# The Singular Points Binomial Method for pricing American path-dependent options

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**Abstract**

We introduce a new numerical approach, called "Singular Points Method", for pricing path-dependent options. This method, based on a continuous representation of the price at every node of the binomial tree, allows to obtain very precise upper and lower bounds of the pure discrete binomial price reducing drastically the time of computation. The method allows also to provide a-priori estimates of the difference between an upper and a lower bound. We apply the method to the case of Asian and lookback American options.

Keywords: option pricing; American options, Asian options, lookback option, tree methods.

## Introduction

A path-dependent option is an option whose payoff depends not only on the value of the stock price at maturity but also on the past history of the underlying asset price. In this paper we are mainly interested in the case of Asian and lookback options.

The pay-off of an Asian option is based on some form of averaging of the underlying asset price over the life of the option. The most common cases are those for which the average is the arithmetic one or the geometric one. Lookback options are options whose payoff depend on the maximum or on the minimum of the underlying asset price reached during the life of the option.

American lookback and American Asian options cannot be valued by closed-form formulae, even in the Black-Scholes model, and require the use of numerical methods.

Here we consider tree methods for pricing this type of options.

It is well known that the Cox-Ross-Rubinstein method applied to Asian options with arithmetic average is problematic since the number of the possible averages increases exponentially with the number of the steps of the tree. For this reason Hull and White ([6]) and in a similar way Barraquand-Pudet ([2]), proposed more feasible approaches. The main idea of this procedure is to restrict the possible arithmetic averages to a set of some representative values. These values are selected in order to span all the possible values of the averages reachable at each node of the tree. The price is then computed by a backward induction procedure where the prices associated to the averages not included in the set of representative values, are obtained by some suitable interpolation methods.

These techniques reduce meaningful the computations by respect to the pure binomial, however they presents some drawbacks related to the precision of the approximation and also to the convergence to the continuous value as observed by Forsyth et al in [7].

Later Chalasani et alt. ([3], [4]) proposed a complete different approach which allows to obtain tine upper and lower bounds on the pure binomial price of American Asian option. Their method requires a forward procedure and a backward induction. This algorithm increases in a meaningful way the precision of the estimates but presents a different problem: the implementation involves a large amount of memory which does not allow to evaluate prices with a large number of tree steps.

In the case of lookback options the complexity of the pure binomial algorithm is of order $O(n^3)$ (where $n$ is the number of tree steps) and the methods proposed in [6] and [2] do not improve the efficiency. Most recently Babbs ([1]), gives a very efficient and accurate solution to the problem for American floating strike lookback options by using a procedure of complexity of order $O(n^2)$. Babbs use a change of “numeraire” approach, that cannot be applied in the fixed strike case.

In this paper we introduce a new discrete procedure which allows to price both American Asian and lookback options. The method provides very precise upper and lower bounds of the pure binomial discrete value and reduces drastically the time of computation with respect to the previous techniques.

The main idea is to give a continuous representation of the option price function as a piecewise linear convex function of the path-dependent variable (average or maximum/minimum). These functions are characterized only by a set of points that we name "singular points". All such functions can be evaluated by backward induction in a straightforward way. So that the method provides an alternative and more efficient approach for evaluating the pure binomial price associated to the path-dependent options. Moreover, the property of convexity of the piecewise linear function representing the price, allows to obtain in a simple and natural way upper and lower bounds of the price. A further appeal of the procedure is that it is possible to fix an a-priori level of precision for the distance between the estimates and the pure binomial value. This can be done in a very efficient way reducing drastically the amount of time and space memory. For example (as reported in the last section) we are able to obtain the price of an Asian average option with 200 steps with a relative error of order $10^{-4}$ with a very few requirement of computational (less than 2 sec) and space memory.

The paper is organized as follows: in Section 1 we present the standard binomial techniques for American Asian and lookback options. Section 2 is devoted to the presentation of the singular points method in the Asian case including the description of the implementation of the algorithm. In Section 3 we present the algorithm for American lookback options. Finally in Section 4 a comparison with the best lattice based method known in literature is offered.

## 1 The pure binomial algorithm

In this paper, we consider a market model where the evolution of a risky asset is governed by the Black-Scholes stochastic differential equation

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dB_t, \quad S_0 = s_0, \tag{1}$$

in which $(B_t)_{0\leq t\leq T}$ is a standard Brownian motion, under the risk neutral measure $Q$. The nonnegative constant $r$ is the force of interest rate, $q$ is the continuous dividend yields and $\sigma$ is the volatility of the risky asset. Then $S_T$ is the value of the underlying asset at maturity $T$:

$$S_T = s_0 e^{(r-q-\frac{\sigma^2}{2})T+\sigma B_T}.$$

We consider two examples of path-dependent options written on the underlying $S_t$: arithmetic Asian options and lookback options.

## 1.1 American Asian options

The price of an American Asian option of initial time 0 and maturity $T$ is:

$$P(0, S_0, A_0) = \sup_{\tau \in \mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau, A_\tau)|S_0 = s_0, A_0 = s_0\right],$$

where: $\mathcal{T}_{0,T}$ is the set of all stopping times with values in $[0, T]$, $\psi$ denotes the payoff function and $A_\tau$ is the arithmetic average of the price of the underlying asset over the period $[0, \tau]$, i.e. $A_\tau = \frac{1}{\tau}\int_0^\tau S_t dt$.

Let $K$ be the strike price. Some examples of payoff functions useful for Asian option pricing are:

- Fixed Asian Call: the payoff is $(A_T - K)_+$
- Fixed Asian Put: the payoff is $(K - A_T)_+$
- Floating Asian Call: the payoff is $(S_T - A_T)_+$
- Floating Asian Put: the payoff is $(A_T - S_T)_+$.

Consider now the binomial approach. Let $n$ be the number of steps of the binomial tree and $\Delta T = \frac{T}{n}$ the corresponding time-step. The lognormal diffusion process $(S_{i\Delta T})_{0 \le i \le n}$ is approximated by the Cox-Ross-Rubinstein binomial process

$$S_i = (s_0 \prod_{j=1}^{i} Y_j)_{0 \le i \le n}$$

where the random variables $Y_1, \ldots, Y_n$ are independent and identically distributed with values in $\{d, u\}$. Let us denote by $\pi = \mathbb{P}(Y_n = u)$. The Cox-Ross-Rubinstein tree corresponds to the choice $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}}$ and

$$\pi = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}$$

In a discrete-time setting, the payoff at maturity $n$ of an Asian option is given by $f(S_n, A_n)$ where

$$A_n = \frac{1}{n+1}\sum_{i=0}^{n} S_i$$

and the average process $(A_i)_{0 \le i \le n}$ is recursively computed by

$$A_{i+1} = \frac{(i+1)A_i + S_{i+1}}{i+2}, A_0 = s_0.$$

In the Cox-Ross-Rubinstein model, the price at time 0 of the American (resp. European) Asian option with payoff function $\psi$ is given by $v(0, s_0, s_0)$ where the functions $v(i, x, y)$ can be computed by the following backward dynamic programming equations

$$\begin{cases} v(n, x, y) = \psi(x, y) \\ v(i, x, y) = \max\left(\psi'(x, y), e^{-r\Delta T}\left[\pi v(i+1, xu, \frac{(i+1)y + xu}{i+2}) + (1-\pi)v(i+1, xd, \frac{(i+1)y + xd}{i+2})\right]\right), \end{cases} \quad (2)$$

where $\psi' = \psi$ in the American case and $\psi' \equiv 0$ in the European case.

The obtained tree is not recombining so that the algorithm is of exponential complexity. The evaluation of $v(0, s_0, s_0)$ requires time computations and memory requirement of the order $O(2^n)$ and this fact shows that the algorithm is completely unfeasible from a practical point of view.

