Robust control of consumption-investment strategies with logarithmic utility and time-consistent penalties

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Motivation and problem formulation

Price process $(S_t)_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, d-dimensional semimartingale
Bond $B_t \equiv 1$, arbitrage-free: $\mathcal{M} \neq \emptyset$

Classical portfolio optimization for risk-averse investor

Find investment strategy ξ with initial capital x that maximizes utility of terminal wealth at T

$$\mathbb{E}\Big[\,U\Big(x + \int_0^T \xi_t \, dS_t\Big)\Big] \longrightarrow \max,$$

based on von Neumann-Morgenstern preferences

$$X \succ Y \qquad \Longleftrightarrow \qquad \mathbb{E}[\,U(X)\,] > \mathbb{E}[\,U(Y)\,]$$

for concave increasing U and a probability measure $\mathbb{P}$.

Problem: $\mathbb{P}$ often unknown, we are facing model uncertainty.

Gilboa/Schmeidler (1989) characterize preferences with aversion against risk and model uncertainty:

$$X \succ Y \quad \Longleftrightarrow \quad \inf_{Q \in \mathcal{Q}} E_Q[\,U(X)\,] > \inf_{Q \in \mathcal{Q}} E_Q[\,U(Y)\,],$$

where $\mathcal{Q}$ is a set of probability measures ("coherent case").

Extension by Maccheroni et al. (2006):

$$X \succ Y \quad \Longleftrightarrow \quad \inf_{Q} \big(E_Q[\,U(X)\,] + \gamma(Q)\big) > \inf_{Q} \big(E_Q[\,U(Y)\,] + \gamma(Q)\big)$$

for a penalty function γ ("concave case").

Resulting problem:

$$\inf_{Q} \big(E_Q[\,U(X_T)\,] + \gamma(Q) \big) \longrightarrow \max \qquad \text{for } X_T = x + \int_0^T \xi_t \, dS_t .$$

(same with consumption strategies)

Possible connection with macro-economic puzzles:

Can empirically detected extreme risk aversion be explained by *uncertainty aversion*?

Typical result:

There exists a measure $\widehat{Q} \in \{\gamma < \infty\}$ such that the robust problem is equivalent to the standard problem for $\widehat{Q}$.

Methods for solution:

- **Robust statistics, Choquet capacity theory:** S. (2005a)
- **Convex duality:** Quenez (2004), S. (2005), Gundel (2005), S. & Wu (2005), S. (2007a), Föllmer & Gundel (2005), Gundel & Weber (2006), Wittmüss (2006)
- **BSDE:** Quenez (2004), Bordigoni, Matoussi, Schweizer (2005), Müller (2005), Bordigoni (2007)
- **Stochastic control:** Hansen & Sargent (2001), Talay & Zheng (2002), Korn and Wilmott (2002), Korn and Menkens (2005), Hernández-Hernández & S. (2006), Hernández-Hernández & S. (2007a, 2007b), S. (2007b), Øksendahl & Sulem (2007)

black = coherent case

green = entropic penalties $\gamma(Q) = H(Q|\mathbb{P})$

blue = general concave case

A stochastic control approach

Stochastic factor model: under $\mathbb{P}$

$$dS_t = S_t \sigma(Y_t)\, dW_t^1 + S_t b(Y_t)\, dt$$

$$dY_t = \rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2 + g(Y_t)\, dt$$

with smooth coefficients, g with linear growth, other coefficients bounded, σ bounded from below, and $-1 \leq \rho \leq 1$.

Incomplete model, i.e., many equivalent martingale measures

$$\frac{dP^*}{d\mathbb{P}} = \mathcal{E}\Big(- \int_0 \theta(Y_t)\, dW_t^1 - \int_0 \nu_t\, dW_t^2 \Big)$$

for ν progressive and

$$\theta(y) = \frac{b(y)}{\sigma(y)}.$$

Defining the robust utility functional

For $Q \ll \mathbb{P}$ there is a process η such that

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int \eta_{1t}\, dW_t^1 + \int \eta_{2t}\, dW_t^2\Big)_T =: D_T^\eta \qquad Q\text{-a.s.}$$

