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(joint work with Dilip B. Madan)

COHERENT ACCEPTABILITY INDICES

MEASURES OF PERFORMANCE

X – cash flow of a portfolio, $X = W_1 - W_0$

Quality of X = Reward/Risk

Sharpe ratio: $\mathbf{E}X/\sigma(X)$

RAROC: $\mathbf{E}X/\mathrm{VAR}(X)$

Gain-Loss ratio: $\mathbf{E}X/\mathbf{E}X^-$, $X^- = \max\{-X,0\}$

Coherent RAROC: $\mathbf{E}X/\rho(X)$, ρ – coherent risk

AXIOMS

Definition. A coherent acceptability index is a map $\alpha(X)$ with values in $[0,\infty]$ such that:

(i) If $\alpha(X) \geq z$ and $\alpha(Y) \geq z$, then $\alpha(X+Y) \geq z$;

(ii) If $X \leq Y$, then $\alpha(X) \leq \alpha(Y)$;

(iii) $\alpha(\lambda X) = \alpha(X)$ for $\lambda > 0$.

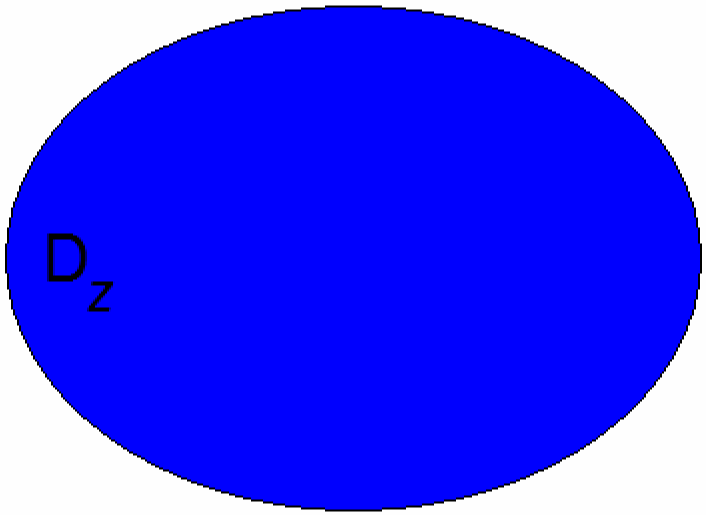
Ex. X = 1, 2 with P = 0.5, Y = 2, 2.1 with P = 0.5.
Then SR(X) > SR(Y).

Ex. X = –10, 1 with P = 0.04, 0,96; Law Y = Law X.
Then RAROC(X)=RAROC(Y)=∞, RAROC(X+Y)<∞.

REPRESENTATION

Theorem. α *is a coherent acceptability index* $\Leftrightarrow$ *there exists a family* $(\mathrm{D}_z)_{z\in[0,\infty)}$ *of sets of probability measures such that* $\mathrm{D}_x \subseteq \mathrm{D}_y$ *for* $x \le y$ *and*

$$\alpha(X) = \inf\left\{z : \inf_{\mathrm{Q}\in\mathrm{D}_z} \mathbf{E}_{\mathrm{Q}} X > 0\right\}.$$



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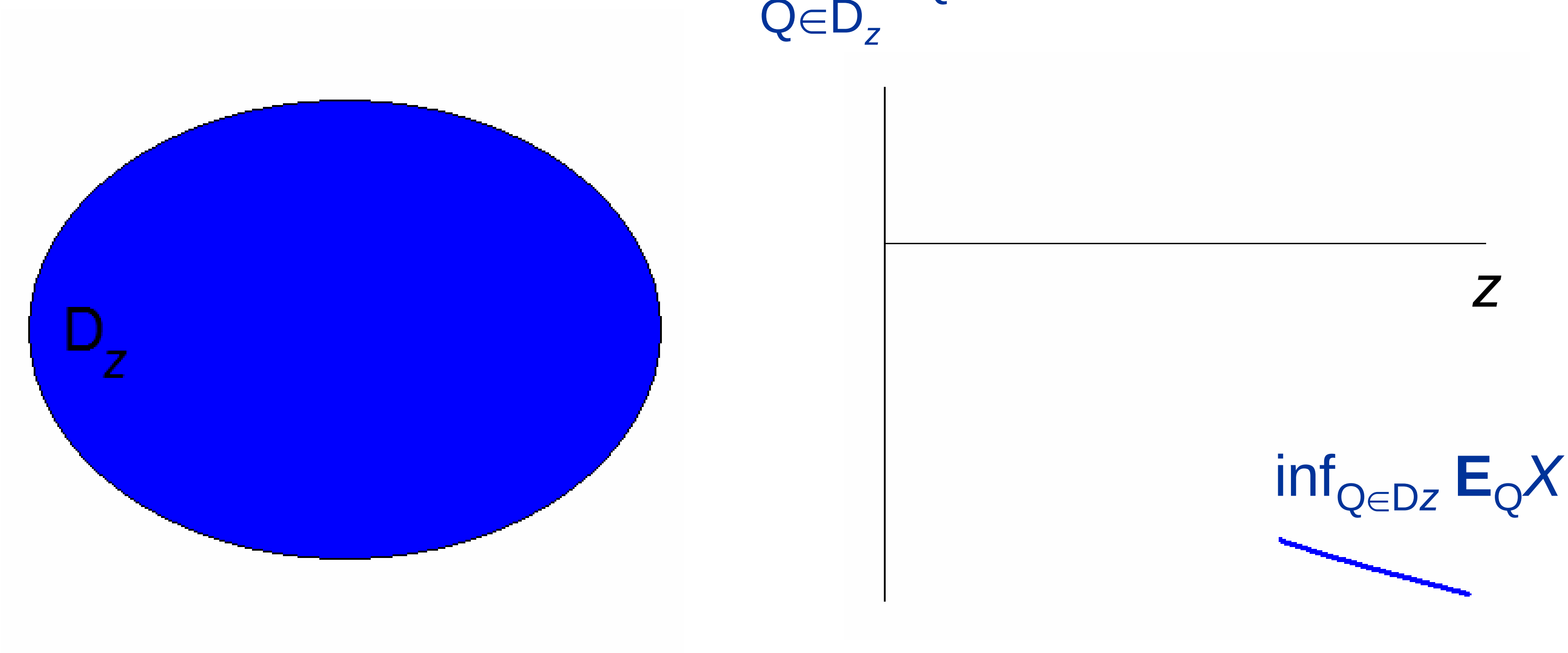

