The Singular Points Binomial method for pricing American path-dependent options

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Outline

- Pure binomial method for American Asian options and lookback options
- Short overview of past literature
- Singular points methods
- Numerical comparisons
- Example
  - American Asian arithmetic average option
  - Binomial algorithm with 200 steps
  - Relative error of order 10^{-4}
  - Very few requirement of computational time (less than 2 sec) and space memory.

American Asian options

The stock price process satisfies the following SDE:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dB_t$$

The price of an American Asian option of initial time 0 and maturity T is:

$$P(0, S_0, A_0) = \sup_{\tau \in \mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau, A_\tau) | S_0 = s_0, A_0 = s_0\right],$$

$$A_\tau = \frac{1}{\tau}\int_0^\tau S_t dt$$

- Fixed Asian Call: the payoff is $(A_T - K)_+$
- Floating Asian Call: the payoff is $(S_T - A_T)_+$

American lookback Options

The price of an American lookback option is:

$$P(0, S_0, S_0^*) = \sup_{\tau \in \mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau, S_\tau^*) | S_0 = s, S_0^* = s\right].$$

where

$$S_\tau^* = M_\tau = \max_{u \in [0,\tau]} S_u \quad or \quad S_\tau^* = m_\tau = \min_{u \in [0,\tau]} S_u$$

- Fixed Lookback Call: the payoff is $(M_T - K)_+$.
- Floating Lookback Call: the payoff is $(S_T - m_T)_+$.

CRR discrete model

Consider now the pure binomial approach. The lognormal diffusion process $(S_{i\Delta T})_{0\leq i\leq n}$ is approximated by the Cox-Ross-Rubinstein binomial process

$$S_i = (s_0 \prod_{j=1}^{i} Y_j)_{0\leq i\leq n}$$

where the random variables $Y_1, \ldots, Y_n$ i.i.d. random variables with value in $\{d, u\}$. The Cox-Ross-Rubinstein tree : $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}}$ and

$$\pi = \mathbb{P}(Y_n = u) = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}$$

Asian case

In a discrete-time setting, the payoff at maturity n of an Asian option is given by $\psi(S_n, A_n)$ where

$$A_n = \frac{1}{n+1} \sum_{i=0}^{n} S_i$$

and the average process $(A_i)_{0 \le i \le n}$ is recursively computed by

$$A_{i+1} = \frac{(i+1)A_i + S_{i+1}}{i+2}, A_0 = s_0.$$

In the Cox-Ross-Rubinstein model, the price is obtained using the following

backward dynamic programming algorithm

Pure binomial algorithm:Asian case

$$\begin{cases} v(n,x,y) = \psi(x,y) \\ v(i,x,y) = \max\left(\psi(x,y),\right. \\ \left. e^{-r\Delta T}\left[\pi v(i+1,xu,\frac{(i+1)y+xu}{i+2}) + (1-\pi)v(i+1,xd,\frac{(i+1)y+xd}{i+2})\right]\right), \end{cases} \tag{1}$$

The algorithm is of exponential complexity ($n > 25$ OUT OF MEMORY).

Lookback case

In a discrete-time setting, the payoff at maturity n of an European lookback option, written on the maximum, is given by $\psi(S_n, M_n)$ where

$$M_n = \max(S_0, ..., S_n)$$

The backward dynamic programming algorithm:

$$\begin{cases} v(n,x,y) = \psi(x,y) \\ v(i,x,y) = \max\left(\psi(x,y),\right. \\ \left. e^{-r\Delta T}\Big[\pi v(i+1, xu, \max(xu,y)) + (1-\pi)v(i+1,xd,y)\Big]\right), \end{cases} \tag{2}$$

The evaluation of $v(0, s_0, s_0)$ requires a number of computations of order n^3.

Past literature

Hull and White (Journal of Derivatives 93) and in a similar way Barraquand-Pudet (Mathematical Finance 96), proposed more feasible approaches:

- The main idea of this procedure is to restrict the possible arithmetic averages to a set of some representative values.

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- The prices associated to the averages not included in the set of representative values, are obtained by interpolation methods.
- Meaningful reduction of the time computation.
- Some drawbacks related to the precision of the approximation and also to the convergence to the continuous value as observed by Forsyth et al.(Review of Derivatives Research 02).

Chalasani et al. method (Journal of Computational Finance 99, Review of Derivatives Research 99)

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- Chalasani et al. obtain upper and lower estimates using an idea of Rogers-Shi.

