\mathcal{D}_x$. The acceptability index $\alpha(X)$ is then defined by the largest level of x such that $X \in \mathcal{A}_x$. Alternatively one considers the smallest level of x such that $X \notin \mathcal{A}_x$ with the understanding that if the set of $x's$ is empty, as would be the case for an arbitrage, then the smallest level of x is ∞. In the theorem below we impose a technical assumption that α is unbounded above.

Theorem 1 *A map $\alpha : L^\infty \to [0, \infty]$ is a coherent acceptability index if and only if there exists a collection $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ of subsets of $\mathcal{P}$ such that $\mathcal{D}_x \subseteq \mathcal{D}_y$ for $x \le y$ and*

$$\alpha(X) = \inf\left\{x \in \mathbb{R}_+ : \inf_{Q\in\mathcal{D}_x} E^Q[X] < 0\right\}, \tag{4}$$

where $\inf \emptyset = \infty$.

Proof. See appendix. ■

The above theorem serves as the counterpart of (2) for the indices as opposed to a single utility. Other notions and results associated with coherent risk measures or utilities also find their counterparts for indices. We have therefore associated with every acceptability index a decreasing collection of acceptability cones and an associated increasing collection of test measures. The proposition below provides the counterpart of the determining set.

Proposition 2 *Let α be a coherent acceptability index. Then there exists the maximal system $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$, for which (4) is true, i.e. for any other system $(\mathcal{D}'_x)_{x\in\mathbb{R}_+}$ satisfying (4), we have $\mathcal{D}'_x \subseteq \mathcal{D}_x$ for any $x \in \mathbb{R}_+$.*

Proof. See Appendix. ■

We will call the collection $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ identified in Proposition 2 the determining system of α. Finding the determining system for an acceptability index is important in understanding the structural relationships between different indices. It is also useful in making judgements on the suitability of an index for a proposed application. The richness of the cone of acceptability is intimately linked to the diversity of measures in the determining set. There should be some diversity to induce a proper conic structure, but yet one does not wish to have so much diversity that we start entertaining completely orthogonal measures that have no common basis.

The lemma below will be used in Section 3 to identify determining sets for various acceptability indices. Its proof is similar to that of Proposition 2. By the L^1-closedness of a subset of $\mathcal{P}$ we mean the L^1-closedness of the set of its Radon-Nikodym derivatives with respect to P.

Lemma 3 *Let $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ be a family of convex $L^1 -$ closed subsets of $\mathcal{P}$ satisfying the condition $\mathcal{D}_x = \cap_{y>x}\mathcal{D}_y$ for any $x \in \mathbb{R}_+$. Define α by (4). Then $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ is the determining system of α.*

The counterpart of the extreme measure is now the whole family of extreme measures $(Q^*_x(X))_{x\in\mathbb{R}_+}$, where $Q^*_x(X)$ is the element of $\mathcal{D}_x$, at which the minimum of expectations $E^Q[X]$ is attained (here $(\mathcal{D}_x)$ is the determining system of α). These might be seen as the family of state-price densities corresponding to various degrees of risk aversion, the latter being measured by x. Of particular importance is the measure $Q^*(X) = Q^*_{x_0}(X)$ corresponding to $x_0 = \alpha(X)$. To see this, denote $u_x(X) = \inf_{Q \in \mathcal{D}_x} E^Q[X]$. If the map $x \mapsto u_x(X)$ is continuous, as is the case for all the examples below, then it follows from (3) that

$$E^{Q^*(X)}[Y] > 0 \implies \exists \delta > 0 : \forall \varepsilon \in (0, \delta),\ \alpha(X + \varepsilon Y) > \alpha(X),$$
$$E^{Q^*(X)}[Y] < 0 \implies \exists \delta > 0 : \forall \varepsilon \in (0, \delta),\ \alpha(X + \varepsilon Y) < \alpha(X),$$

so that $E^{Q^*(X)}[Y]$ is the analog of the reservation price.

The analogy with reservation prices and classical or monetary utility should not be taken too literally. One must recognize that an acceptability index when viewed from a preference perspective is special in that it is like a preference ordering where the entire positive orthant serves like a bliss point with infinite acceptability. The order is only defined on the half space with positive expectation and the level sets contain rays joining points to the origin. Yet the extreme measures may be used to identify directions of greater efficiency, higher levels of acceptability, or closeness to arbitrages.

The set of extreme measures obtained on varying the random variables X are also important as they equivalently describe the cone of acceptability as the other measures are just in the convex hull of the extreme measures. In this sense they describe the extremities or edges of the cone.

To conclude the section, let us also remark that the second basic representation result in the theory of coherent utilities is the representation of law invariant coherent utilities provided by Kusuoka (2001). Subsection 3.5 provides its counterpart for coherent indices and in particular shows that law invariance is equivalent to second order monotonicity. As for the remaining two properties of indices, i.e. arbitrage and expectation consistency, it can be shown that they admit an interpretation in terms of the determining system: arbitrage consistency is equivalent to the property that the closure of $\bigcup_x \mathcal{D}_x$ coincides with $\mathcal{P}$, while expectation consistency is equivalent to the property $\mathcal{D}_0 = \{P\}$.

In the next section we study the suitability of several coherent indices by analyzing the structure of the corresponding determining system and particularly of the stream of extreme measures.

3 Performance Measures and Coherent Acceptability Indices: Some Examples

We consider in this section a variety of performance measures and their properties. Some of these measures correspond to coherent acceptability indices, while others do not. When we have an example of a coherent acceptability index, we shall identify explicitly the associated determining set and comment on its structure. We consider each measure and its properties and structure in a separate subsections.

We will identify in the following measures from $\mathcal{P}$ typically denoted by Q with their densities with respect to P typically denoted by Z.

3.1 Sharpe Ratio $SR(X)$

The first measure we consider is the Sharpe ratio of the cash flow X defined by the ratio of the mean to the standard deviation $\sigma(X)$:

$$SR(X) = \frac{E[X]}{\sigma(X)}.$$

This measure is clearly scale invariant, law invariant, and expectation consistent. It also has the Fatou property. Furthermore, it is consistent with second order monotonicity and convexity.

It is well known, however, that the Sharpe ratio does not satisfy the monotonicity property. To observe the violation it is sufficient to compare the tosses of fair coins with outcomes a_1, a_2 and b_1, b_2, respectively, where $0 < a_1 < a_2 < b_1 < b_2$ with a_1, a_2 very close to each other so that the first coin has a small standard deviation that inflates its Sharpe ratio above that of the second coin. The Sharpe ratio is also not consistent with arbitrage as was noted explicitly in an example by Bernardo and Ledoit (2000) taking a positive random variable with an infinite variance to produce a nonnegative cash flow with a zero Sharpe ratio. It is therefore not a coherent acceptability index.