Time-consistent penalty function

$$\gamma(Q) := E_Q\Big[\int_0^T h(\eta_t)\, dt\Big]$$

for $h : \mathbb{R}^2 \longrightarrow [0, \infty]$ convex, lsc., $h(0) = 0$, C^1 in dom h, and

$$h(x) \geq \kappa_1 |x|^2 - \kappa_2.$$

Problem formulation: For $U(x) = \log x$,

$$\inf_Q \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big) \longrightarrow \max \qquad \text{for } X_T = x + \int_0^T \xi_t\, dS_t.$$

Let

$$\psi(y,x) := \inf_{\eta\in\mathbb{R}^2} \left\{ \eta\cdot x + \frac{1}{2}(\eta_1 + \theta(y))^2 + h(\eta) \right\},$$

and for $\overline{\rho} := \sqrt{1-\rho^2}$

$$\phi(y,z) := \psi(y,(\rho,\overline{\rho})z).$$

Assumption: ϕ has superlinear growth in z and ψ satisfies a radial growth condition in direction $(\rho,\overline{\rho})$:

$$\begin{aligned} |\nabla_x \psi(y,(\rho,\overline{\rho})z)| &\le C(1 + |\partial_z \psi(y,(\rho,\overline{\rho})z)|) \\ &= C(1 + |\partial_z \phi(y,z)|) \end{aligned}$$

(use superdifferentials if ψ is not differentiable)

Theorem: There exists a classical solution $v(t, y)$ of

$$\begin{cases} v_t = \frac{1}{2} v_{yy} + \phi(v_y) + g v_y \\ v(0, \cdot) = 0 \end{cases}$$

which is unique (under suitable growth conditions). Let

$$\eta^*(t, y) := \nabla_x \psi\big(y, (\rho, \overline{\rho}) v_y(t, y)\big).$$

Then

$$\widehat{\xi}_t = X_t \cdot \frac{\eta_1^*(T-t, Y_t) + \theta(Y_t)}{\sigma(Y_t)}$$

is an optimal strategy, and a 'worst-case measure' is given by

$$\frac{d\widehat{Q}}{d\mathbb{P}} := \mathcal{E}\Big(\int_0 \eta^*(T-t, Y_t)\, dW_t \Big)_T.$$

Sketch of proof

Main difficulties:

(a) Maximin problem leads to Hamilton-Jacobi-Bellman-Isaacs PDE

(b) Side condition: η must come from a probability measure $Q \ll \mathbb{P}$

Tackling (a) via convex duality

Idea by Quenez (2004): For

$$\widetilde{U}(y) := \sup_{x>0} \big(U(x) - xy \big) = -\log y - 1$$

the dual problem is

$$\mathbb{E}\Big[D_T^{\eta} \widetilde{U}\Big(\frac{\widehat{z} Z_T^{\nu}}{D_T^{\eta}} \Big) \Big] + \mathbb{E}\Big[D_T^{\eta} \int_0^T h(\eta_t)\, dt \Big] \longrightarrow \min$$

where $x = u'(\widehat{z})$ and

$$\begin{aligned} Z^{\nu} &= \mathcal{E}\Big(-\int_0 \theta(Y_t)\, dW_t^1 - \int_0 \nu_t\, dW_t^2 \Big) \\ D^{\eta} &= \mathcal{E}\Big(\int \eta_{1t}\, dW_t^1 + \int \eta_{2t}\, dW_t^2 \Big). \end{aligned}$$

If minimizer $(\widehat{\eta}, \widehat{\nu})$ exists, then the optimal terminal wealth is

$$\widehat{X}_T = -\widetilde{U}'\Big(\frac{\widehat{z} Z_T^{\widehat{\nu}}}{D_T^{\widehat{\eta}}}\Big).$$

To translate dual problem into control problem, we need to show

$$E_Q\Big[\log \frac{D_T^{\eta}}{Z_T^{\nu}}\Big] = \frac{1}{2} E_Q\Big[\int_0^t (\eta_{1s} + \theta(Y_s))^2 + (\eta_{2s} + \nu_s)^2 \, ds\Big]$$

If $\mathbb{E}[\, Z_T^{\nu}\,] = 1$, then

$$\text{l.h.s.} = H(Q|Z_T^{\nu}.\mathbb{P}) = \text{r.h.s.}$$

Otherwise use *Föllmer measure* $\overline{P}_\nu$ on $\Omega \times (0, \infty]$ defined via

$$\overline{P}_\nu[\, A \times (t, \infty]\,] = \mathbb{E}[\, Z_t \mathrm{I}_A\,] \qquad \text{for } A \in \mathcal{F}_t.$$

• PDE is HJB equation for dual problem

• Existence of a solution v is obtained by approximation with compactly supported h^n and PDE methods.