### 1.2 Lookback options

The price of an American lookback option is:

$$P(0, S_0, S_0^*) = \sup_{\tau \in \mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau, S_\tau^*)|S_0 = s, S_0^* = s\right].$$

where $\psi$ denotes the payoff function of the option and

$$S_\tau^* = M_\tau = \max_{u\in[0,\tau]} S_u \quad or \quad S_\tau^* = m_\tau = \min_{u\in[0,\tau]} S_u$$

Let $K$ be the strike. Some examples of payoff function useful in lookback option pricing are:

- Fixed Lookback Call: the payoff is $(M_T - K)_+$.
- Fixed Lookback Put: the payoff is $(K - m_T)_+$.
- Floating Lookback Call: the payoff is $(S_T - m_T)_+$.
- Floating Lookback Put: the payoff is $(M_T - S_T)_+$.

In a discrete-time setting, the payoff at maturity $n$ of an European lookback option, written on the maximum, is given by $\psi(S_n, M_n)$ where

$$M_n = \max(S_0, ..., S_n)$$

and the maximum process $(M_i)_{0\leq i\leq n}$ can be computed recursively by

$$M_{i+1} = \max(M_i, S_{i+1}), M_0 = s_0.$$

In the Cox-Ross-Rubinstein model, the price at time 0 of the corresponding American lookback option is given by $v(0, s_0, s_0)$ where the functions $v(i, x, y)$ can be computed by the following backward dynamic programming equations

$$\begin{cases} v(n,x,y) = \psi(x,y) \\ v(i,x,y) = \max\left(\psi(x,y), e^{-r\Delta T}\left[\pi v(i+1, xu, \max(xu,y)) + (1-\pi)v(i+1, xd, y)\right]\right), \end{cases} \tag{3}$$

where $\psi(x, y)$ is the payoff function. The evaluation of $v(0, s_0, s_0)$ requires a number of computations of order $n^3$.

## 2 The Singular Points Method

In this section we introduce a new backward procedure. The main idea is to give a continuous representation of the price as a piecewise linear function at every node of the tree, describing the path-dependent nature of the option. Such representation depends only by a finite number of points (i.e. the points were the slope of the function changes) called "the singular points".

## 2.1 Piecewise linear convex functions and singular points

Henceforth we will use the following notations:

**Definition 1** *Given a set of points: $(x_1, y_1), ..., (x_n, y_n)$, such that $a = x_1 < x_2 < ... < x_n = b$ and*

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} < \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, \qquad i = 2, ..., n-1, \tag{4}$$

*let us consider the function $f(x)$, $x \in [a, b]$, obtained by interpolating linearly the given points. The points $(x_1, y_1), ..., (x_n, y_n)$ (which characterize the piecewise linear function $f$), will be called the singular points of $f$, while $x_1, ..., x_n$ will be called the singular values of $f$.*

**Remark 1** *In the previous definition we consider only piecewise linear functions with strictly increasing slopes, this implies that the resulting function $f$ is convex.*

*In the following we shall consider only piecewise linear functions which are continuous and convex on the interval $[a, b]$. For every such function we may find a set of singular points characterizing it and satisfying equation (4).*

The following results, which have a very simple geometrical interpretation (see Figure 1 and 2), allow to construct upper and lower bounds of the option price.

**Lemma 1** *Let $f$ be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points.*

*Removing a point $(x_i, y_i)$, $i = 2, ..., n-1$, from the set $C$, the resulting piecewise linear function $\widetilde{f}$, whose set of singular points is $C \setminus \{(x_i, y_i)\}$, is again convex in $[a, b]$ and we have:*

$$f(x) \leq \widetilde{f}(x), \quad \forall x \in [a, b].$$

*Proof :* The previous inequality and the convexity of $\widetilde{f}$ follow from the fact that $\widetilde{f}$ is the maximum between $f$ and the function given by the straight line joining the points $(x_{i-1}, y_{i-1})$, $(x_{i+1}, y_{i+1})$. $\diamondsuit$

**Remark 2** *From the previous Lemma it follows that every piecewise linear function $\widetilde{f}$ whose singular points are a subset of $C$ (containing the first and the last singular point) is still convex and satisfies $\widetilde{f} \geq f$.*

**Lemma 2** *Let $f$ be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points. Moreover let $(\overline{x}, \overline{y})$ be the intersection between the straight line joining $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$ and the one joining $(x_{i+1}, y_{i+1})$, $(x_{i+2}, y_{i+2})$, $2 \leq i \leq n-2$.*

*If we consider the new set of $n-1$ singular points*

$$\{(x_1, y_1), ..., (x_{i-1}, y_{i-1}), (\overline{x}, \overline{y}), (x_{i+2}, y_{i+2}), ..., (x_n, y_n)\},$$

*the associated piecewise linear function $\widetilde{f}$ is again convex on $[a, b]$ and we have:*

$$f(x) \geq \widetilde{f}(x), \quad \forall x \in [a, b].$$

*Proof :* The singular points of $f$ satisfy the property of increasing slopes (4). The set of slopes associated to the singular points of $\widetilde{f}$ are obtained removing the slope of the line joining $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$, hence (4) is again satisfied and $\widetilde{f}$ is convex. The inequality $f \geq \widetilde{f}$ is trivial. $\diamondsuit$

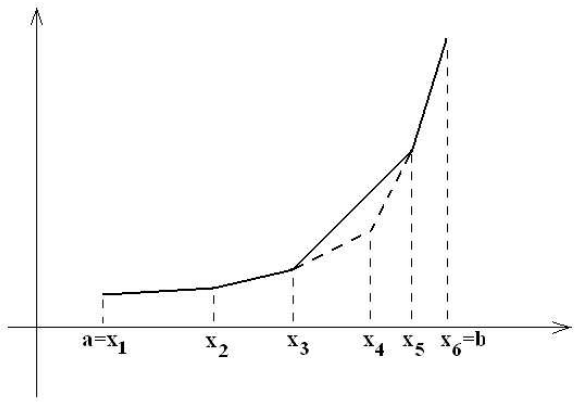


Figure 1: Upper estimate: $x_4$ has been removed.

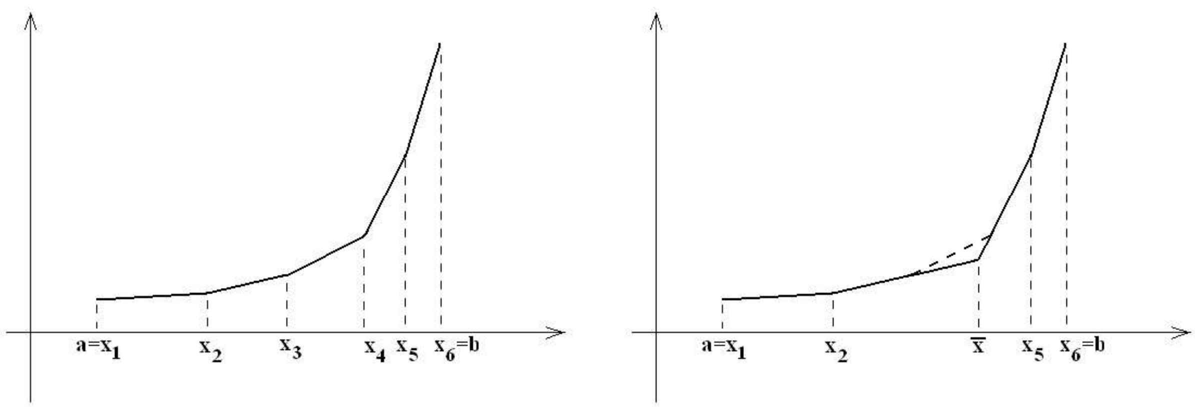


Figure 2: Lower estimate: $x_3$ and $x_4$ have been removed, $\overline{x}$ has been inserted.

### 2.2 Fixed strike European Asian options

We first describe the proposed algorithm in the framework of a fixed strike European Asian call option.

In this case the method consists in evaluating at every node of the tree the price of the option for every possible choice of the average at that point; not only for those which are effectively achievable, but for all possible averages between the minimum and maximum realized at that point. In this way we will show that it is possible to give a continuous representation of the price function as a piecewise linear convex function of the average. This function is characterized only by its singular points.