3.2 Gain-Loss Ratio $GLR(X)$

The Gain-Loss ratio is defined as

$$GLR(X) = \begin{cases} \frac{E[X^+]}{E[X^-]} - 1 & \text{if } E[X] > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This measure is clearly law and scale invariant, arbitrage and expectation consistent, and it is monotonic. It also satisfies the Fatou property, and we observe that it satisfies convexity. Hence, it is a coherent acceptability index.

To observe that it satisfies convexity suppose that $GLR(X) \geq x$ and $GLR(Y) \geq x$, where $x > 0$. We then have equivalently that $E[X] \geq xE[X^-]$ and $E[Y] \geq xE[Y^-]$. By the convexity of the function x^-, we get that

$$xE\left[(\lambda X + (1-\lambda)Y)^-\right] \leq x(\lambda E[X^-] + (1-\lambda)E[Y^-]),$$

and so $GLR(\lambda X + (1-\lambda)Y) \geq x$.

The Gain-Loss ratio also satisfies second order monotonicity. To observe this property consider X, Y such that Y dominates X in the second order. Suppose $GLR(X) = x > 0$. Then $E[X] = xE[X^-]$. As $-x^-$ is increasing and concave, we deduce that $E[Y^-] \leq E[X^-]$. Then

$$E[Y] \geq E[X] = xE[X^-] \geq xE[Y^-]$$

or equivalently $GLR(Y) \geq x$.

The Gain-Loss ratio therefore satisfies all the properties discussed in Section 1, and we may identify its determining set. For this purpose we define

$$\mathcal{D}_x = \{c + Y : c \in \mathbb{R}_+,\ 0 \leq Y \leq cx,\ E(c+Y) = 1\}, \quad x \in \mathbb{R}_+.$$

Then

$$\inf_{Z\in\mathcal{D}_x} E[ZX] \geq 0 \Leftrightarrow \inf_{0\leq Y\leq x} E\left[(1+Y)X\right] \geq 0, \quad x\in\mathbb{R}_+.$$

Furthermore, it is clear that the latter infimum is attained at $Y = x\mathbf{1}_{X\leq 0}$ and is equal to $E[X] - xE[X^-]$. It follows that

$$\inf_{Z\in\mathcal{D}_x} E[ZX] \geq 0 \Leftrightarrow E[X^+] \geq (x+1)E[X^-] \Leftrightarrow GLR(X) \geq x, \quad x\in\mathbb{R}_+.$$

Thus $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ represents GLR. By Lemma 3, it is the determining system.

For X with a continuous distribution, the density of the extreme measure $Q_x^*(X)$ is of the form $c(1+x\mathbf{1}_{X\leq b})$, but there is no explicit analytic formula for b, c, and in fact these values might not be unique. Not having an explicit formula for the state-price density is a big disadvantage. Another disadvantage is that a density $c(1+x\mathbf{1}_{X\leq b})$ exaggerates losses uniformly. However, from economic perspectives we are accustomed to measure changes that exaggerate large losses more than small losses. These considerations lead us to formulate a measure more closely related to such economic theoretic measure changes.

3.3 Tilt Coefficient $TC(X)$

The tilt coefficient $TC(X)$ for a random cash flow X may be thought of as the highest level of absolute risk aversion for exponential utility such that the cash flow is still attractive to such a utility at the margin. Alternatively, it is the smallest level of absolute risk aversion such that the cash flow is strictly unattractive to such a utility. We then have

$$TC(X) = \inf\left\{\lambda\in\mathbb{R}_+ : E\left[Xe^{-\lambda X}\right] < 0\right\},$$

where $\inf\emptyset = \infty$. It is easy to see that TC is monotone, law invariant and has the Fatou property. It is arbitrage and expectation consistent. However, it is not convex or scale invariant, and hence, it is not a coherent acceptability index.

To observe the absence of convexity we may find a number $a > 0$ such that the function xe^{-x} is convex on $[a,\infty)$. We then choose a set A with $0 < P(A) < 1$ and define the random variables X, Y such that *a.e.* on A, $X, Y \in [a,\infty)$ and $X \neq Y$ on A. On the complement of A we take $X = Y = c$, where c is the constant such that $E[Xe^{-X}] = E\left[Ye^{-Y}\right] = 0$. We have thus organized that $TC(X) = TC(Y) = 1$. However, when we consider the average, we have

$$\frac{X+Y}{2}\exp\left(-\frac{X+Y}{2}\right) < \frac{Xe^{-X}+Ye^{-Y}}{2} \text{a.e. on } A,$$

and so $TC\left(\frac{X+Y}{2}\right) < 1$.

With respect to scale invariance we note that $TC(\theta X) = TC(X)/\theta$, and so one may consider a revised measure in the form $TC(X)E[X]$ when $E[X] > 0$ with the value 0 if $E[X] \leq 0$. We would then preserve scale invariance but not convexity.

The tilt coefficient also fails to satisfy second order monotonicity. To observe this violation consider X as above with $TC(X) = 1$, but now define Y to equal X on the complement of A and set it equal to $b = E[X\mathbf{1}_A]/P(A)$ on A. Then Y dominates X in second order, but due to the convexity of xe^{-x} on A, we have $E[Ye^{-Y}] < E[Xe^{-X}] = 0$. Hence, $TC(Y) < 1$.

As the results of the preceding section show, a coherent acceptability index is the analog of the tilt coefficient with the family of classical utilities replaced by a family of coherent utilities. Below we study several examples of coherent indices corresponding to several families of coherent utilities. These are firstly $TVAR$, tail value at risk also known as conditional value at risk or expected shortfall. $TVAR$ is a particular case of the class of risk measures $WVAR$, weighted value at risk, which is our second example. We then specialize the weighted value at risk to four particular weight functions that lead us to four coherent risk measures, which we denote $MINVAR$, $MAXVAR$, $MINMAXVAR$, and $MAXMINVAR$. Finally, we consider coherent-based risk-adjusted return on capital ($RAROC$), which might be considered as the coherent counterpart of the Sharpe ratio.

3.4 Acceptability Index Based on TVAR $AIT(X)$

Tail Value at Risk or $TVAR$ is a coherent risk measure defined as $TVAR(X) = -u^{\lambda}(X)$, where

$$u^{\lambda}(X) = \inf_{Q \in \mathcal{D}^{\lambda}} E^{Q}[X]$$

and $\mathcal{D}^{\lambda}$ is the set of probability measures absolutely continuous with respect to P such that $dQ/DP \leq \lambda^{-1}$, where $\lambda \in (0,1]$ is a parameter. In particular, if X has a continuous distribution, then the above infimum is attained at the measure $Q^*(X)$ with

$$\frac{dQ^*(X)}{dP} = \lambda^{-1}\mathbf{1}_{X \leq q^{\lambda}(X)},$$

where q^{λ} denotes the λ-quantile of X. From this it is clear that $u^{\lambda}(X) = E[X|X \leq q^{\lambda}(X)]$, which motivates the term $TVAR$.