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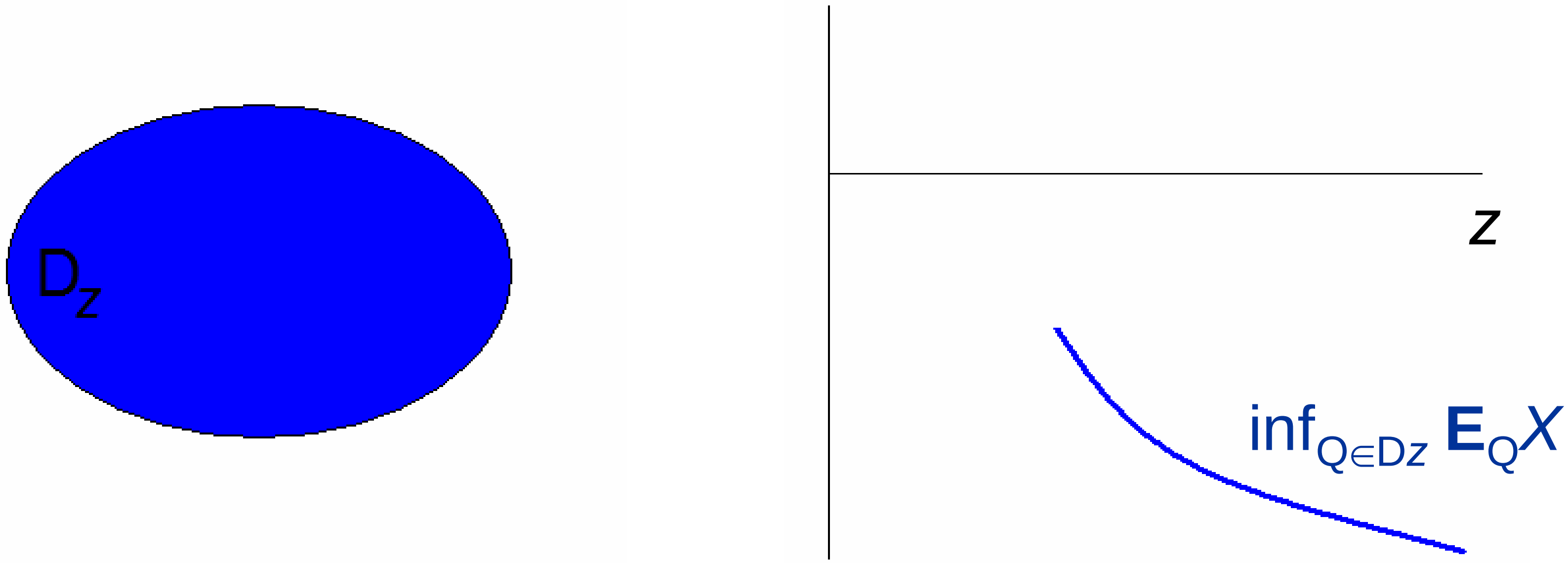