• Verification lemma: $v =$ value function

• A priori estimates yields

  - v_t is bounded

  - v is bounded

Tackling (b), the admissibility of η^*

Gradient estimate for v:

v bounded $\Rightarrow$ v_y has to attain local maxima.
There we have $v_{yy} = 0$, thus

$$v_t = \phi(v_y) + g v_y.$$

As v_t is bounded and $|g(y)| \leq c_1(1 + |y|)$,

$$\left| \frac{\phi(v_y)}{v_y} \right| \leq c_2(1 + |y|).$$

Hence,

$$|\phi'(v_y)| \leq c_3(1 + |y|) \qquad \text{for } |v_y| \geq 1.$$

By the radial growth condition:

$$|\eta^*(t, y)| \leq c_4(1 + |y|).$$

Therefore

$$\sup_{0\le t\le T} \mathbb{E}\Big[\, e^{\varepsilon|\eta^*(T-t,Y_t)|^2} \,\Big] \le c \cdot \sup_{0\le t\le T} \mathbb{E}\Big[\, e^{\delta|Y_t|^2} \,\Big] < \infty.$$

This yields a local Novikov condition....

Coherent case with HARA utility:

$$U(x) = \frac{x^\alpha}{\alpha} \qquad \text{for } \alpha < 0.$$

For $\Gamma \subset \mathbb{R}^2$ compakt and convex:

$$\mathcal{Q} = \left\{ Q \,\middle|\, \frac{dQ}{d\mathbb{P}} = \mathcal{E}\left(\int \eta_{1t}\, dW_t^1 + \int \eta_{2t}\, dW_t^2 \right),\ \eta_t \in \Gamma \right\}.$$

Theorem: We have

$$\sup_{\xi} \inf_{Q \in \mathcal{Q}} E_Q \Big[U \Big(x + \int_0^T \xi_t \, dS_t \Big) \Big] = \frac{x^\alpha}{\alpha} \exp \big[(1-\alpha) w(T, Y_0) \big]$$

where w is the unique bounded classical solution of the PDE

$$w_t = \frac{1}{2} w_{yy} + (g - \beta\rho\theta) w_y + \frac{1}{2} \cdot \frac{1-\beta\rho^2}{1-\beta} w_y^2 + $$
$$+ \max_{\eta \in \Gamma} \Big[\rho(1-\beta)\eta_1 w_y - \frac{\beta(1-\beta)}{2} (\theta + \eta_1)^2 + \eta_2 \overline{\rho} w_y \Big]$$

with initial condition $w(0, \cdot) \equiv 0$.

If $\eta^*(t, y) \in \Gamma$ is a maximizer, then the optimal strategy is given by

$$\widehat{\xi}_t = X_t \pi^*(T - t, Y_t)$$

where $\beta = -\alpha/(1-\alpha)$ and

$$\pi^* = \sigma^{-1} \big[(1-\beta)(\eta_1^* + \theta) + \rho w_y \big].$$

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$$\sup_{\xi} \inf_{Q \in \mathcal{Q}} E_Q\Big[U\Big(x + \int_0^T \xi_t \, dS_t\Big)\Big] = \frac{x^\alpha}{\alpha} \exp\left[(1-\alpha)w(0, Y_0)\right]$$

where w is the unique bounded classical solution of the PDE

$$w_t = \frac{1}{2} w_{yy} + (g - \beta\rho\theta) w_y + \frac{1}{2} \cdot \frac{1 - \beta\rho^2}{1 - \beta} w_y^2 +$$

$$+ \max_{a \le \eta_1 \le b} \left[\rho(1-\beta)\eta_1 w_y - \frac{\beta(1-\beta)}{2}(\theta + \eta_1)^2\right]$$

with initial condition $w(T, \cdot) \equiv 0$.