The acceptability index based on $TVAR$ is AIT and is defined by

$$AIT(X) = \left(\inf\left\{\lambda \in (0,1] : u^{\lambda}(X) \geq 0\right\}\right)^{-1} - 1,$$

where $\inf \emptyset = 1$.

As the map $\lambda \to u^{\lambda}(X)$ is continuous in λ, we have that $AIT(X) \geq x$ if and only if $u^{\frac{1}{x+1}}(X) \geq 0$, and hence, AIT has the convexity property. It is also monotone, scale invariant and has the Fatou property. The law invariance of $TVAR$ is obvious if X has a continuous distribution and is seen from Föllmer and Schied (2002), Lemma 4.46 in the general case. Second order monotonicity is also inherited from the same property of u^{λ}; see Föllmer and Schied (2002).

Arbitrage and expectation consistency properties follow from the relations

$$\lim_{\lambda \downarrow 0} u^\lambda(X) = \operatorname*{essinf}_{\omega} X(\omega) = \sup\{c \in \mathbb{R} : X \geq c \text{ a.s.}\},$$
$$\lim_{\lambda \uparrow 1} u^\lambda(X) = E[X].$$

Thus we see that AIT is a coherent acceptability index. Clearly, it is represented by the system

$$\mathcal{D}_x = \{Z : 0 \leq Z \leq x+1,\ E[Z] = 1\}, \quad x \in \mathbb{R}_+,$$

and, by Lemma 3, it is the determining system.

For X with a continuous distribution, the extreme measures or coherent state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = (x+1)\mathbf{1}_{X \leq q^{1/(x+1)}(X)}, \quad x \in \mathbb{R}_+.$$

These measures are even more extreme than those supporting GLR. Here we ignore gains altogether employing measures that are zero for gains and uniform with respect to the size of losses. From the perspectives of economic considerations these are unreasonable measures and probably more so than those associated with GLR. This problem can be overcome by the AIW index proposed below.

3.5 Acceptability Indices Based on WVAR $AIW(X)$

Weighted value at risk or $WVAR$ is a coherent risk measure defined as $WVAR(X) = -u^\mu(X)$, where

$$u^\mu(X) = \int_{(0,1]} u^\lambda(X)\mu(d\lambda) \tag{5}$$

and μ is a probability measure on $(0,1]$.

Before introducing the associated acceptability index we describe this risk measure in an alternative way that readily yields the tractable examples we later construct. For this purpose we introduce the function

$$\Psi^\mu(y) = \int_0^y \int_{(0,z]} \lambda^{-1}\mu(d\lambda)dz, \quad y \in [0,1].$$

In fact, we have in $\mu \to \Psi^\mu$ a one-to-one correspondence between probability measures μ on $(0,1]$ and concave distribution functions on $(0,1]$. Indeed, differentiation of Ψ^μ establishes that it is monotone and concave. Changing the order of integration shows that $\Psi^\mu(1) = 1$, while clearly $\Psi^\mu(0) = 0$. The inverse map is given by $\mu(dy) = -y(\Psi^\mu)''(dy)$, where $(\Psi^\mu)''$ is the second derivative in the sense of distributions, i.e. it is the probability measure on $(0,1]$ given by $(\Psi^\mu)''((a,b]) = (\Psi^\mu)'_+(b) - (\Psi^\mu)'_+(a)$, where $(\Psi^\mu)'_+$ is the right-hand derivative of Ψ^μ with the convention $(\Psi^\mu)'_+(1) = 0$.

According to Föllmer and Schied (2004), Theorem 4.64,

$$u^\mu(X) = \int_{\mathbb{R}} yd(\Psi^\mu(F_X(y)), \tag{6}$$

where F_X is the distribution function of X. Note that the right-hand side of the above equality is just the expectation of a random variable having $\Psi^\mu(F_X)$ as the distribution function. The representation (6) shows that the coherent utility associated with weighted value at risk is equivalent to computing expectations under a concave distortion with the concave distortion being provided by the function Ψ^μ.This representation is very convenient in constructing particular representatives of the class $WVAR$, and below we employ it to construct $MAXVAR$, $MINVAR$, $MAXMINVAR$, and $MINMAXVAR$. It is also at the core of defining the AIW index. In particular, it is seen from (6) that $u^\mu \leq u^{\widetilde{\mu}}$ if and only if $\Psi^\mu \geq \Psi^{\widetilde{\mu}}$.

The function u^μ is a coherent utility as all the necessary properties are inherited from u^λ. According to Cherny (2006), its determining set is given by

$$\mathcal{D}^\mu = \left\{ Z : Z \geq 0,\ E[Z] = 1,\ \text{and } E[(Z-y)^+] \leq \Phi^\mu(y)\ \forall y \in \mathbb{R}_+ \right\}, \tag{7}$$

where Φ^μ is the convex conjugate of Ψ^μ:

$$\Phi^\mu(y) = \sup_{z \in [0,1]} (\Psi^\mu(z) - yz), \quad y \in \mathbb{R}_+.$$

For X with a continuous distribution, the minimum of expectations $E^Q[X]$ over $Q \in \mathcal{D}^\mu$ is attained at the measure $Q^*(X)$ given by

$$\frac{dQ^*(X)}{dP} = (\Psi^\mu)'_-(F_X(X)), \tag{8}$$

where $(\Psi^\mu)'_-$ is the left-hand derivative; see Cherny (2006). We also refer to this paper for more information of $WVAR$.

Representation (6) allows us to define the $WVAR$ acceptability index AIW by

$$AIW(X) = \inf\left\{ x \in \mathbb{R}_+ : \int_{\mathbb{R}} yd\left(\Psi_x(F_X(y))\right) < 0 \right\} \tag{9}$$

(we set $\inf \emptyset = \infty$), where $(\Psi_x)_{x \in \mathbb{R}_+}$ is a collection of concave distribution functions on $[0,1]$ that pointwise increase in x. One may replace in (9) the function Ψ_x by the function $\Psi_x^- = \lim_{\varepsilon \downarrow 0} \Psi_{x-\varepsilon}$. As the map $x \mapsto \int_{\mathbb{R}} yd(\Psi_x^-(F_X(y)))$ is left-continuous, we may then write

$$AIW(X) \geq x \Longleftrightarrow \int_{\mathbb{R}} yd(\Psi_x^-(F_X(y))) \geq 0,$$

from which it is clear that AIW satisfies convexity and the Fatou property. Monotonicity and scale invariance are obvious. Law invariance and second order monotonicity follow from the same properties of u^μ, which, in turn, are inherited from u^λ. For arbitrage consistency one needs that $\Psi_x(y)$ tends to unity pointwise on $(0, 1]$ as x increases to infinity. Expectation consistency requires that $\Psi_x(y)$ tends pointwise to y as x tends to zero.