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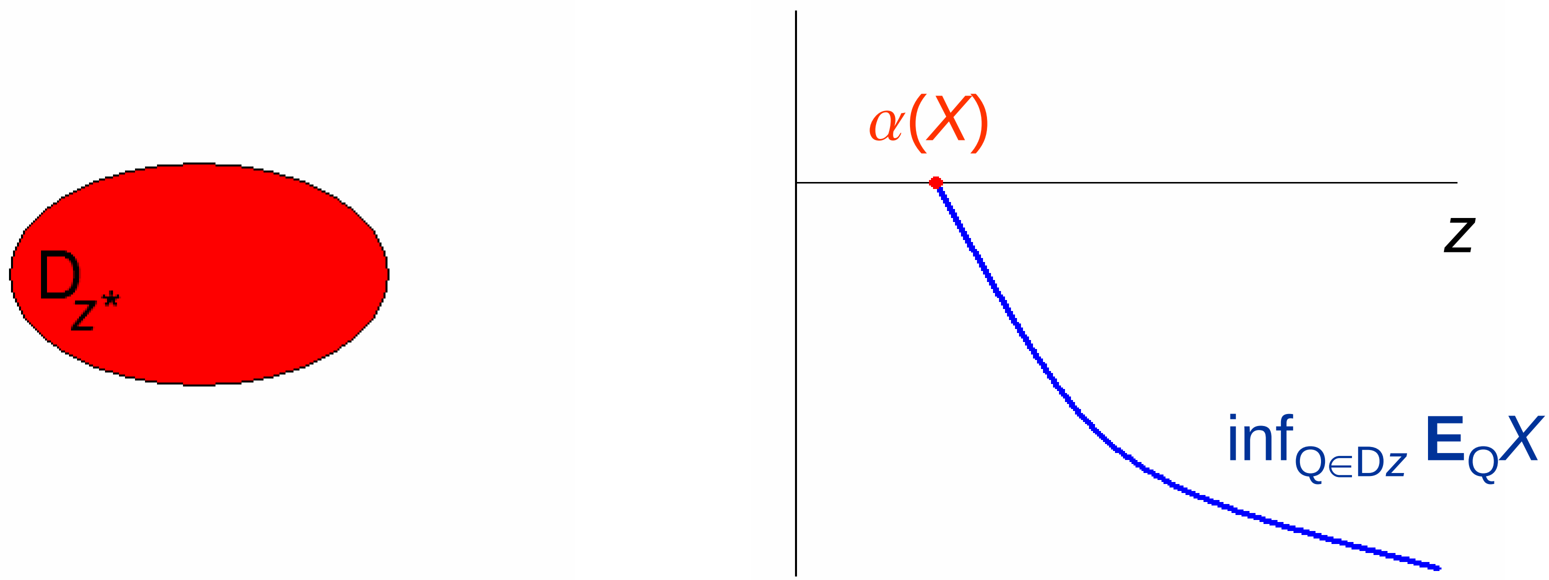
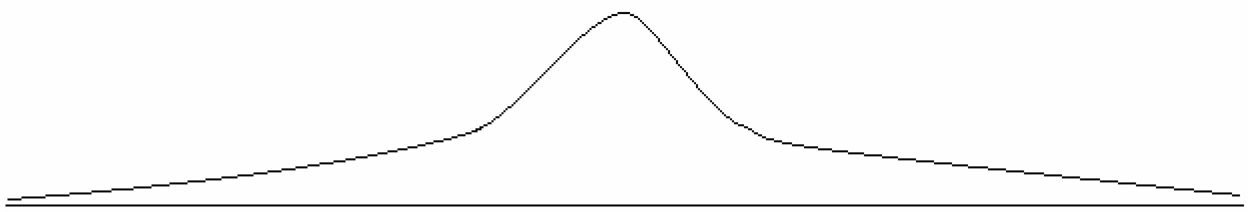


CVAR

$\rho(X) = -\inf_{Q \in D} \mathbf{E}_Q X$ – coherent risk measure (Artzner, Delbaen, Eber, Heath, 1997)

Conditional Value at Risk (CVAR):

$D = \{Q : dQ/dP \leq 1/\lambda\}, \ \lambda \in (0,1).$



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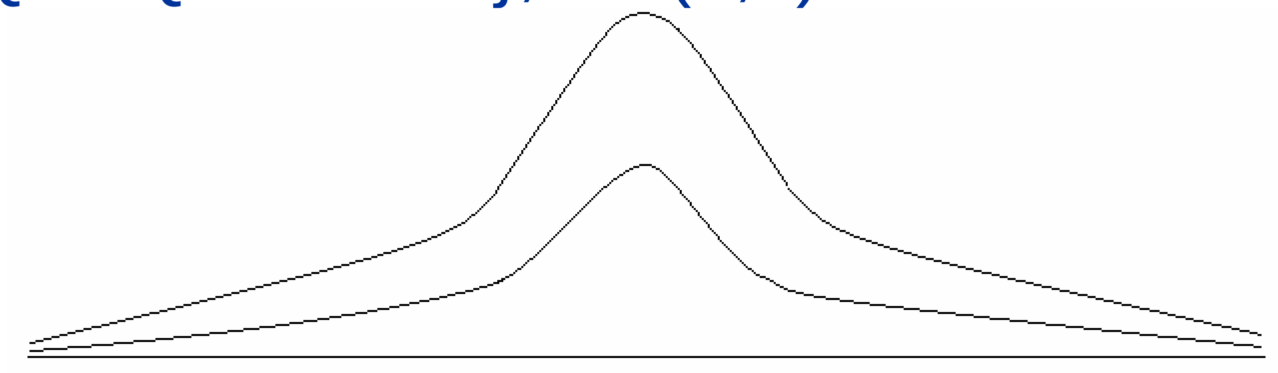