Let us introduce some more notations.

Let us denote by $N_{i,j}$ the node of the tree whose underlying is $S_{i,j} = s_0 u^{2j-i}$, $i = 0, ..., n$, $j = 0, ..., i$. We will associate to each node $N_{i,j}$ a set of singular points, whose number is $L_{i,j}$. The singular points will be denoted by

$$(A^l_{i,j}, P^l_{i,j}), \quad l = 1, ..., L_{i,j}.$$

Dealing with Asian options, the singular values $A^l_{i,j}$ are called singular averages and $P^l_{i,j}$ are called singular prices.

Consider first the nodes $N_{n,j}$, $j = 0, ..., n$, of the tree at maturity. At every node the average values vary between a minimum average $A^{min}_{n,j}$ and a maximum average $A^{max}_{n,j}$ which are easily valuable:

$$A^{min}_{n,j} = \frac{s_0}{n+1}\left(\frac{1-d^{n-j+1}}{1-d} + d^{n-j}\left(\frac{1-u^{j+1}}{1-u} - 1\right)\right),$$

$$A^{max}_{n,j} = \frac{s_0}{n+1}\left(\frac{1-u^{j+1}}{1-u} + u^{j}\left(\frac{1-d^{n-j+1}}{1-d} - 1\right)\right).$$

For every $A \in [A^{min}_{n,j}, A^{max}_{n,j}]$ the price of the option can be continuously defined by $v_{n,j}(A) = (A-K)_+$ (remark that $v_{n,j}(A) \equiv v(n, S_{n,j}, A)$ where $v(n, x, y)$, is the price function introduced in Section 1.1).

Note that the function $v_{n,j}(A)$ is a piecewise linear function satisfying Definition 1 whose singular points are valuable in a straightforward way. In fact:

- if $K \in (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 3 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2, 3$ (hence $L_{n,j} = 3$), where

$$\begin{array}{ll} A^1_{n,j} = A^{min}_{n,j}, & P^1_{n,j} = 0; \\ A^2_{n,j} = K, & P^2_{n,j} = 0; \\ A^3_{n,j} = A^{max}_{n,j}, & P^3_{n,j} = A^{max}_{n,j} - K. \end{array} \tag{5}$$

.

- if $K \notin (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 2 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2$, $(L_{n,j} = 2)$, where

$$\begin{array}{ll} A^1_{n,j} = A^{min}_{n,j}, & P^1_{n,j} = (A^{min}_{n,j} - K)_+ \ ; \\ A^2_{n,j} = A^{max}_{n,j}, & P^2_{n,j} = (A^{max}_{n,j} - K)_+ \ . \end{array} \tag{6}$$

- In the case $j = 0$ and $j = n$ the minimum and maximum of the averages coincide and $L_{n,j} = 1$.

Therefore we can conclude

**Lemma 3** *At every node at maturity the function $v_{n,j}(A)$ which provides the price of the option, is a piecewise linear function in interval $[A^{min}_{n,j}, A^{max}_{n,j}]$. Moreover such a function is convex in its domain.*

Consider now the step $i$, $0 \le i \le n-1$. At the node $N_{i,j}$ we can evaluate recursively the minimum and the maximum of the averages, respectively

$$A^{min}_{i,j} = \frac{(i+2)A^{min}_{i+1,j+1} - S_{i+1,j+1}}{i+1}, \qquad A^{max}_{i,j} = \frac{(i+2)A^{max}_{i+1,j} - S_{i+1,j}}{i+1}.$$

**Lemma 4** *At every node $N_{i,j}$, $i = 0, ..., n$, $j = 0, ..., i$, the function $v_{i,j}(A)$ which provides the price of the option as function of the average $A$, is piecewise linear and convex in the interval $[A^{min}_{i,j}, A^{max}_{i,j}]$.*

*Proof :* The claim is true at step $i = n$ (at maturity) by Lemma 3. At step $i = n-1$, the price function $v_{i,j}(A)$, with $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, is obtained by considering the discounted expectation value:

$$v_{i,j}(A) = e^{-r\Delta T}[\pi v_{i+1,j+1}(A') + (1-\pi)v_{i+1,j}(A'')], \tag{7}$$

where

$$A' = \frac{(i+1)A + s_0 u^{2j-i+1}}{i+2}, \qquad A'' = \frac{(i+1)A + s_0 u^{2j-i-1}}{i+2}. \tag{8}$$

As $v_{n,j}(A)$ is piecewise linear and convex in its domain and $h_1(A) = v_{i+1,j+1}(\frac{(i+1)A+s_0u^{2j-i+1}}{i+2})$ is the composite function of a linear function of $A$ and a piecewise linear convex one, $h_1(A)$ is piecewise linear and convex as function of $A$. The same holds true for $h_2(A) = v_{i+1,j}(\frac{(i+1)A+s_0u^{2j-i-1}}{i+2})$. We can conclude that $v_{i,j}(A)$ is piecewise linear and convex in its domain.

The claim of the Lemma now follows by backward induction. ♢

Consider again the step $i = n-1$ and the node $N_{i,j}$. By Lemma 4, $v_{i,j}(A)$ is piecewise linear and convex, hence it is characterized by his singular points.

The evaluation of the singular points can be done recursively by a backward algorithm which will be described in the sequel.

Each average $A^l_{i+1,j}$, $l = 1,...,L_{i+1,j}$, associated to a singular point of the node $N_{i+1,j}$ is projected in a new average value $B^l$ at the node $N_{i,j}$ by the relation

$$B^l = \frac{(i+2)A^l_{i+1,j} - s_0u^{2j-i-1}}{i+1}. \tag{9}$$

Note that $B^l$ is the average evaluated at the node $N_{i,j}$ which becomes $A^l_{i+1,j}$ after a down movement of the underlying.

Observe that $B^l$ is increasing with respect to $l$ , $B^{L_{i+1,j}} = A^{max}_{i,j}$ for all $j$, and $B^1 \notin [A^{min}_{i,j}, A^{max}_{i,j}]$ if $0 < j < i$. Each $B^l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ becomes the first coordinate of a singular point associated to the node $N_{i,j}$.

In order to evaluate the price $v_{i,j}(B^l)$ associated to the singular average $B^l \in [A^{min}_{i,j}, A^{max}_{i,j}]$, we remark that after a down movement of the underlying, $B^l$ transforms into $A^l_{i+1,j}$, which price is $P^l_{i+1,j}$. Consider now an up movement of the underlying. In this case $B^l$ transforms into the average: $B^l_{up} = \frac{(i+1)B_l+s_0u^{2j-i+1}}{i+2}$. This average clearly could not belong to the set of singular averages associated to the node $N_{i+1,j}$. Therefore we need to evaluate the index $s$ such that $B^l_{up} \in [A^s_{i+1,j+1}, A^{s+1}_{i+1,j+1}]$. Since in this interval the price function is linear, we have

$$v_{i+1,j+1}(B^l_{up}) = \frac{P^{s+1}_{i+1,j+1} - P^s_{i+1,j+1}}{A^{s+1}_{i+1,j+1} - A^s_{i+1,j+1}}(B^l_{up} - A^s_{i+1,j+1}) + P^s_{i+1,j+1}.$$

We can evaluate the price associated to the singular average $B^l$ evaluating the discounted expectation value:

$$v_{i,j}(B^l) = e^{-r\Delta T}[\pi v_{i+1,j+1}(B^l_{up}) + (1-\pi)v_{i+1,j}(A^l)]. \tag{10}$$

In a similar way each singular average $A^l_{i+1,j+1}$, $l = 1,...,L_{i+1,j+1}$ associated to the node $N_{i+1,j+1}$ is projected in a new average $C^l$ at the node $N_{i,j}$ by the relation

$$C^l = \frac{(i+2)A^l_{i+1,j+1} - s_0u^{2j-i+1}}{i+1}. \tag{11}$$

Now $C^1 = A^{min}_{i,j}$ for all $j$, and $C^{L_{i+1,j+1}} \notin [A^{min}_{i,j}, A^{max}_{i,j}]$ if $0 < j < i$. For each $C^l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ we can evaluate the corresponding price $v_{i,j}(C^l)$ in a similar way as before.