In order to identify the determining system, introduce the right modification $\Psi_x^+ = \lim_{\varepsilon \downarrow 0} \Psi_{x+\varepsilon}$ and define the dual functions

$$\Phi_x(y) = \sup_{z \in [0,1]} (\Psi_x^+(z) - yz), \quad x, y \in \mathbb{R}_+.$$

It is then clear from (7) that the system

$$\mathcal{D}_x = \left\{ Z : Z \geq 0,\ E[Z] = 1,\ \text{and } E[(Z - y)^+] \leq \Phi_x(y)\ \forall y \in \mathbb{R}_+ \right\}, \quad x \in \mathbb{R}_+$$

defines AIW. By Lemma 3, it is the determining system.

According to (8), for X with a continuous distribution, the extreme measures or coherent state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = (\Psi_x^+)'_-(F_X(X)), \quad x \in \mathbb{R}_+,$$

where the differential is taken in y.

We observe further that acceptability indices based on weighted value at risk are essentially related to law invariant acceptability indices. This is established in the following theorem. The notion of dilatation monotonicity appearing below was introduced for coherent risks by Leitner (2004) and is also motivated by the notion of factor risks introduced by Cherny and Madan (2006). In the theorem below we assume that the probability space is atomless. This condition is automatically satisfied if it supports a random variable with a continuous distribution.

Theorem 4 *For a coherent acceptability index α, the following conditions are equivalent:*

(a) α is law invariant;

(b) α is monotone with respect to second order;

(c) α is dilatation monotone, i.e. for any $X \in L^\infty$ and any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}$, we have $\alpha(E[X|\mathcal{G}]) \geq \alpha(X)$;

(d) the determining system $(\mathcal{D}_x)_{x \in \mathbb{R}_+}$ of α is law invariant, i.e. for any $x \in \mathbb{R}_+$ and any $Z \overset{law}{=} Z'$, we have that if $Z \in \mathcal{D}_x$, then $Z' \in \mathcal{D}_x$;

(e) there exists a family $(\alpha^\gamma)_{\gamma \in \Gamma}$ of AIW indices such that

$$\alpha(X) = \inf_{\gamma \in \Gamma} \alpha^\gamma(X), \quad X \in L^\infty.$$

Proof. See Appendix. ■

Below we study three particular representatives of the class AIW. We employ the notation

$$u_x(X) = \int_{\mathbb{R}} y d\Psi_x(F_X(y))$$

for the stream of coherent utilities associated with the index.

3.6 Acceptability Index Based on MINVAR $AIMIN(X)$

A particularly attractive and very intuitive concave distortion is given by the function

$$\Psi_x(y) = 1 - (1-y)^{x+1}, \quad x \in \mathbb{R}_+,\ y \in [0,1]. \tag{10}$$

For integer x we have $u_x(X) = E[Y]$, where

$$Y \overset{law}{=} \min\{X_1, \ldots, X_{x+1}\}$$

and $X_1, \ldots, X_{x+1}$ are independent draws of X. We call the risk measure $\rho_x(X) = -u_x(X)$ *MINVAR* and denote the associated acceptability index as $AIMIN$. It is then the largest number x such that the expectation of the minimum of $x+1$ draws from the cash flow distribution is still positive. By working with real values for x in distortion (10), we allow for a real number of draws.

The convex dual of the concave distortion Ψ_x may be evaluated to be

$$\Phi_x(y) = \begin{cases} 1 - y + x\left(\frac{y}{x+1}\right)^{1+\frac{1}{x}} & \text{if } 0 \le y \le x+1, \\ 0 & \text{if } y \ge x+1. \end{cases}$$

The densities from $\mathcal{D}_x$ therefore have an upper bound of $x+1$ on the amount by which large losses may be exaggerated.

For X with a continuous distribution, the state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = (x+1)(1 - F_X(X))^x, \quad x \in \mathbb{R}_+.$$

A potential drawback of $AIMIN$ is that this density tends to a finite value $x+1$ at $-\infty$.

3.7 Acceptability Index Based on MAXVAR $AIMAX(X)$

Another concave distortion is to consider

$$\Psi_x(y) = y^{\frac{1}{x+1}}, \quad x \in \mathbb{R}_+,\ y \in [0,1].$$

For integer x we have $u_x(X) = E[Y]$, where Y is a random variable with the property

$$\max\{Y_1, \ldots, Y_{x+1}\} \overset{law}{=} X,$$

where $Y_1, \ldots, Y_{x+1}$ are independent draws of Y. We call the risk measure $\rho_x(X) = -u_x(X)$ *MAXVAR* and denote the associated acceptability index as $AIMAX$.

The convex dual of Ψ_x may be evaluated as

$$\Phi_x(y) = \begin{cases} 1 - y & \text{if } y \le \frac{1}{x+1}, \\ \frac{x}{x+1}\left(\frac{1}{(x+1)y}\right)^{\frac{1}{x}} & \text{if } y \ge \frac{1}{x+1}. \end{cases}$$

The densities from $\mathcal{D}_x$ allow for arbitrary large levels by which losses may be exaggerated as $\Phi_x(y)$ is never zero.

For X with a continuous distribution, the state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = \frac{1}{x+1}(F_X(X))^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}_+.$$

This density tends to $+\infty$ at $-\infty$ but has another potential drawback: it tends to a strictly positive value $1/(x+1)$ at $+\infty$. This corresponds to an asymptotically linear utility for large gains and is a potentially realistic perspective. Many classical state-price densities, however, tend to zero at positive infinity.

3.8 Acceptability Index Based on MAXMINVAR $AIMAXMIN(X)$

Combining $MINVAR$ and $MAXVAR$, we consider the distortion

$$\Psi_x(y) = (1-(1-y)^{x+1})^{\frac{1}{x+1}}, \quad x \in \mathbb{R}_+,\ y \in [0,1].$$

For integer x we have $u_x(X) = E[Y]$, where Y is a random variable with the property:

$$\max\{Y_1, \ldots, Y_{x+1}\} \overset{law}{=} \min\{X_1, \ldots, X_{x+1}\},$$

where $X_1, \ldots, X_{x+1}$ are independent draws of X, while $Y_1, \ldots, Y_{x+1}$ are independent draws of Y. We call the risk measure $\rho_x(X) = -u_x(X)$ $MAXMINVAR$ and denote the associated acceptability index as $AIMAXMIN$ in recognition of the fact that we construct the worst case scenario first using a $MINVAR$ perspective followed by a $MAXVAR$ perspective.