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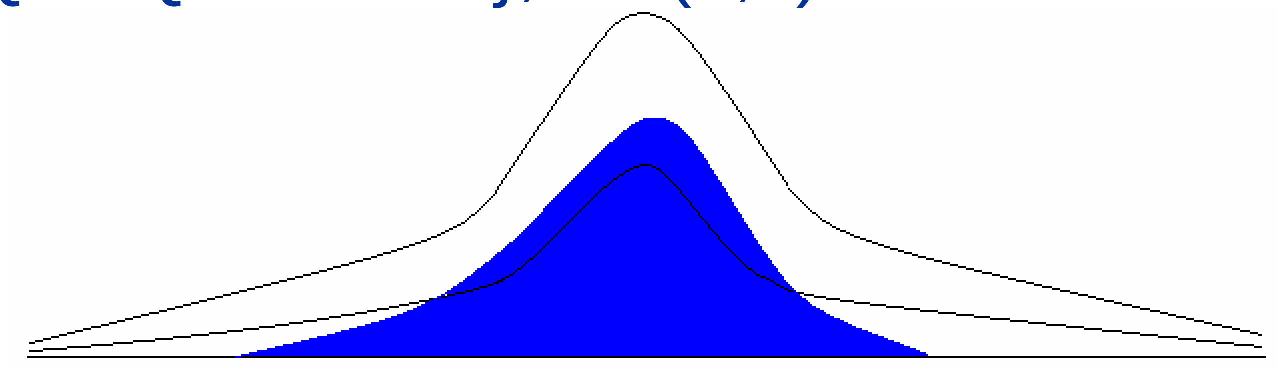

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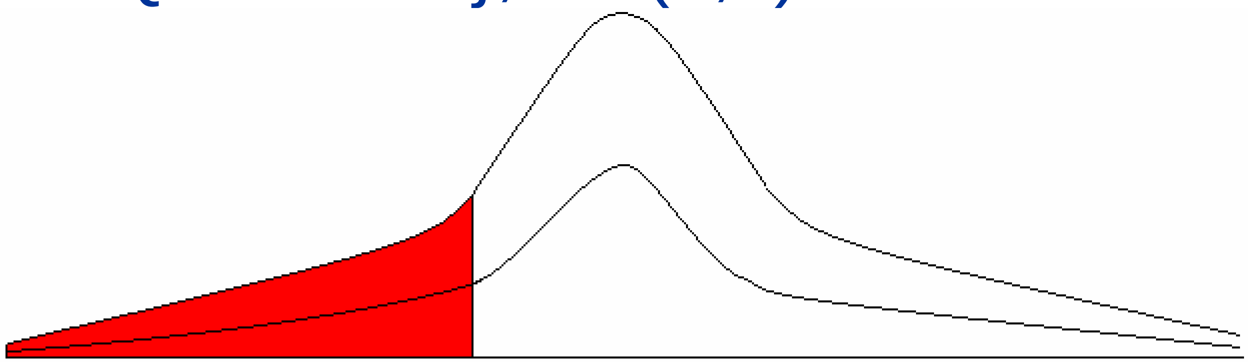