Finally we proceed by a sorting of the averages $B^l$ and $C^l$ belonging to $[A^{min}_{i,j}, A^{max}_{i,j}]$, obtaining an ordered set $\{(A^l_{i,j}, P^l_{i,j}), ..., (A^{L_{i,j}}_{i,j}, P^{L_{i,j}}_{i,j})\}$ of singular points at the node $N_{i,j}$. By the previous construction these are exactly all the singular points associated to this node. Remark that $L_{i,j} \leq L_{i+1,j} + L_{i+1,j+1} - 2$.

The previous argument can be applied at every step $i = n-1, ..., 0$ and holds for all $j = 1, ..., i-1$. At the nodes $N_{i,i}$, $N_{i,0}$, there is only a singular point whose price is given by

$$P^1_{i,0} = e^{-r\Delta T}[\pi P^1_{i+1,0} + (1-\pi)P^1_{i+1,1}], \qquad P^1_{i,i} = e^{-r\Delta T}[\pi P^1_{i+1,i+1} + (1-\pi)P^{L_{i+1,i}}_{i+1,i}]; \tag{12}$$

so that we get a complete description of the price function $v_{i,j}(A)$ at every node of the tree.

The value $P^1_{0,0}$ is exactly the binomial price relative to the tree with $n$ steps of the fixed strike European Asian call option. In fact the method provides the price corresponding to every possible average at each node, in particular for the averages which are effectively realized in the binomial tree.

### 2.3 Fixed strike American Asian options

Consider now the American case. At maturity we have the same situation as in the European case. The price function is $v_{n,j}(A) = (A-K)_+$ for $A \in [A^{min}_{n,j}, A^{max}_{n,j}]$, and it is characterized by the same singular points.

Consider the step $i = n-1$. At the node $N_{i,j}$ we first compute, by using the procedure described in the previous section, the singular points associated to this node, obtaining in this way the continuation value function $v^c_{i,j}(A)$.

To taking into account the American feature, as usual, the price function $v_{i,j}(A)$ is obtained by comparing the continuation value with the early exercise:

$$v_{i,j}(A) = \max\{v^c_{i,j}(A), A-K\}.$$

Let us remark that $v_{i,j}(A)$, $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, is still a piecewise linear convex function. For this reason we can characterize it again by its singular points.

In order to compute the singular points associated to the American case we first remark that the slopes characterizing the piecewise linear convex function $v^c_{i,j}(A)$ are all less than 1. In fact this follows immediately by virtue of equations (7), (8) and by differentiating $v^c_{i,j}(A)$ in the open intervals $(A^l_{i,j}, A^{l+1}_{i,j})$, $l = 1, ..., L_{i,j}$. Therefore there are two possible cases:

1. $A^{max}_{i,j} - K \leq v^c_{i,j}(A^{max}_{i,j})$ then $v_{i,j} \equiv v^c_{i,j}$, so the singular points do not change;
2. $A^{max}_{i,j} - K > v^c_{i,j}(A^{max}_{i,j})$. Here we have two subcases:
   - $A^{min}_{i,j} - K \geq v^c_{i,j}(A^{min}_{i,j})$ then $v_{i,j}(A) = A-K$ for all $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, so the set of singular points consists only of the two points
   $$(A^{min}_{i,j}, A^{min}_{i,j} - K),\ (A^{max}_{i,j}, A^{max}_{i,j} - K).$$
   - $A^{min}_{i,j} - K < v^c_{i,j}(A^{min}_{i,j})$ then there exist an unique average $\overline{A}$ where the continuation value is equal to the early exercise. Let $j_0$ be the largest index such that $A^{j_0}_{i,j} < \overline{A}$. The new set of singular points becomes (see also Figure 3):
   $$\{(A^1_{i,j}, P^1_{i,j}), ..., (A^{j_0}_{i,j}, P(A^{j_0}_{i,j})), (\overline{A}, \overline{A} - K), (A^{max}_{i,j}, A^{max}_{i,j} - K)\}.$$

The same argument can be applied at every step $i = n-2, ..., 0$. This allows to compute $P^1_{0,0}$ which provides the pure American binomial price relative to the tree of $n$ steps.

**Remark 3** *The number of singular points associated to a node could decrease in the American case, so the American procedure could be faster than the European one.*

**Remark 4** *In the case of Asian put option the procedure is similar.*

**Remark 5** *In the floating strike case the procedure is modified as follows: at maturity the singular points depend not more on the strike $K$ but on the underlying value at each node $S_{i,j}$. Therefore the new singular points are obtained by replacing $K$ by $S_{i,j}$. The backward procedure is the same as before, taking into account properly the new intrinsic values.*

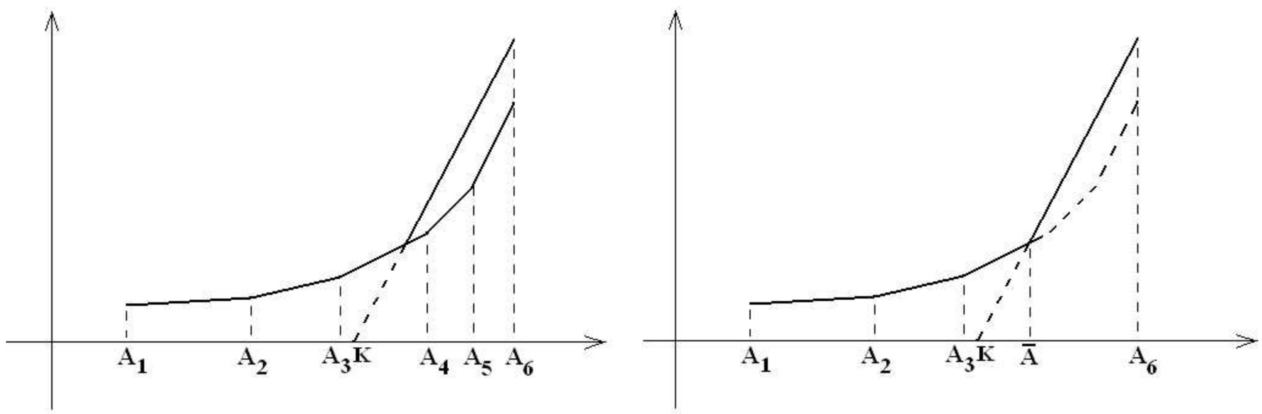


Figure 3: The point $\overline{A}$ has been inserted, $A_4$ and $A_5$ have been removed.

## 2.4 Upper and lower bound

In the previous subsections we have introduced a new method in order to evaluate the pure binomial price in a discrete setting of an European or American Asian option.

As $L_{i,j} \leq L_{i+1,j} + L_{i+1,j+1} - 2$, the resulting algorithm can be of exponential complexity as the standard binomial technique.

The main advantage of our technique is that it allows to obtain easily an upper and a lower bound of the binomial price, reducing drastically the amount of time computation and the memory requirement. Moreover a further appeal is given by the possibility to obtain an a-priori control of the distance of the estimates from the pure binomial price. Actually all such results are simple consequences of Lemma 1 and Lemma 2 previously introduced.

More precisely, in order to get an upper bound of the pure binomial price we just remove some singular points at every node. Lemma 1 ensures that the value obtained in such way is an upper estimate of the pure binomial price.