In this case the convex dual $\Phi_x(y)$ of Ψ_x does not have a closed form. However, it may be computed, and we present a graph for a sample of x values. We observe that this dual function goes to unity as y tends to zero and tends to zero as y goes to infinity.

For X with a continuous distribution, the state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = (1-F_X(X))^x(1-(1-F_X(X))^{x+1})^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}_+.$$

In particular, this density tends to $+\infty$ at $-\infty$ and to 0 at $+\infty$.

3.9 Acceptability Index Based on MINMAXVAR $AIMINMAX(X)$

Another way to combine $MINVAR$ and $MAXVAR$ is to consider

$$\Psi_x(y) = 1-(1-y^{\frac{1}{x+1}})^{x+1}, \quad x \in \mathbb{R}_+,\ y \in [0,1].$$

For integer x we have $u_x(X) = E[Y]$, where Y is a random variable with the property:

$$Y \overset{law}{=} \min\{Z_1, \ldots, Z_{x+1}\},$$
$$\max\{Z_1, \ldots, Z_{x+1}\} \overset{law}{=} X,$$

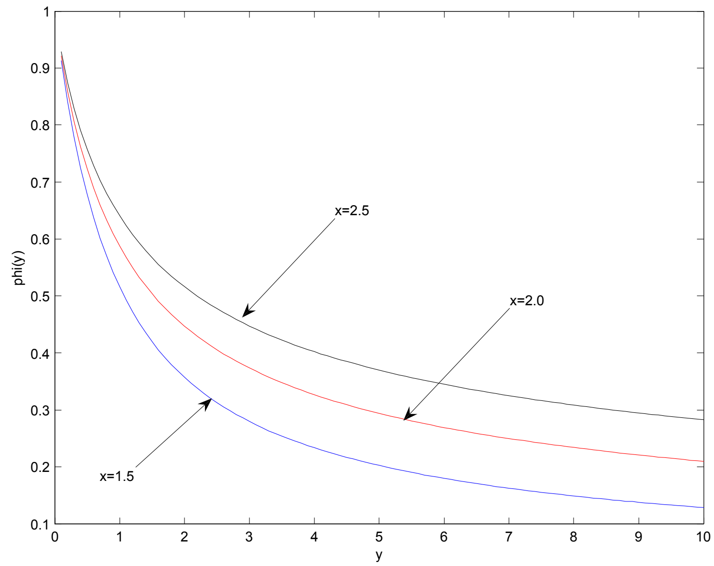

Figure 1: The Convex Dual $\Phi_x(y)$ for a sample of x values for MAXMINVAR.

where $Z_1, \ldots, Z_{x+1}$ are independent draws of Z. We call this risk measure $\rho_x(X) = -u_x(X)$ $MINMAXVAR$ and denote the associated acceptability index as $AIMINMAX$ in recognition of the fact that we construct the worst case scenario first using a $MAXVAR$ perspective followed by a $MINVAR$ perspective.

The convex dual in this case also lacks a closed form, and we graph the computed dual for a sample of x values.

For X with a continuous distribution, the state-price densities are given by

$$\frac{dQ_x^*(X)}{dP} = (1 - F_X(X)^{\frac{1}{x+1}})^x F_X(X)^{-\frac{x}{x+1}}, \quad x \in \mathbb{R}_+.$$

In particular, this density also tends to $+\infty$ at $-\infty$ and to 0 at $+\infty$.

3.10 Risk Adjusted Return on Capital $RAROC(X)$

Our last index generalizes the Sharpe ratio by changing the risk measure to be coherent. Suppose $\rho(X)$ is any coherent risk measure with a determining set $\mathcal{D}$. We assume that $P \in \mathcal{D}$; for example, this is automatically satisfied if ρ is law invariant as seen from the result of Kusuoka (2001). We define

$$RAROC(X) = \begin{cases} \frac{E[X]}{\rho(X)} & \text{if } E[X] > 0, \\ 0 & \text{otherwise.} \end{cases}$$

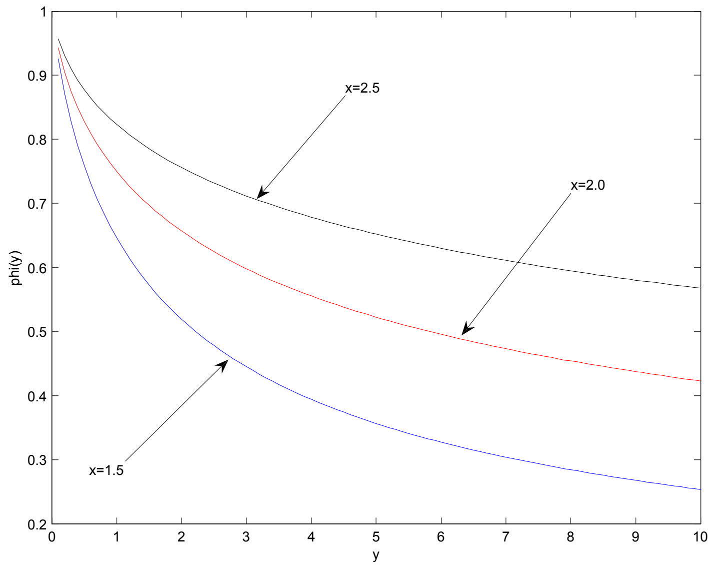

Figure 2: The Convex Dual $\Phi_x(y)$ for a sample of x values for MINMAXVAR.

with the convention $RAROC(X) = +\infty$ if $\rho(X) < 0$. Obviously, $RAROC$ satisfies all the properties of a coherent acceptability index. It is law invariant if and only if ρ is law invariant. By Theorem 4, this is further equivalent to second order monotonicity. The expectation consistency follows from the condition $P \in \mathcal{D}$. However, $RAROC$ has an essential drawback: it is not arbitrage consistent as can be noted by considering X with $P(X<0)>0$ and $\inf_{Q\in\mathcal{D}} E^Q[X] > 0$.