$\text{argmin}_{Q \in D} \mathbf{E}_Q X = P(\,\bullet \mid X \leq q_\lambda)$

$\Longrightarrow \rho(X) = -\mathbf{E}(X \mid X \leq q_\lambda)$

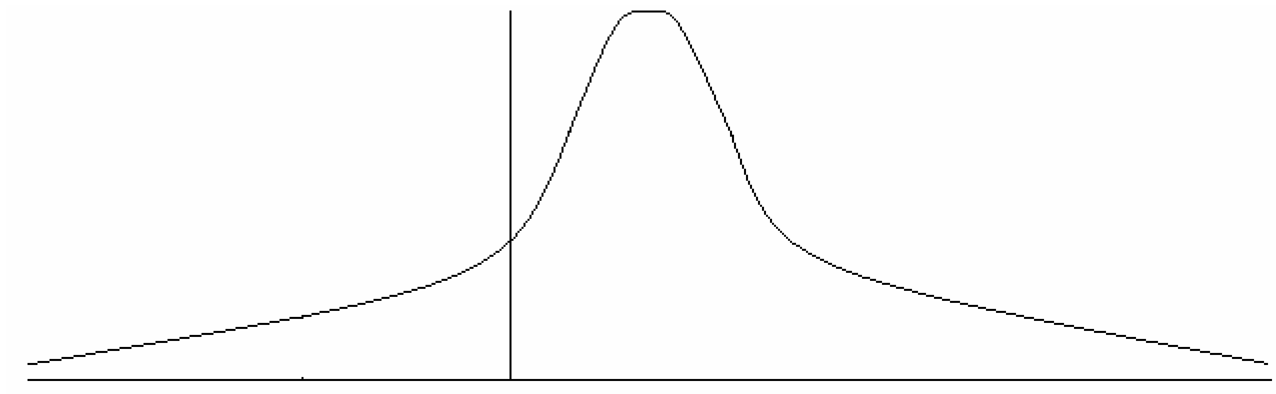
AIC

CVAR Acceptability Index (AIC):

$D_z = \{Q : dQ/dP \leq 1+z\}$.

Then $\alpha(X) = 1/\lambda^* - 1$, where

$$\lambda^* = \inf\{\lambda : \mathbf{E}(X \mid X \leq q_\lambda) \geq 0\}.$$



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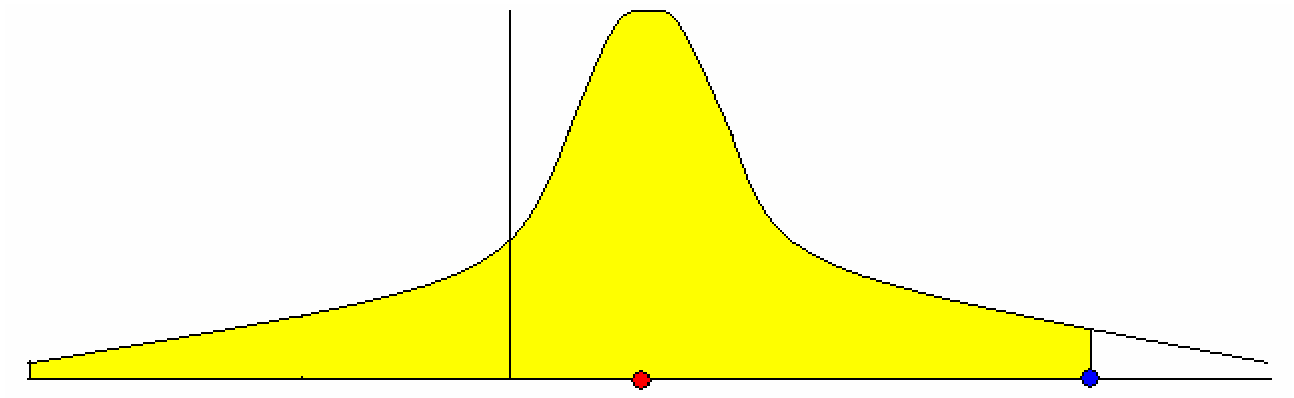

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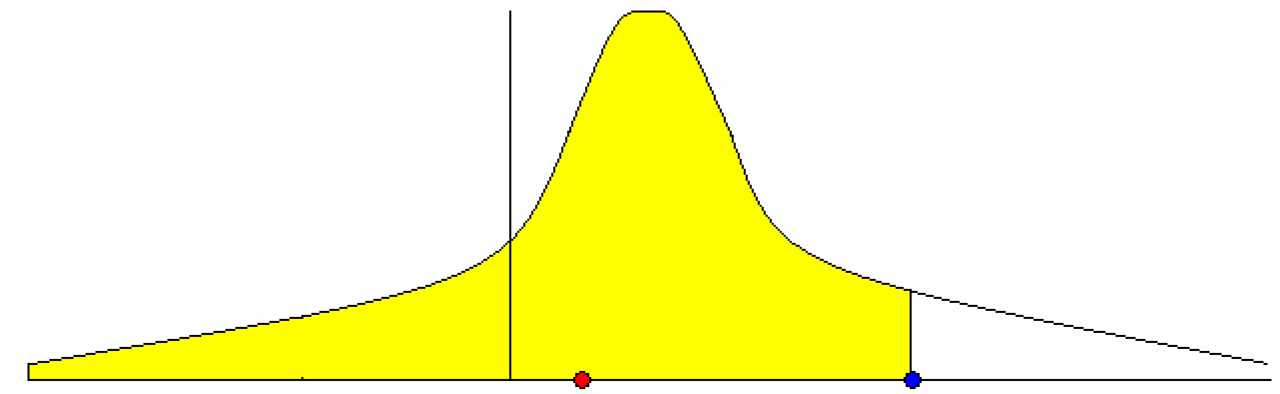