There are several possible criteria to remove the singular points. Here we propose the following: Consider the set of singular points $C = \{(A^1_{i,j}, P^1_{i,j}), ..., (A^L_{i,j}, P^L_{i,j})\}$ ($L = L_{i,j}$), associated to the node $N_{i,j}$ and the corresponding price value function $v_{i,j}(A)$. Let $v'_{i,j}(A)$ the price value function obtained by removing a point $(A^l_{i,j}, P^l_{i,j})$ from $C$. We have

$$|v_{i,j}(A) - v'_{i,j}(A)| \leq \epsilon_l, \qquad \forall A \in [A^{min}_{i,j}, A^{max}_{i,j}] \tag{13}$$

where

$$\epsilon_l = v_{i,j}(A^l_{i,j}) - v'_{i,j}(A^l_{i,j}) = \frac{P^{l+1}_{i,j} - P^{l-1}_{i,j}}{A^{l+1}_{i,j} - A^{l-1}_{i,j}}(A^l_{i,j} - A^{l-1}_{i,j}) + P^{l-1}_{i,j} - P^l_{i,j}. \tag{14}$$

Therefore, given a real number $h > 0$ we choose to remove the point $(A^l_{i,j}, P^l_{i,j})$ if $\epsilon_l < h$. Repeating this procedure iteratively at every node of the tree we can conclude that the obtained upper estimate differs from the pure binomial value at most for $nh$.

The algorithm for the computation of the lower bound is similar and follows by Lemma 2. Removing the points $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$, $(A^l_{i,j}, P^l_{i,j})$, $l = 2, ..., L-2$, and adding the point $(\overline{x}, \overline{y})$ (see Lemma 2) the difference between the values of the associated piecewise linear functions is less or equal to $\delta_l$, where

$$\delta_l = \frac{P^l_{i,j} - P^{l-1}_{i,j}}{A^l_{i,j} - A^{l-1}_{i,j}}(\overline{x} - A^{l-1}_{i,j}) + P^{l-1}_{i,j} - \overline{y}. \tag{15}$$

This replacement will take place only if $\delta_l < h$. Inductively we get that the obtained lower estimate differs again from the pure binomial value at most for $nh$.

### 2.5 Sketch of the algorithm in the American Asian case

Let us finally summarize the algorithm in order to obtain an upper and a lower bound of the pure binomial price for a fixed strike American Asian call option, with an error less that $nh$ ($h > 0$).

- **STEP n**
  - Compute the singular points at maturity by using (5) and (6).
- **STEP i**, for $i = n-1, ..., 0$
  - Evaluate $P^1_{i,0}$, $P^1_{i,i}$ by comparing the continuation values given in (12) with the early exercise.
  - For each node $N_{i,j}$, $j = 1, ..., i-1$, compute the set of the singular points by the following steps:

    1. for each average $A^l_{i+1,j}$, $l = 1, ..., L_{i+1,j}$ compute $B_l$ by (9),
    2. for $B_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ compute $v^c_{i,j}(B_l)$ by (10),
    3. for each average $A^l_{i+1,j+1}$, $l = 1, ..., L_{i+1,j}$ compute $C_l$ by (11),
    4. for $C_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ compute $v^c_{i,j}(C_l)$,
    5. sort the set of the singular averages $B_l$ and $C_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ obtaining the set of $L_{i,j}$ singular points associated to the node $N_{i,j}$,
    6. compute the American price according to Case 1 or 2 of Section 2.3 getting a new set of singular points with a new cardinality denoted again by $L_{i,j}$,
    7. compute upper and lower bounds with error less than $h$

       **upper bound:** remove sequentially all the singular points $(A^l_{i,j}, P^l_{i,j})$, $l = 2, ..., L_{i,j} - 1$, for which $\epsilon_l < h$ (see (14)) obtaining a new set with a new cardinality denoted again by $L_{i,j}$,

       **lower bound:** For every $l$, $l = 2, ..., L_{i,j} - 2$, for which $\delta_l < h$ (see (15)), remove the points $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$, $(A^l_{i,j}, P^l_{i,j})$ and add the point $(\overline{x}, \overline{y})$ given by the intersection between the two straight lines joining $(A^{l-2}_{i,j}, P^{l-2}_{i,j})$, $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$ and $(A^l_{i,j}, P^l_{i,j})$, $(A^{l+1}_{i,j}, P^{l+1}_{i,j})$, respectively. We obtain again a new set of singular points with a new cardinality $L_{i,j}$.

$P^1_{0,0}$ is the upper [lower] estimate of the pure binomial price with error less that $nh$.

# 3 Lookback American options

We can apply the same procedure described in the Asian case, to the lookback options. Actually, in this case the algorithm admits several simplifications.

Consider a fixed strike lookback American call option. At the nodes $N_{n,j}$, $j = 0, ..., n$ (at maturity) the maximum of the underlying varies between a minimum value $M_{n,j}^{min}$ and a maximum value $M_{n,j}^{max}$ given by

$$M_{n,j}^{min} = \max\{S_{n,j}, s_0\}, \qquad M_{n,j}^{max} = s_0 u^j.$$

For every $M \in [M_{n,j}^{min}, M_{n,j}^{max}]$ the price of the option can be continuously defined by $v_{n,j}(M) = (M - K)_+$.

As in the Asian case, the function $v_{n,j}(M)$ is piecewise linear and its singular points are valuable using relations (5), (6) where $M$ replaces $A$.

Consider now the step $i$, $0 \leq i \leq n-1$. At the node $N_{i,j}$ we can evaluate recursively the minimum and the maximum values of the maximum of the underlying by the relations

$$M_{i,j}^{min} = \max\{s_0, M_{i+1,j+1}^{min}/u\}, \qquad M_{i,j}^{max} = M_{i+1,j}^{max}. \tag{16}$$

**Lemma 5** *At every node $N_{i,j}$, $i = 0, ..., n$, $j = 0, ..., i$, $v_{i,j}(M)$ is a piecewise linear and convex function in the interval $[M_{i,j}^{min}, M_{i,j}^{max}]$.*

*Proof :* The claim is true at step $i = n$. Consider now the step $i = n - 1$. We extend the function $v_{i+1,j+1}$ to the interval $[M_{i+1,j+1}^{min}/u, M_{i+1,j+1}^{max}]$ setting $v_{i+1,j+1}(M) = v_{i+1,j+1}(M_{i+1,j+1}^{min})$ for $M \in [M_{i+1,j+1}^{min}/u, M_{i+1,j+1}^{min})$. With such an extension the continuation value price function $v_{i,j}^c(M)$, becomes

$$v_{i,j}^c(M) = e^{-r\Delta T}[\pi v_{i+1,j+1}(M) + (1-\pi)v_{i+1,j}(M)]. \tag{17}$$

As $v_{i+1,j+1}(M)$ and $v_{i+1,j}(M)$ are piecewise linear and convex we can conclude that the same holds true for $v_{i,j}^c(M)$. Moreover $v_{i,j}(M) = \max\{v_{i,j}^c(M), M - K\}$, therefore $v_{i,j}(M)$ is still piecewise linear and convex. Inductively we have the claim. $\diamondsuit$

By the previous lemma, the price of an American lookback option can be obtained by computing only the singular points of the price function at every node. For this purpose we could use the same algorithm already described in the case of Asian options. So the procedure for American lookback options consists in evaluating first the singular points $(M_{i,j}^1, P_{i,j}^1), ..., (M_{i,j}^L, P_{i,j}^L)$ of $v_{i,j}^c(M)$. Then we can get the singular points of $v_{i,j}(M)$ in an easy way:

- if $M_{i,j}^{max} - K \leq v_{i,j}^c(M_{i,j}^{max})$ then the sets of singular points of $v_{i,j}(M)$ and $v_{i,j}^c(M)$ coincide;
- if $M_{i,j}^{min} - K \geq v_{i,j}^c(M_{i,j}^{min})$ then the set of singular points is composed only by two points: $(M_{i,j}^{min}, M_{i,j}^{min} - K)$, $(M_{i,j}^{max}, M_{i,j}^{max} - K)$;
- if $M_{i,j}^{min} - K < v_{i,j}^c(M_{i,j}^{min})$ and $M_{i,j}^{max} - K > v_{i,j}^c(M_{i,j}^{max})$ then there exists an unique critical value $\overline{M}_{i,j} \in (M_{i,j}^{min}, M_{i,j}^{max})$ were the continuation value coincides with the early exercise value. Then the set of singular points of $v_{i,j}$ is composed by all the singular points of $v_{i,j}^c$ whose singular value belongs to $[M_{i,j}^{min}, \overline{M}_{i,j})$ with the addition of the points: $(\overline{M}_{i,j}, \overline{M}_{i,j} - K)$, $(M_{i,j}^{max}, M_{i,j}^{max} - K)$.