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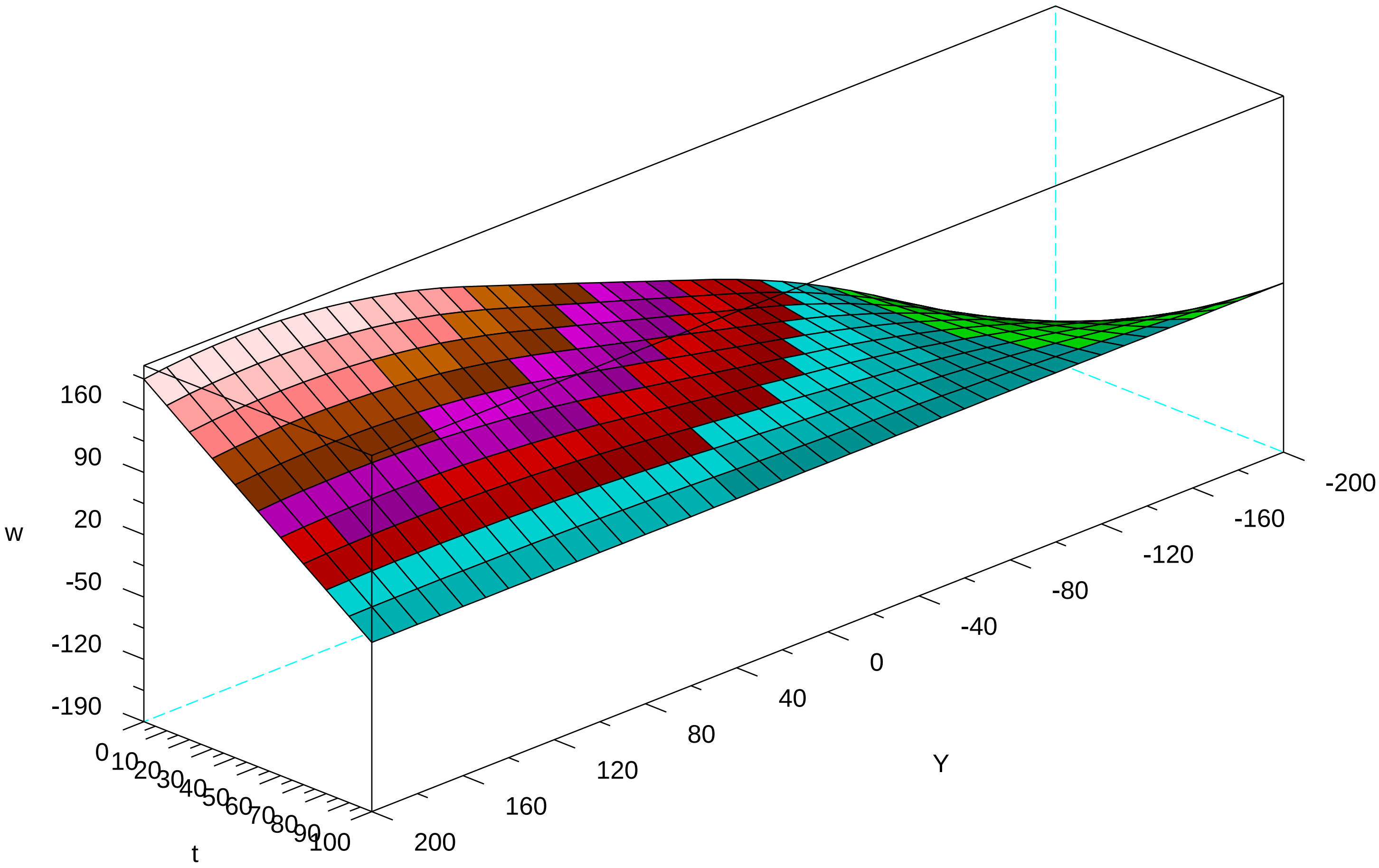
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Advantages and disadvantages of the method

++ Numerically computable solution in realistic models, including incomplete models and local volatility models

– Becomes much more complicated if assumptions on model parameters are relaxed (e.g. $\sigma(S_t, Y_t)$)

– Requires specific utility functions

– Requires time-consistent penalties

Thank you for your attention