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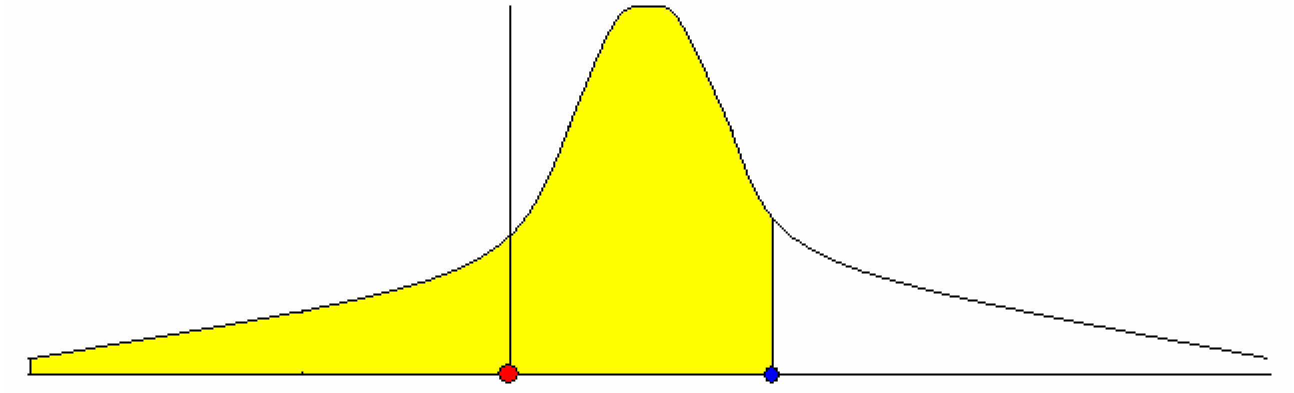


WVAR

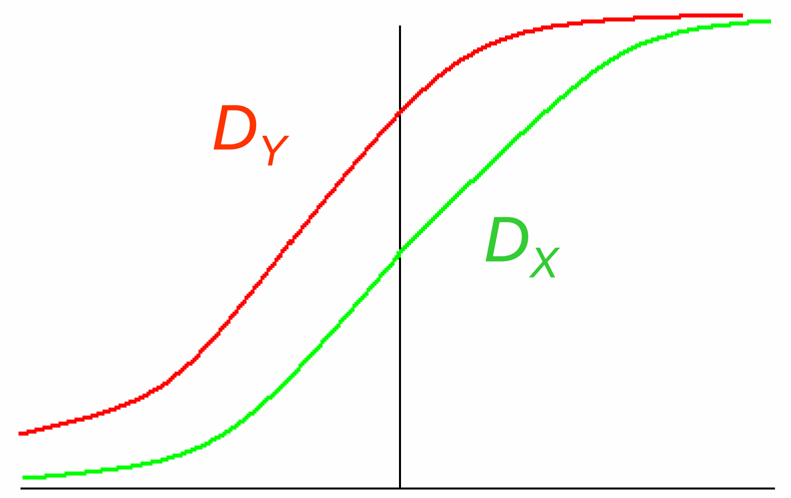
Weighted Value at Risk (WVAR):

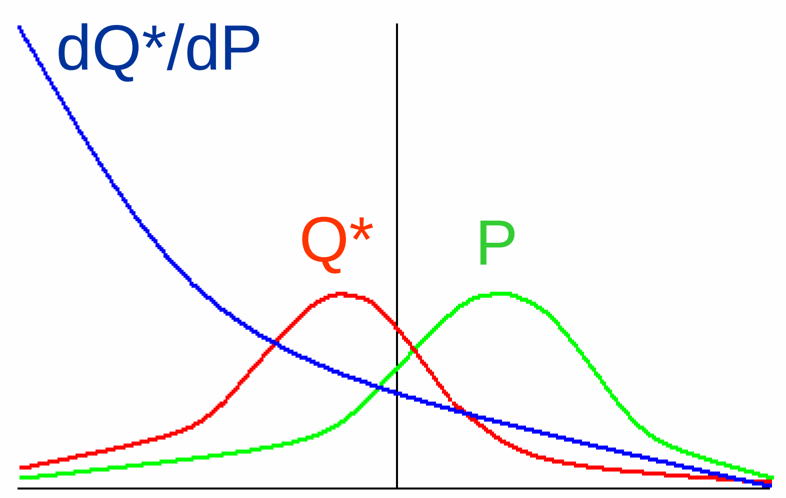
$\rho(X) = \int_0^1 \rho_\lambda(X)\,\mu(d\lambda)$, μ – probability measure on (0,1]

$\rho(X) = -\int_R x\, d\Psi(D_X(x))$, $\Psi(x) = \int_0^x \int_y^1 \lambda^{-1}\,\mu(d\lambda)dy$