It is important to note that the particular structure of the tree in the lookback case allows to obtain a simpler and more efficient procedure for the evaluation of the singular points of $v_{i,j}$. This procedure, described in the next Proposition 1, is based on the possibility of computing the singular points in a direct way avoiding the sorting procedure. For this purpose we first need some properties which are strictly related to the lookback case:

**Lemma 6** *The price value function $v_{i,j}(M)$, $M \in [M_{i,j}^{min}, M_{i,j}^{max}]$ has the following properties:*

*a) if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ then $v_{i,j}(M)$ is constant in $[M_{i,j}^{min}, K]$,*

*b) if $M \in [M_{i,j}^{min}, M_{i,j-1}^{max}]$ and $v_{i,j}(M) = M - K$ then $v_{i,j-1}(M) = M - K$,*

*c) if $M \in [M_{i+1,j+1}^{min}, M_{i,j}^{max}]$ and $v_{i+1,j+1}(M) = M - K$ then $v_{i,j}(M) = M - K$,*

*d) assume that $x_1 = s_0 u^l$, $x_2 \in (s_0 u^l, s_0 u^{l+1})$, $x_3 = s_0 u^{l+1}$, are singular values of $v_{i,j}(M)$. If we delete the singular point $(x_2, v_{i,j}(x_2))$ then $v_{0,0}(s_0)$ does not change.*

*Proof :* The first two properties follow easily by induction on the tree. Property (c) is a consequence of (b). Remark that the value of the option on the nodes $N_{i,0}, N_{i,i}$, $i = 0, ..., n-1$, depends only on the values that $v_{i+1,j}$ assumes at the nodal stock values of the tree. Therefore the claim of (d) follows again by backward induction. $\diamondsuit$

By Lemma 6(d) it follows that every singular value which lies between two consecutive nodal stock values which are singular values as well, can be removed. This implies that we can delete the critical value $\overline{M}_{i,j}$ during the backward procedure if it lies between two consecutive nodal singular values.

In the next proposition we shall see that the set of internal singular values of $v_{i,j}$ at every node can be reduced to a sequence of consecutive nodal singular values with the eventual addition of $K$. $\overline{M}_{i,j}$ lies always between two consecutive nodal singular values, so that it is not necessary to compute it in the backward procedure. In order to state the claim we denote by $l_0$ the smallest index such that

$$s_0 u^{l_0} > \max\{K, M_{i,j}^{min}\}.$$

**Proposition 1** *At every node of the binomial tree, the set of singular values of $v_{i,j}$ can be reduced to: $M_{i,j}^{min}, M_{i,j}^{max}$, $K$ if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ and a set (eventually empty) of consecutive nodal stock values $\{s_0 u^{l_0}, s_0 u^{l_0+1}, ..., s_0 u^{l_0+k}\}$. If $M = s_0 u^{l_0+k} < \frac{M_{i,j}^{max}}{u}$, then $v_{i,j}(M) = M - K$.*
*Moreover all the singular values of $v_{i,j}$ belonging to $(K, M_{i,j}^{max})$ are singular values of $v_{i+1,j+1}$ as well.*

*Proof :* Consider the case $i = n-1$. Take first $j \geq int[\frac{i}{2}]$ and $j < n-1$ (the case $j = n-1$ is trivial).

At the node $N_{i,j}$ the singular values of $v_{i,j}^c$ are $M_{i,j}^{min}$, $M_{i,j}^{max}$, $K$ if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ and eventually $uM_{i,j}^{min} = M_{i+1,j+1}^{min}$. By Lemma 6(a) $uM_{i,j}^{min}$ is a singular value of $v_{i,j}^c$ if and only if $uM_{i,j}^{min} \geq K$.

Consider now the value function $v_{i,j}$. The possible singular points of $v_{i,j}$ are the same of $v_{i,j}^c$ with the possible addiction of $\overline{M}_{i,j}$ (when it exists). If $\overline{M}_{i,j}$ exists then necessarily $uM_{i,j}^{min}$ is a singular value and $K < uM_{i,j}^{min}$. Hence $v_{i+1,j+1}(uM_{i,j}^{min}) = uM_{i,j}^{min} - K$ and, by Lemma 6(c) $v_{i,j}(uM_{i,j}^{min}) = uM_{i,j}^{min} - K$. Hence $\overline{M}_{i,j} \in [M_{i,j}^{min}, uM_{i,j}^{min}]$ and by Lemma 6(d) it can be removed. Hence the claims hold.

In the case $j < int[\frac{i}{2}]$ there are no singular values in $(K, M_{i,j}^{max})$ so the claim is trivial.

Consider now the general case $i < n-1$ and take $0 < j < n-i$ (the cases $j = 0$, $j = n$ are trivial).

All the singular values of $v_{i+1,j+1}$ belonging to $[M_{i,j}^{min}, M_{i,j}^{max}]$ are singular values of $v_{i,j}^c$ as well. We claim that $v_{i,j}^c$ has no other internal singular values except possible the strike price $K$. In fact if $M > \min\{K, M_{i,j}^{min}\}$ is a singular of $v_{i+1,j}$ then, by induction, it is a singular value of $v_{i+2,j+1}$, therefore it is a singular value of $v_{i+1,j+1}^c$. By Lemma 6(b) we can conclude that it is a singular value of $v_{i+1,j+1}$ as well. Therefore the set of all the singular values of $v_{i,j}^c$ is composed by $M_{i,j}^{min}$, $M_{i,j}^{max}$, eventually $K$ and a sequence of consecutive nodal values $\{s_0 u^{l_0}, s_0 u^{l_0+1}, ..., s_0 u^{l_0+k}\}$ which are singular values of $v_{i+1,j+1}$.

Consider now $v_{i,j}$.

If $v^{i,j}(M_{i,j}^{max}) \geq M_{i,j}^{max} - K$ then $v_{i,j} \equiv v_{i,j}^c$ and their singular points coincide. If $s_0 u^{l_0+k} < \frac{M_{i,j}^{max}}{u}$ then $s_0 u^{l_0+k+1}$ is not a singular value of $v_{i+1,j+1}$. By induction $v_{i+1,j+1}(s_0 u^{l_0+k}) = s_0 u^{l_0+k} - K$. By Lemma 6(c) $v_{i,j}(s_0 u^{l_0+k}) = s_0 u^{l_0+k} - K$.

Assume $v^{i,j}(M_{i,j}^{max}) < M_{i,j}^{max} - K$. If $v_{i,j}(M_{i,j}^{min}) \leq M_{i,j}^{min} - K$ then there no singular points in $(M_{i,j}^{min}, M_{i,j}^{max})$ and the claims hold.
If $v_{i,j}(M_{i,j}^{min}) > M_{i,j}^{min} - K$ then $\overline{M}_{i,j}$ exists. Let $l_1$ be the largest index such that $s_0 u^{l_1} \in (K, M_{i,j}^{max})$ and $s_0 u^{l_1}$ is a singular point of $v_{i,j}^c$. If $s_0 u^{l_1} = M_{i,j}^{max}/u$ then the sequence of singular values of $v_{i,j}^c$ include all the nodal values from $s_0 u^{l_0}$ to $M_{i,j}^{max}$. Denoting by $L$ the smallest index such that $\overline{M}_{i,j} \leq s_0 u^L$, we have that the singular values $s_0 u^{L+1}, ..., s_0 u^{l_1}$ can be removed, hence the claims hold. If $s_0 u^{l_1} < M_{i,j}^{max}/u$ by the induction hypothesis $v_{i+1,j+1}(s_0 u^{l_1}) = s_0 u^{l_1} - K$. By Lemma 6(c) $\overline{M}_{i,j} \leq s_0 u^{l_1}$. Again $s_0 u^{L+1}, ..., s_0 u^{l_1}$ can be removed and $v_{i,j}(s_0 u^L) = s_0 u^L - K$, proving the claims.
$\diamondsuit$

**Remark 6** *As in the case of Asian options, our procedure allows to obtain an upper and a lower bound of the price in a simple way. In this case however the singular points are very few and their distance is much more relevant with respect to the Asian case.*

*For this reason is not useful to compute upper and lower bounds unless we need to consider an extremely large number of time steps.*

### 3.1 Sketch of the algorithm in the American lookback case

Let us summarize the algorithm in order to obtain the pure binomial price for a fixed strike American lookback call option.