The determining system for $RAROC$ is given by

$$\mathcal{D}_x = \frac{1}{1+x}\{P\} + \frac{x}{1+x}\mathcal{D}, \quad x \in \mathbb{R}_+.$$

If $\inf_{Q\in\mathcal{D}} E^Q[X]$ is attained at a unique measure Q^*, then the set of extreme measures for X is given by

$$Q_x^*(X) = \frac{1}{1+x}\{P\} + \frac{x}{1+x}Q^*, \quad x \in \mathbb{R}_+.$$

Let us finally note that if ρ corresponds to $WVAR$, then $RAROC$ is a particular case of the AIW index with

$$\Psi_x(y) = \frac{1}{1+x}y + \frac{x}{1+x}\Psi^\mu(y), \quad x \in \mathbb{R}_+, \quad y \in (0,1].$$

4 Testing Performance Measures

We evaluate the performance measures described above for some simple strategies. This is useful for a variety of purposes. First, we get an idea about the numerical magnitudes of measures and the types of values one may expect to see for them. We also develop some understanding of the relative values of the various measures as they are all computed for the same cash flow streams. However, as many of the measures represent coherent levels of law invariant acceptability, they are also useful as pricing devices under the physical measure. For example, if we are to quote on put options on the net asset value of a fund of hedge funds, and we have access to just the time series of net asset values, then we may work out the price that attains a particular level of acceptability. From such a perspective the acceptability level attained in liquid options markets is of interest. We could then price put options on the fund of hedge funds asset value to attain at a level of acceptability comparable to that observed in the exchange traded options markets.

It is also instructive to have an assessment of these measures with a view to testing for high alpha strategies or detecting market anomalies. As mentioned in the introduction, much of the literature takes a half space view of alpha or the existence of an anomaly. Were the cash flows to be evaluated for performance using our indices they may be found wanting from a perspective that is law invariant and conic with respect to inclusion of the positive orthant.

We present in two subsections the data used and the results obtained.

4.1 Data

We focus attention on the performance measures for cash flows related to writing options, holding the position unhedged and paying out the necessary cash flow at the option maturity. One expects these cash flows to have a positive mean for the writer as the contracts represent a sale of insurance. We studied the measures for these cash flows in two major world indices, these are options on the $S\&P$ 500 index and options on the $FTSE$. In each case we considered 7 strike buckets and 4 maturity buckets and wrote all out of the money options that were puts for strikes below the spot and calls for strikes above the spot. The options were held to maturity and the cash flow paid out. As a result we had 28 cash flows for each strike/maturity bucket, where the number of trades in each bucket was around 4500 for the period beginning in December of 2000 till December of 2005.

4.2 Results for Cash Flows on Option Trades

For each of the series of 28 cash flows we compute eight measures of trade performance. These are the Sharpe ratio, the tilt coefficient, the Gain-Loss ratio, the $RAROCX10$, where the risk is measured by the expectation of the minimum of 10 draws from the distribution. Finally we evaluate the four new acceptability indices based on $MINVAR$ and $MAXVAR$, $MAXMINVAR$ and $MINMAXVAR$. The results are presented in two tables, one for the SPX and the other for the $FTSE$. Each table contains eight subtables with 7 rows for the strike buckets and 4 columns for the maturity buckets. The eight subtables display the eight performance measures.

The TC are generally the smallest followed by $AIMINMAX$, $AIMAXMIN$, $AIMAX$, $RAROCX10$, SR, $AIMIN$, and GLR. For the SPX all Sharpe ratios are positive, but for the $FTSE$ at the money option sales and put options in the $.5 - .75$ maturity range had a negative expected value.

For the SPX we observe that $AIMIN$, $AIMAX$, $AIMAXMIN$, and $AIMINMAX$ are generally below 1 with the exception of short maturity deep out of the money calls. All the measures display a U-shaped structure with respect to strike at each maturity. The sharpness of the U-shape on the downside first decreases with maturity but then tends to rise. On the upside the U-shape flattens with respect to maturity. The upside calls generally have a sharper lift than the downside puts.

For the $FTSE$ short maturity calls do attain an $AIMIN$ above 1. The U-shape in the strike direction is maintained for all maturities. The sharpness of the U-shape with respect to maturity varies. The upside calls have a higher level of acceptability than comparable downside puts.

The values of $AIMINMAX$, $AIMAXMIN$ are comparable to each other and are below the values for $AIMIN$, $AIMAX$, and this is to be expected as the measures now form worst cases in two ways inclusive of both the $MINVAR$ and $MAXVAR$ strategies. The order does not seem to matter much, at least, for the cash flows associated with option writes held to maturity.

5 Conclusion

We treat the absence of arbitrage as a zero level of efficiency and develop measures of efficiency via the ability of trading strategies to approximate or attain arbitrages. The degree of efficiency is measured the level of acceptability of a cash flow and arbitrages have an infinite level of acceptability. We then axiomatize the notion of a coherent acceptability index as a new measure of the degree of acceptability of cash flows. Connections between our theory of indices and classical expected utility theory are elaborated and contrasted. In particular, an index is linked to a decreasing family of coherent utility functions, which, in turn, serves as a coherent counterpart of a family of expected utilities with increasing degrees of risk aversion.

We also introduce the family of extreme measures associated with an index. As seen from the given examples, these serve as the coherent analogs of the family of state-price densities with increasing degrees of risk aversion. There is, however, a basic difference between the coherent state-price densities and the classical ones: the latter depend on the absolute performance levels, while the former depend on the quantile levels. This difference stems from the difference between the Von Neumann–Morgenstern axiomatization of expected utilities and the Artzner–Delbaen–Eber–Heath axiomatization of coherent utilities.

We further consider eight concrete indices SR, GLR, TC, AIT, $AIMIN$, $AIMAX$, $AIMAXMIN$, and $AIMINMAX$. In addition we consider two classes of indices AIW and $RAROC$. Of these indices SR fails to satisfy monotonicity and arbitrage consistency; TC fails to satisfy convexity, scale invariance, and second order monotonicity; $RAROC$ fails to satisfy arbitrage consistency. Furthermore, GLR does not allow for an analytic formula for the state-price density; in fact, explicit analytic formulas for coherent state-price densities are available only for the $WVAR$ risk measure and the corresponding AIW index. For AIT, state-price densities are zero above a certain level and are constant below this level, which is not economically reasonable. The state-price densities for $AIMIN$ and $AIMAX$ are smooth, but each has a potential drawback: that for $AIMIN$ tends to a finite value at $-\infty$, while that for $AIMAX$ tends to a strictly positive value at $+\infty$. In contrast, the state-price densities of $AIMAXMIN$ and $AIMINMAX$ exhibit the behavior of tending to ∞ and 0 at $-\infty$ and $+\infty$, respectively. Needless to say, these indices also satisfy all the 8 desirable properties outlined in Section 2 of the paper.

Computations of all measures are illustrated for simple strategies of writing options, holding them to maturity unhedged and paying out the required cash flows. The underliers used are the SPX and the $FTSE$. It is observed that all measures have a U-shape in the strike direction for all maturities. Furthermore, the SPX has a higher level of acceptability on the downside, while the $FTSE$ has a higher acceptability on the upside.