$\Psi:[0,1]\to[0,1]$ is concave, increasing, $\Psi(0)=0$, $\Psi(1)=1$

$\rho(X) = \mathbf{E}_{Q^*}X$, $dQ^*/dP = \Psi'(D_X(X))$



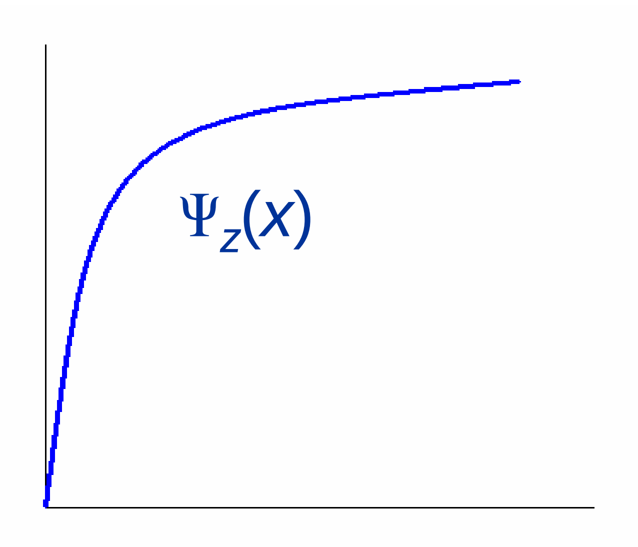


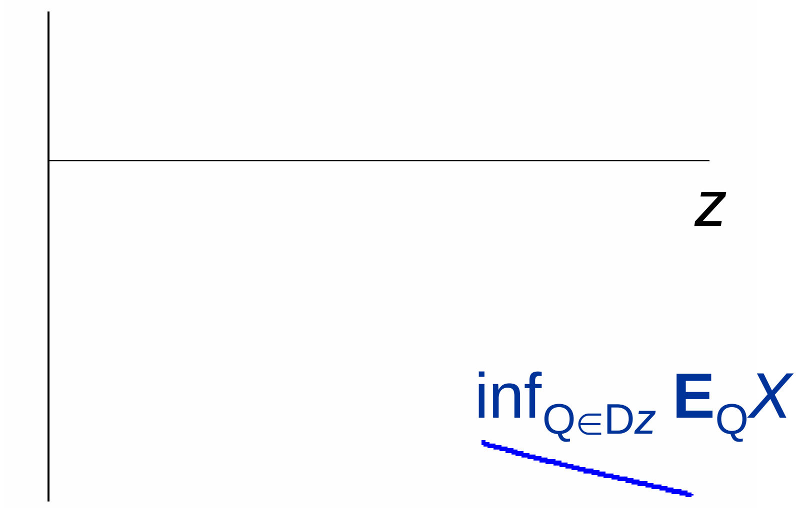
AIW

WVAR Acceptability Index (AIW):

$$\alpha(X) = \inf\left\{z : \int_{\mathbb{R}} x \, d\Psi_z(D_X(x)) > 0\right\},$$

$(\Psi_z)_{z\in[0,\infty)}$ – family of concave distortions increasing in z





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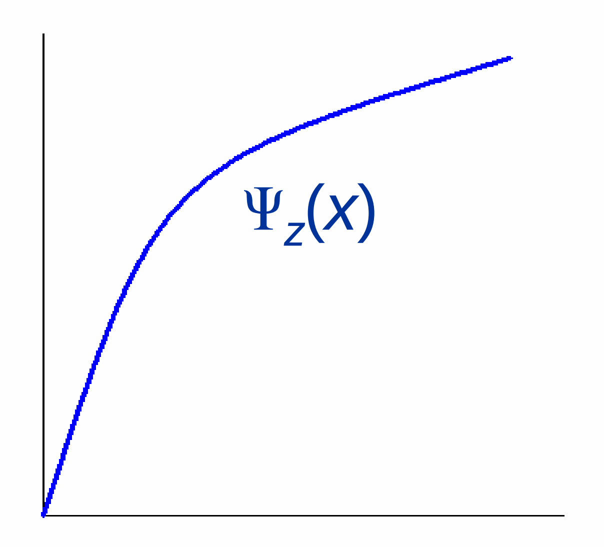