- **STEP n**
  - Compute the singular points at maturity by using (5) and (6) where $M$ replace $A$.

- **STEP i**, for $i = n-1, ..., 0$
  - compute $P_{i,0}^1$, $P_{i,i}^1$ by comparing the continuation values given in (12) with the early exercise,
  - for each node $N_{i,j}$, $j = 1, ..., i-1$, compute the set of the singular points by the following steps:

    1. evaluate $v_{i,j}^c(M_{i,j}^{min})$, $v_{i,j}^c(M_{i,j}^{max})$,
       If $v_{i,j}^c(M_{i,j}^{min}) \leq M_{i,j}^{mim} - K$ then there are only two singular points: $(M_{i,j}^{min}, M_{i,j}^{min} - K)$, $(M_{i,j}^{max}, M_{i,j}^{max} - K)$ and the computation is concluded; otherwise insert $(M_{i,j}^{min}, v_{i,j}(M_{i,j}^{min}))$, $(M_{i,j}^{max}, v_{i,j}(M_{i,j}^{max}))$,
    2. if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ then insert $(K, v_{i,j}(K))$,
    3. for every singular value $M$ of the node $N_{i+1,j+1}$ belonging to $(K, M_{i,j}^{max})$ add $(M, v_{i,j}^c(M))$.
       If $v_{i,j}^c(M_{i,j}^{max}) \geq M_{i,j}^{max} - K$ then $v_{i,j}^c$ and $v_{i,j}$ coincide so the computation is concluded. Otherwise (in this case $\overline{M}_{i,j}$ exists) remove all the singular points with singular value internal to $[M_{i,j}^{min}, M_{i,j}^{max}]$ and singular price given by the early exercise, except for the one which has the smallest singular value.

# 4 Numerical Comparisons

In this section we illustrate numerically the efficiency of the singular points method already introduced for pricing fixed Asian and lookback options in the American case.
We assume that the initial value of the stock price is $s_0 = 100$, the maturity $T = 1$, the force of interest rate $r = 0.1$, the continuous dividend yield $q = 0.03$. We consider two choices for the volatility $\sigma = 0.2$, $\sigma = 0.4$ and two choices for the strike $K = 90$ and $K = 110$.

All the computations have been performed in double precision on a PC with a processor Pentium IV Centrino at 1.6 Ghz with 512 Mb of RAM.

## 4.1 Fixed strike American Asian call options

In order to check the behavior of the singular point algorithm we compare

1. the pure binomial model,
2. the Hull-White method (HW) with $h = 0.005$ (see [6]),
3. the forward shooting grid method (FSG) of Barraquand-Pudet with $\rho = 0.1$, (see [2]),
4. the Chalasani et al. method (CJEV) that provides an upper and a lower bound (see [3]),
5. the singular points method providing an upper and a lower bound with error less than $nh$, for two different choices of $h$: $h = 10^{-4}$ ($SP_1$), $h = 10^{-5}$ ($SP_2$).

We consider different time steps $n = 25, 50, 100, 200, 400, 800$ and we report the price estimates and the corresponding time of computation (in parenthesis). The pure binomial method is available only for $n = 25$, while CJEV is available only for $n = 25, 50, 100$ because of problem of out of memory. In the case of CJEV the global computational time in order to get the upper and lower estimates (which are obtained by an unique procedure) has been reported. For the SP methods the two estimates are obtained separately.

| | n | HW | FSG | CJEV | | $SP_1$ | | $SP_2$ | | PURE BIN |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | down | up | down | up | down | up | |
| $\sigma = 0.2$ | 25 | 14.60279 | 14.25496 | 14.24607 | 14.24723 | 14.24602 | 14.24630 | 14.24614 | 14.24617 | 14.24616 |
| | | (0.028) | (0.032) | | (0.012) | (0.009) | (0.008) | (0.011) | (0.009) | |
| | 50 | 14.49228 | 14.48210 | 14.47778 | 14.47887 | 14.47756 | 14.47839 | 14.47786 | 14.47794 | - |
| | | (0.15) | (0.20) | | (0.20) | (0.03) | (0.02) | (0.06) | (0.05) | |
| | 100 | 14.64524 | 14.62562 | 14.62141 | 14.62220 | 14.62081 | 14.62296 | 14.62148 | 14.62167 | - |
| | | (0.78) | (1.64) | | (3.02) | (0.14) | (0.09) | (0.36) | (0.22) | |
| | 200 | 14.74383 | 14.71127 | - | - | 14.70569 | 14.71052 | 14.70723 | 14.70769 | - |
| | | (4.70) | (13.20) | | | (0.58) | (0.37) | (1.66) | (1.08) | |
| | 400 | 14.80389 | 14.76035 | - | - | 14.75270 | 14.76295 | 14.75632 | 14.75734 | - |
| | | (37.94) | (105.82) | | | (2.53) | (1.50) | (7.55) | (4.74) | |
| | 800 | 14.83690 | 14.78716 | - | - | 14.77506 | 14.79573 | 14.78299 | 14.78510 | - |
| | | (201.86) | (836,80) | | | (11.78) | (6.34) | (35.91) | (21.14) | |
| $\sigma = 0.4$ | 25 | 18.32871 | 18.32871 | 17.84535 | 17.85148 | 17.84660 | 17.84689 | 17.84671 | 17.84674 | 17.84672 |
| | | (0.046) | (0.031) | | (0.013) | (0.009) | (0.008) | (0.012) | (0.009) | |
| | 50 | 18.21421 | 18.21659 | 18.20337 | 18.20847 | 18.20418 | 18.20504 | 18.20446 | 18.20455 | - |
| | | (0.28) | (0.20) | | (0.20) | (0.03) | (0.03) | (0.08) | (0.06) | |
| | 100 | 18.46687 | 18.46077 | 18.44834 | 18.45253 | 18.44905 | 18.45117 | 18.44973 | 18.44994 | - |
| | | (1.78) | (1.64) | | (3.01) | (0.20) | (0.13) | (0.52) | (0.34) | |
| | 200 | 18.62676 | 18.60731 | - | - | 18.59517 | 18.60009 | 18.59673 | 18.59720 | - |
| | | (9.51) | (13.23) | | | (0.97) | (0.64) | (2.81) | (1.83) | |
| | 400 | 18.72696 | 18.69170 | - | - | 18.67808 | 18.68853 | 18.68160 | 18.68264 | - |
| | | (50.70) | (104.60) | | | (5.30) | (3.28) | (16.16) | (10.66) | |
| | 800 | 18.78597 | 18.73771 | - | - | 18.72036 | 18.74151 | 18.72803 | 18.73020 | - |
| | | (302.72) | (825,66) | | | (37.23) | (22.30) | (114.26) | (74.84) | |