SPX Options

Strike Ranges	SR Maturity Range 0-.25	.25-.5	.5-.75	.75-1.0	TC Maturity Range 0-.25	.25-.5	.5-.75	.75-1.0
.85-.9	0.3166	0.1790	0.2136	0.4585	0.0520	0.0247	0.0345	0.1346
.9-.95	0.1970	0.1234	0.1748	0.2802	0.0266	0.0128	0.0246	0.0600
.95-1.0	0.1281	0.0741	0.0857	0.1043	0.0132	0.0050	0.0067	0.0100
1.0-1.05	0.0697	0.0203	0.0127	0.0566	0.0044	0.0004	0.0002	0.0031
1.05-1.1	0.1761	0.2732	0.2879	0.2347	0.0284	0.0683	0.0763	0.0520
1.1-1.15	0.3544	0.3423	0.4538	0.3667	0.0846	0.0939	0.1536	0.1084
1.15-1.2	0.5336	0.4849	0.4950	0.4409	0.1262	0.1393	0.1574	0.1366

	GLR				RAROCX10			
.85-.9	2.9789	0.8292	0.8997	2.0282	0.3328	0.1073	0.1244	0.2954
.9-.95	1.2354	0.4962	0.6636	0.9792	0.1437	0.0682	0.0961	0.1596
.95-1.0	0.6234	0.2528	0.2768	0.2921	0.0773	0.0380	0.0439	0.0551
1.0-1.05	0.2498	0.0578	0.0345	0.1444	0.0361	0.0100	0.0063	0.0302
1.05-1.1	0.5665	0.9487	1.0264	0.7847	0.1070	0.1861	0.2036	0.1612
1.1-1.15	2.1914	1.5448	2.1740	1.4472	0.2734	0.2344	0.3170	0.2389
1.15-1.2	6.8759	3.8486	2.7580	2.0353	0.7889	0.4555	0.3674	0.2935

	AIMIN				AIMAX			
.85-.9	0.9115	0.3461	0.4710	1.1245	0.3397	0.1667	0.2059	0.4718
.9-.95	0.4708	0.2762	0.2889	0.4581	0.2003	0.1306	0.1341	0.2304
.95-1.0	0.2508	0.0763	0.1169	0.1884	0.1169	0.0398	0.0593	0.1075
1.0-1.05	0.0644	0.0077	0.0216	0.0245	0.0348	0.0044	0.0123	0.0152
1.05-1.1	0.2461	0.3635	0.4193	0.2764	0.1823	0.2791	0.3238	0.2254
1.1-1.15	0.6849	0.4636	0.8665	0.6080	0.3699	0.3050	0.4894	0.3805
1.15-1.2	1.2659	1.1474	0.9899	0.8882	0.5485	0.5218	0.4718	0.4546

	AIMAXMIN				AIMINMAX			
.85-.9	0.2249	0.1076	0.1359	0.2994	0.2145	0.1049	0.1315	0.2780
.9-.95	0.1330	0.0856	0.0885	0.1458	0.1291	0.0838	0.0866	0.1404
.95-1.0	0.0772	0.0259	0.0388	0.0669	0.0759	0.0257	0.0384	0.0657
1.0-1.05	0.0224	0.0028	0.0078	0.0093	0.0223	0.0028	0.0078	0.0093
1.05-1.1	0.1010	0.1500	0.1727	0.1193	0.0983	0.1440	0.1649	0.1155
1.1-1.15	0.2207	0.1721	0.2823	0.2170	0.2095	0.1648	0.2629	0.2050
1.15-1.2	0.3362	0.3166	0.2871	0.2726	0.3127	0.2944	0.2678	0.2546

FTSE Options

	SR				TC			
Strike	Maturity Range				Maturity Range			
Ranges	0-.25	.25-.5	.5-.75	.75-1.0	0-.25	.25-.5	.5-.75	.75-1.0
.85-.9	0.2505	0.0650	-0.0475	0.0988	0.0396	0.0038	0.0000	0.0090
.9-.95	0.1103	0.0064	-0.1085	-0.0858	0.0100	0.0000	0.0000	0.0000
.95-1.0	0.0462	-0.0214	-0.1615	-0.2076	0.0020	0.0000	0.0000	0.0000
1.0-1.05	-0.0149	-0.0541	-0.2304	-0.2873	0.0000	0.0000	0.0000	0.0000
1.05-1.1	0.5915	0.4471	0.4948	0.4469	0.2599	0.1775	0.2244	0.1822
1.1-1.15	0.6761	0.6869	0.7332	0.5958	0.2326	0.3341	0.4149	0.2851
1.15-1.2	0.5219	0.7185	1.0434	0.8054	0.1167	0.2588	0.5540	0.4188

	GLR				RAROCX10			
.85-.9	1.8315	0.2645	0.0000	0.2802	0.2089	0.0356	0.0000	0.0528
.9-.95	0.5638	0.0203	0.0000	0.0000	0.0686	0.0033	0.0000	0.0000
.95-1.0	0.1802	0.0000	0.0000	0.0000	0.0245	0.0000	0.0000	0.0000
1.0-1.05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.05-1.1	3.2350	1.9633	2.0273	1.8022	0.4959	0.3532	0.4291	0.3623
1.1-1.15	9.1868	5.7715	4.2449	2.7973	1.0758	0.7685	0.6863	0.4782
1.15-1.2	17.4153	23.4815	16.0821	5.1221	2.0014	2.8906	2.3423	0.7614

	AIMIN				AIMAX			
.85-.9	0.4579	0.0269	0.0000	0.1385	0.2314	0.0157	0.0000	0.0849
.9-.95	0.1657	0.0000	0.0000	0.0000	0.0867	0.0000	0.0000	0.0000
.95-1.0	0.0526	0.0000	0.0000	0.0000	0.0289	0.0000	0.0000	0.0000
1.0-1.05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.05-1.1	1.1565	0.6869	0.8379	0.7031	0.6760	0.5104	0.6533	0.5422
1.1-1.15	1.7939	1.3843	1.5108	1.0274	0.7746	0.7789	0.9108	0.6721
1.15-1.2	2.1375	2.8520	3.2708	1.8586	0.9083	1.1088	1.2811	0.9130

	AIMAXMIN				AIMINMAX			
.85-.9	0.1447	0.0099	0.0000	0.0517	0.1401	0.0098	0.0000	0.0509
.9-.95	0.0556	0.0000	0.0000	0.0000	0.0549	0.0000	0.0000	0.0000
.95-1.0	0.0185	0.0000	0.0000	0.0000	0.0184	0.0000	0.0000	0.0000
1.0-1.05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.05-1.1	0.3770	0.2675	0.3313	0.2805	0.3429	0.2492	0.3028	0.2599
1.1-1.15	0.4557	0.4271	0.4886	0.3619	0.4120	0.3852	0.4314	0.3288
1.15-1.2	0.5162	0.6262	0.7293	0.5220	0.4655	0.5498	0.6187	0.4591