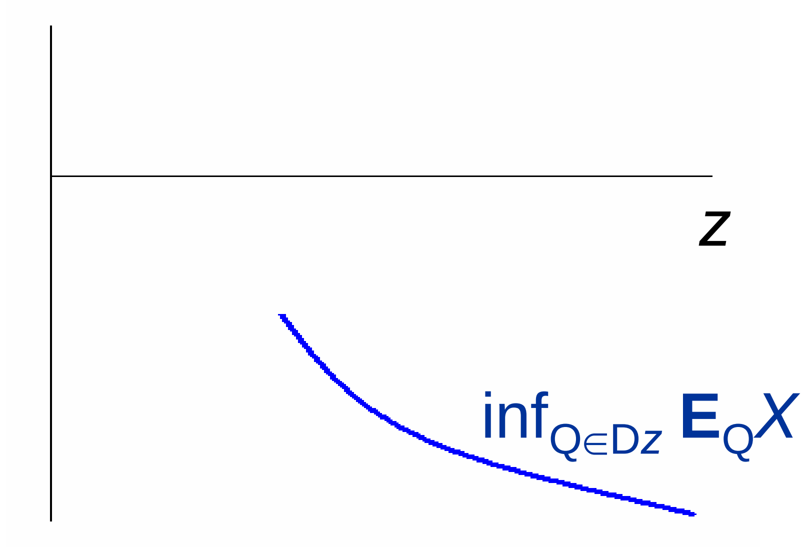

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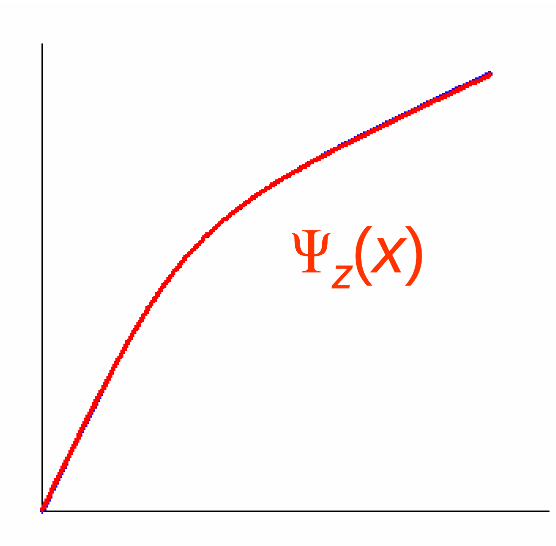

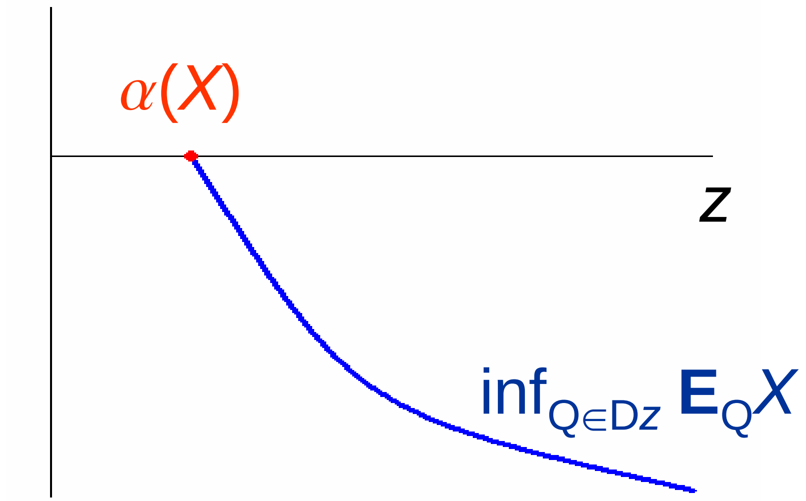

Practical calculation: If $X = x_1,\ldots,x_n$ with P $= 1/n$, then

$$\int_{\mathrm{R}} x \, d\Psi_z(D_X(x)) = -\sum_i x_{(i)} \left[\Psi_z(i/n) - \Psi_z((i-1)/n)\right].$$

COMPARISON

$\alpha(X)$ – performance measure

Conv. $\alpha(X) \geq z,\ \alpha(Y) \geq z \Longrightarrow \alpha(X+Y) \geq z$

Mon. $X \leq Y \Longrightarrow \alpha(X) \leq \alpha(Y)$

Scal. $\alpha(\lambda X) = \alpha(X)$ for $\lambda > 0$

Arb. $\alpha(X) = +\infty \Leftrightarrow X$ is an arbitrage

Expl. Explicit expression for $u_z(X) = \inf_{Q \in Dz} \mathbf{E}_Q X$

Unb. Densities dQ*/dP are unbounded, where

$Q^* = \operatorname{argmin}_{Q \in Dz} \mathbf{E}_Q X$.

COMPARISON

Measure	Conv.	Mon.	Scal.	Arb.	Expl.	Unb.
SR	+		+			
RAROC		+	+			
GLR	+	+	+	+		
CRAROC	+	+	+		+	+
AIC	+	+	+	+	+	
AIW	+	+	+	+	+	+

COMPARISON

Measure	Conv.	Mon.	Scal.	Arb.	Expl.	Unb.
SR	+		+			
RAROC		+	+			
GLR	+	+	+	+		
CRAROC	+	+	+		+	+
AIC	+	+	+	+	+	
AIW	+	+	+	+	+	+

SUMMARY

- Axiomatization of coherent acceptability indices.
  SR, RAROC are not coherent indices.
- Representation of coherent indices through a
  family of coherent risks with various risk aversion.
- New indices AIC and AIW.
- AIW has the best properties.