Table 1: Fixed strike American Asian call options with $T = 1$, $s_0 = 100$, $r = 0.1$, $q = 0.03$ and $K = 90$

| | n | HW | FSG | CJEV | | $SP_1$ | | $SP_2$ | | PURE BIN |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | down | up | down | up | down | up | |
| $\sigma = 0.2$ | 25 | 2.21580 | 2.21376 | 2.20952 | 2.21154 | 2.20971 | 2.21007 | 2.20982 | 2.20984 | 2.20983 |
| | | (0.031) | (0.031) | | (0.013) | (0.008) | (0.008) | (0.011) | (0.009) | |
| | 50 | 2.25529 | 2.24769 | 2.24348 | 2.24475 | 2.24345 | 2.24460 | 2.24383 | 2.24395 | - |
| | | (0.14) | (0.20) | | (0.20) | (0.03) | (0.02) | (0.07) | (0.05) | |
| | 100 | 2.28191 | 2.26623 | 2.26213 | 2.26290 | 2.26156 | 2.26455 | 2.26243 | 2.26272 | - |
| | | (0.78) | (1.64) | | (3.03) | (0.12) | (0.09) | (0.31) | (0.20) | |
| | 200 | 2.29734 | 2.27597 | - | - | 2.27010 | 2.27684 | 2.27215 | 2.27284 | - |
| | | (4.70) | (13.07) | | | (0.55) | (0.34) | (1.50) | (0.95) | |
| | 400 | 2.30536 | 2.28080 | - | - | 2.27242 | 2.28595 | 2.27693 | 2.27841 | - |
| | | (37.92) | (104.59) | | | (2.34) | (1.42) | (6.98) | (4.42) | |
| | 800 | 2.30944 | 2.28292 | - | - | 2.26954 | 2.29539 | 2.27893 | 2.28179 | - |
| | | (201.59) | (828,03) | | | (11.33) | (6.13) | (34.36) | (20.22) | |
| $\sigma = 0.4$ | 25 | 6,78517 | 6.79136 | 6.77940 | 6.78672 | 6.78103 | 6.78135 | 6.78114 | 6.78119 | 6.78116 |
| | | (0.047) | (0.031) | | (0.013) | (0.009) | (0.008) | (0.012) | (0.009) | |
| | 50 | 6.88872 | 6.89089 | 6.87917 | 6.88433 | 6.88080 | 6.88200 | 6.88117 | 6.88128 | - |
| | | (0.28) | (0.20) | | (0.20) | (0.04) | (0.03) | (0.10) | (0.06) | |
| | 100 | 6.95445 | 6.94958 | 6.93817 | 6.94173 | 6.93930 | 6.94226 | 6.94023 | 6.94052 | - |
| | | (1.78) | (1.63) | | (3.03) | (0.19) | (0.14) | (0.50) | (0.33) | |
| | 200 | 6.99555 | 6.98150 | - | - | 6.97038 | 6.97707 | 6.97245 | 6.97311 | - |
| | | (9.47) | (13.12) | | | (0.95) | (0.61) | (2.72) | (1.78) | |
| | 400 | 7.01909 | 6.99776 | - | - | 6.98464 | 6.99848 | 6.98925 | 6.99070 | - |
| | | (50.53) | (104.75) | | | (5.23) | (3.25) | (15.86) | (10.50) | |
| | 800 | 7.03155 | 7.00538 | - | - | 6.98769 | 7.01420 | 6.99746 | 7.00039 | - |
| | | (301.80) | (827,48) | | | (37.25) | (22.41) | (114.25) | (74.92) | |

Table 2: Fixed strike American Asian call options with $T = 1$, $s_0 = 100$, $r = 0.1$, $q = 0.03$ and $K = 110$

| | $\sigma = 0.2,\quad K = 90$ | | | $\sigma = 0.4,\quad K = 90$ | | | $\sigma = 0.2,\quad K = 110$ | | | $\sigma = 0.4,\quad K = 110$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | CJEV | $SP_1$ | $SP_2$ | CJEV | $SP_1$ | $SP_2$ | CJEV | $SP_1$ | $SP_2$ | CJEV | $SP_1$ | $SP_2$ |
| 25 | .00016 | .00028 | .00003 | .00613 | .00029 | .00003 | .00202 | .00036 | .00002 | .00732 | .00032 | .00005 |
| 50 | .00109 | .00083 | .00008 | .00510 | .00096 | .00009 | .00127 | .00115 | .00012 | .00516 | .00120 | .00011 |
| 100 | .00079 | .00215 | .00019 | .00419 | .00212 | .00021 | .00077 | .00299 | .00029 | .00356 | .00296 | .00029 |
| 200 | - | .00483 | .00046 | - | .00492 | .00047 | - | .00674 | .00069 | - | .00669 | .00066 |
| 400 | - | .01025 | .00102 | - | .01045 | .00104 | - | .01353 | .00148 | - | .01384 | .00145 |
| 800 | - | .02067 | .00211 | - | .02115 | .00217 | - | .02585 | .00283 | - | .02651 | .00293 |

Table 3: Difference between the upper and lower estimates for CJEV, SP methods

The numerical results obtained in Table 1 and 2 confirm the reliability of the singular points method. In comparison with the Chalasani et al. method (see also Table 3) we obtain an effective improvement of the precision for the up and lower bounds in better CPU times avoiding all the memory requirement problems. With respect to Hull-White and forward shooting grid method the improvements seem to be meaningful.

### 4.2 American fixed strike lookback call options

In the lookback case we just compare our technique with an optimized version of the pure binomial method. As already observed there are very few singular points involved in the computation, so that the evaluation of an upper and a lower bound is not interesting. Therefore the price obtained with the use of singular points method coincides with the pure binomial, but with an improvement of the computational time (see Table 4 and 5), hence of the efficiency.

Clearly in the lookback case the improvement with respect to the previous literature is less pronounced in comparison with the Asian case, nevertheless it confirms the power of the method and its relevance for pricing American path-dependent options.

| $\sigma$ | n | Bin | SP |
|---|---|---|---|
| 0.2 | 100 | 27.73002 | 27.73002 |
| | | (0.004) | (0.003) |
| | 200 | 28.02747 | 28.02747 |
| | | (0.025) | (0.016) |
| | 400 | 28.24333 | 28.24333 |
| | | (0.184) | (0.077) |
| | 800 | 28.39866 | 28.39866 |
| | | (1.28) | (0.30) |
| | 1600 | 28.51033 | 28.51033 |
| | | (10.75) | (1.55) |

| $\sigma$ | n | Bin | SP |
|---|---|---|---|
| 0.4 | 100 | 44.31762 | 44.31762 |
| | | (0.004) | (0.003) |
| | 200 | 45.00766 | 45.00766 |
| | | (0.026) | (0.017) |
| | 400 | 45.50900 | 45.50900 |
| | | (0.183) | (0.080) |
| | 800 | 45.87045 | 45.87045 |
| | | (1.47) | (0.42) |
| | 1600 | 46.12961 | 46.12961 |
| | | (12.02) | (2.11) |

Table 4: Fixed strike American lookback call options with $K = 90$ for binomial method and SP method

| $\sigma$ | n | Bin | SP |
|---|---|---|---|
| 0.2 | 100 | 11.06517 | 11.06517 |
| | | (0.004) | (0.003) |
| | 200 | 11.27996 | 11.27996 |
| | | (0.025) | (0.016) |
| | 400 | 11.43759 | 11.43759 |
| | | (0.184) | (0.077) |
| | 800 | 11.55096 | 11.55096 |
| | | (1.45) | (0.38) |
| | 1600 | 11.63192 | 11.63192 |
| | | (12.02) | (2.00) |

| $\sigma$ | n | Bin | SP |
|---|---|---|---|
| 0.4 | 100 | 27.28512 | 27.28512 |
| | | (0.004) | (0.003) |
| | 200 | 27.85271 | 14.3245 |
| | | (0.024) | (0.017) |
| | 400 | 28.26777 | 28.26777 |
| | | (0.183) | (0.081) |
| | 800 | 28.57206 | 28.57206 |
| | | (1.46) | (0.43) |
| | 1600 | 28.79142 | 28.79142 |
| | | (11.98) | (2.16) |

Table 5: Fixed strike American lookback call options with $K = 110$ for binomial method and SP method

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