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Appendix

Proof of Theorem 1. The "if" part is obvious, so we prove just the "only if" part. For $x \in \mathbb{R}_+$ consider

$$\mathcal{A}_x = \{X \in L^\infty : \alpha(X) \geq x\}, \tag{11}$$

$$u_x(X) = \sup\{m \in \mathbb{R} : X - m \in \mathcal{A}_x\}. \tag{12}$$

Assume that $\mathcal{A}_x$ is a proper subset of L^∞. In this case $\mathcal{A}_x$ does not contain negative constants. Indeed, if it contained a negative constant, then, by the monotonicity and the scale invariance of α, it would be L^∞. It follows that u_x takes finite values on L^∞. Furthermore, one can verify that u_x is a coherent utility function. (In order to verify the scale invariance property of u_x for $\lambda = 0$, note that the scale invariance of α plus the Fatou property combined with the unboundedness of α show that $\alpha(0) = \infty$; hence, $u_x(0) \geq 0$; furthermore, as $\mathcal{A}_x$ does not contain negative constants, we have $u_x(0) \leq 0$.)

Denote by $\mathcal{D}_x$ the determining set of u_x in the case when $\mathcal{A}_x$ is a proper subset of L^∞, and we set $\mathcal{D}_x = \emptyset$ if $\mathcal{A}_x = L^\infty$. It follows from the monotonicity property of α that, for $x \leq y$, $\mathcal{A}_x \supseteq \mathcal{A}_y$, so that $u_x \geq u_y$, which obviously implies the inclusion $\mathcal{D}_x \subseteq \mathcal{D}_y$. It follows from the equivalence $\alpha(X) \geq x \Leftrightarrow u_x(X) \geq 0$ that (4) holds.

Proof of Proposition 2. Set $u_x^+ = \lim_{\varepsilon \downarrow 0} u_{x+\varepsilon}$, where u_x is given by (12). As shown above, u_x^+ is either identical to ∞ or is a coherent utility function. Denote by $\mathcal{D}_x$ its determining set (if $u_x \equiv \infty$, we take $\mathcal{D}_x = \emptyset$). Using the equalities

$$\alpha(X) = \inf\{x \in \mathbb{R}_+ : u_x(X) < 0\} = \inf\left\{x \in \mathbb{R}_+ : u_x^+(X) < 0\right\},$$

we deduce that $(\mathcal{D}_x)_{x \in \mathbb{R}_+}$ represents α.

Let now $(\mathcal{D}'_x)_{x \in \mathbb{R}_+}$ be another representing system and suppose that there exists $x \in \mathbb{R}_+$ such that $\mathcal{D}'_x \subsetneq \mathcal{D}_x$. As $u_x^+(X) = \sup_{y > x} u_y(X)$, it is easy to check that $\mathcal{D}_x = \bigcap_{y > x} \mathcal{D}_y^0$, where $\mathcal{D}_x^0$ is the determining set of u_x. Then there exists $y > x$ such that $\mathcal{D}'_x \subsetneq \mathcal{D}_y^0$. Clearly, $\mathcal{D}_y^0$ is convex and L^1-closed, so, by the Hahn-Banach theorem, there exists $X \in L^\infty$ such that

$$\inf_{Q \in \mathcal{D}'_x} E^Q[X] < 0 < \inf_{Q \in \mathcal{D}_y^0} E^Q[X].$$

The first inequality implies that $\alpha(X) \leq x$, while the second one implies that $\alpha(X) \geq y$, which is a contradiction.

Proof of Theorem 4. (a)$\Longrightarrow$(d) Let u_x be given by (12) (each u_x either equals ∞ or is a coherent utility function). Clearly, each u_x is law invariant. It follows from Föllmer and Schied (2004), Theorem 4.54 that the determining set $\mathcal{D}_x^0$ of u_x is law invariant for any $x \in \mathbb{R}_+$. Hence, the same is true for $\mathcal{D}_x = \bigcap_{y > x} \mathcal{D}_y^0$.

(d)$\Longrightarrow$(a) It follows from Föllmer and Schied (2004), Theorem 4.54 that each map $u_x(X) = \inf_{Q\in\mathcal{D}_x} E^Q[X]$ is law invariant. Hence, the same is true for $\alpha(X) = \inf\{x \in \mathbb{R}_+ : u_x(X) < 0\}$.

(a)$\Longrightarrow$(e) Let u_x be given by (12). According to Kusuoka's theorem (see Kusuoka (2001) or Föllmer and Schied (2004), Theorem 4.57), for any $x \in \mathbb{R}_+$, there exists a set $\mathcal{M}_x^0$ of probability measures on $(0, 1]$ such that

$$u_x(X) = \inf_{\mu\in\mathcal{M}_x^0} u^\mu(X),$$

where u^μ is given by (5) (if $u_x = \infty$, then $\mathcal{M}_x^0 = \emptyset$). Let $\mathcal{M}_x$ be the largest set for which this representation is true, so that $\mathcal{M}_x \subseteq \mathcal{M}_y$ for $x \le y$. Define $\Gamma = \{(z, \mu) : z \in \mathbb{R}_+, \mu \in \mathcal{M}_z\}$ and

$$\Psi_x^{(z,\mu)}(y) = \begin{cases} y & \text{if } x \le z, \\ \Psi_\mu(y) & \text{if } x \ge z. \end{cases}$$

Then

$$u_x(X) = \inf_{\mu\in\mathcal{M}_x} u^\mu(X) = \inf_{z\le x,\, \mu\in\mathcal{M}_z} u^\mu(X) = \inf_{\gamma\in\Gamma} \int_{\mathbb{R}} y d\Psi_x^\gamma(F_X(y)),$$

so that

$$\begin{aligned} \alpha(X) &= \inf\{x \in \mathbb{R}_+ : u_x(X) < 0\} \\ &= \inf_{\gamma\in\Gamma} \inf\left\{x \in \mathbb{R}_+ : \int_{\mathbb{R}} y d\Psi_x^\gamma(F_X(y)) < 0\right\}, \quad X \in L^\infty. \end{aligned}$$

(e)$\Longrightarrow$(b) This implication follows from the second order monotonicity of AIW.

(b)$\Longrightarrow$(c) To prove this implication, it is sufficient to notice that $E[X|\mathcal{G}]$ dominates X in the second order.

(c)$\Longrightarrow$(a) Let u_x be given by (12). Then, for any $x > 0$, u_x has the dilatation monotonicity property: for any $X \in L^\infty$ and any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}$, we have $u_x(E[X|\mathcal{G}])) \ge u_x(X)$. According to Cherny and Grigoriev (2006), u_x is law invariant. Hence, the same is true for $\alpha(X) = \inf\{x \in \mathbb{R}_+ : u_x(X) < 0\}$.