Singular points method

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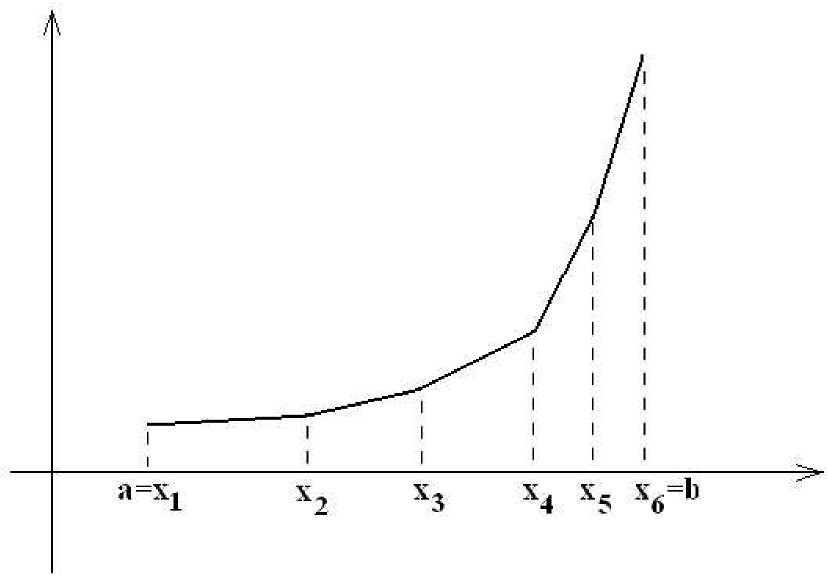
- These functions are characterized only by a set of points that we name singular points.
- The property of convexity allows to obtain in a simple way upper and lower bounds of the price.

Singular points

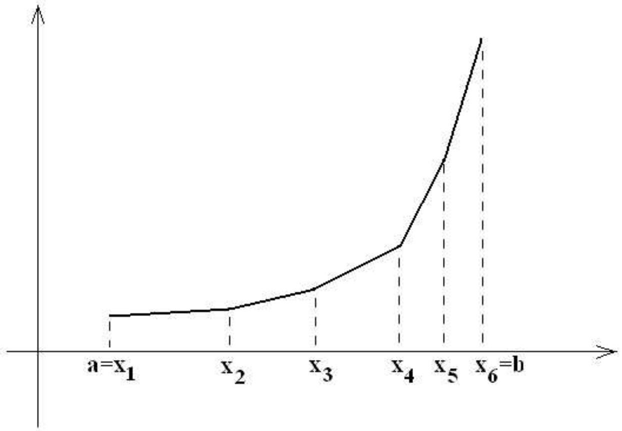
Given a set of points: $(x_1, y_1), ..., (x_n, y_n)$, such that $a = x_1 < x_2 < ... < x_n = b$ and

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} < \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, \qquad i = 2, ..., n-1, \tag{3}$$

let us consider the function $f(x)$, $x \in [a, b]$, obtained by interpolating linearly the given points.



We consider only piecewise linear functions with strictly increasing slopes, so that the function f is convex



The points $(x_1, y_1), ..., (x_n, y_n)$ (which characterize f), will be called the singular points of f.

UPPER BOUND

Lemma 1 *Let f be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points.*

Removing a point (x_i, y_i) from the set C, the resulting piecewise linear function $\widetilde{f}$, whose set of singular points is $C \setminus \{(x_i, y_i)\}$, is again convex in $[a, b]$ and we have:

$$f(x) \leq \widetilde{f}(x), \quad \forall x \in [a, b].$$

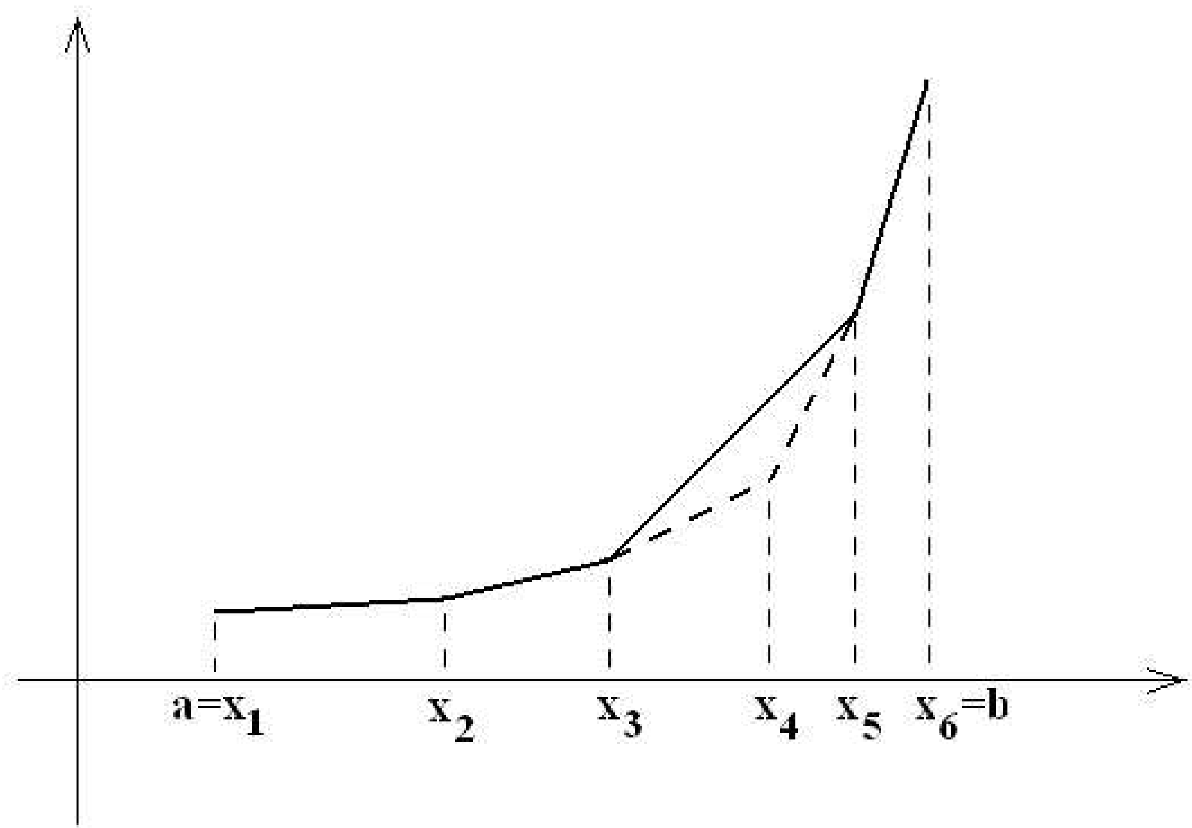


Figure 1: Upper estimate: x_4 has been removed.

LOWER BOUND

Lemma 2 *Let f be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points.*

Let $(\overline{x}, \overline{y})$ be the intersection between the straight line joining (x_{i-1}, y_{i-1}), (x_i, y_i) and the one joining (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}).
If we consider the new set of $n-1$ singular points

$$\{(x_1, y_1), ..., (x_{i-1}, y_{i-1}), (\overline{x}, \overline{y}), (x_{i+2}, y_{i+2}), ..., (x_n, y_n)\},$$

the associated piecewise linear function $\widetilde{f}$ is again convex on $[a, b]$ and we have:

$$f(x) \geq \widetilde{f}(x), \quad \forall x \in [a, b].$$

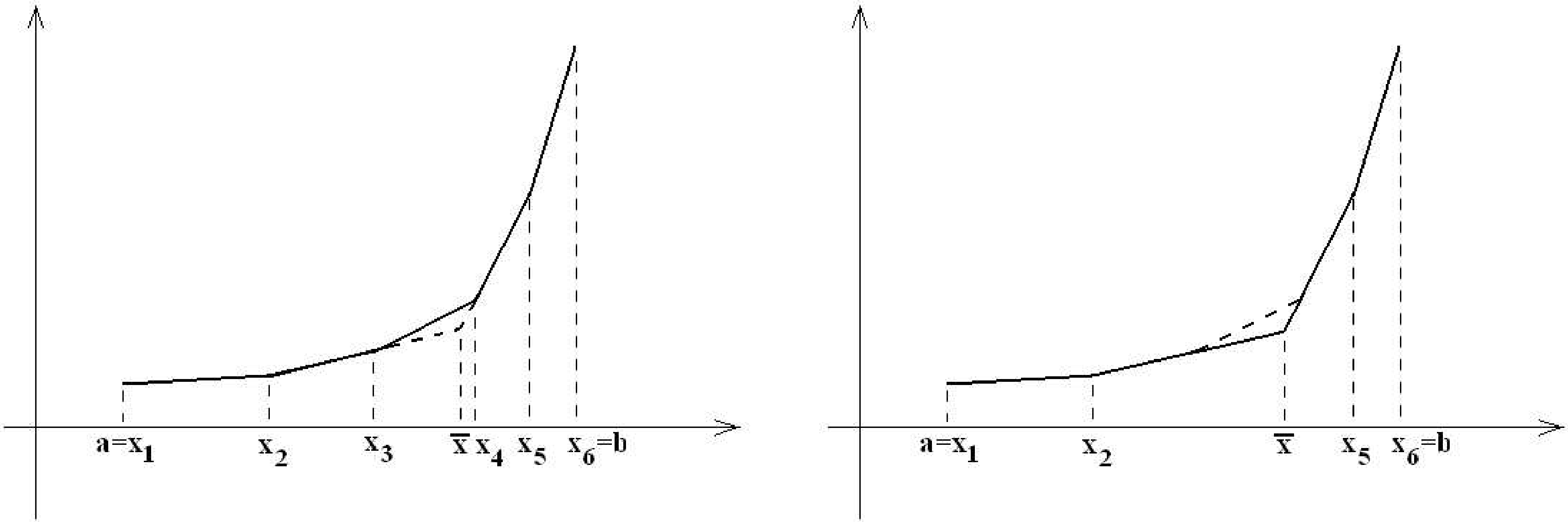


Figure 2: Lower estimate: x_3 and x_4 have been removed, $\overline{x}$ has been inserted.

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- Backward induction algorithm.

Notations

- Let us denote by $N_{i,j}$ the node of the tree whose underlying is $S_{i,j} = s_0 u^{2j-i}$, $i = 0, ..., n$, $j = 0, ..., i$.
- We will associate to each node $N_{i,j}$ a set of singular points, whose number is $L_{i,j}$. The singular points will be denoted by

$$(A^l_{i,j}, P^l_{i,j}), \quad l = 1, ..., L_{i,j}.$$

Backward algorithm: at maturity n

- At every node the average values vary between a minimum average $A_{n,j}^{min}$ and a maximum average $A_{n,j}^{max}$.
- For every $A \in [A_{n,j}^{min}, A_{n,j}^{max}]$ the price of the option can be continuously defined by $v_{n,j}(A) = (A - K)_+$.
- The function $v_{n,j}(A)$ is a piecewise linear and convex function whose singular points are easily valuable.

Critical points at maturity n

- if $K \in (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 3 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2, 3$ ($L_{n,j} = 3$), where

$$\begin{aligned} A^1_{n,j} &= A^{min}_{n,j}, & P^1_{n,j} &= 0; \\ A^2_{n,j} &= K, & P^2_{n,j} &= 0; \\ A^3_{n,j} &= A^{max}_{n,j}, & P^3_{n,j} &= A^{max}_{n,j} - K. \end{aligned} \quad (4)$$

.

- if $K \notin (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 2 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2$, ($L_{n,j} = 2$), where

$$\begin{aligned} A^1_{n,j} &= A^{min}_{n,j}, & P^1_{n,j} &= (A^{min}_{n,j} - K)_+ \ ; \\ A^2_{n,j} &= A^{max}_{n,j}, & P^2_{n,j} &= (A^{max}_{n,j} - K)_+ \ . \end{aligned} \quad (5)$$

- In the case $j = 0$ and $j = n$ the minimum and maximum of the averages coincide and $L_{n,j} = 1$.

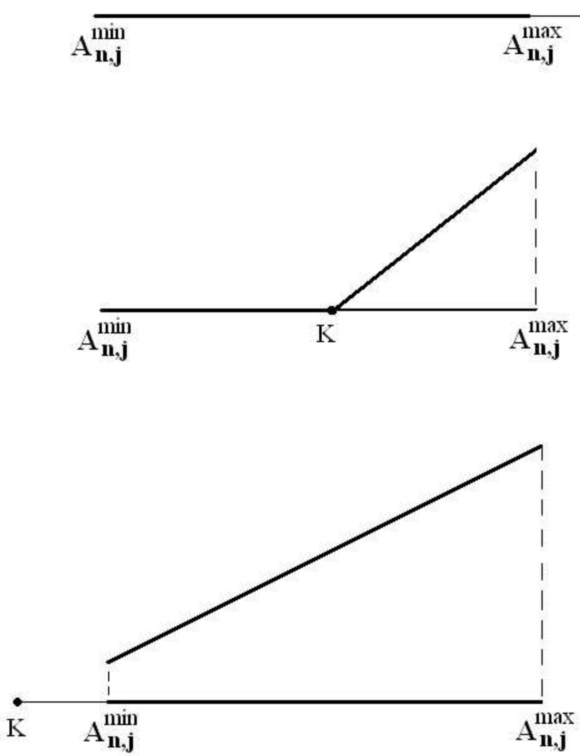

Figure 3: Singular points at maturity

Backward algorithm

Consider now the step i, $0 \leq i \leq n-1$.

Lemma 3 *At every node $N_{i,j}$, $i = 0, ..., n$, $j = 0, ..., i$, the function $v_{i,j}(A)$ which provides the price of the option as function of the average A, is piecewise linear and convex in the interval $[A_{i,j}^{min}, A_{i,j}^{max}]$.*

The evaluation of the singular points can be done recursively by a backward algorithm.

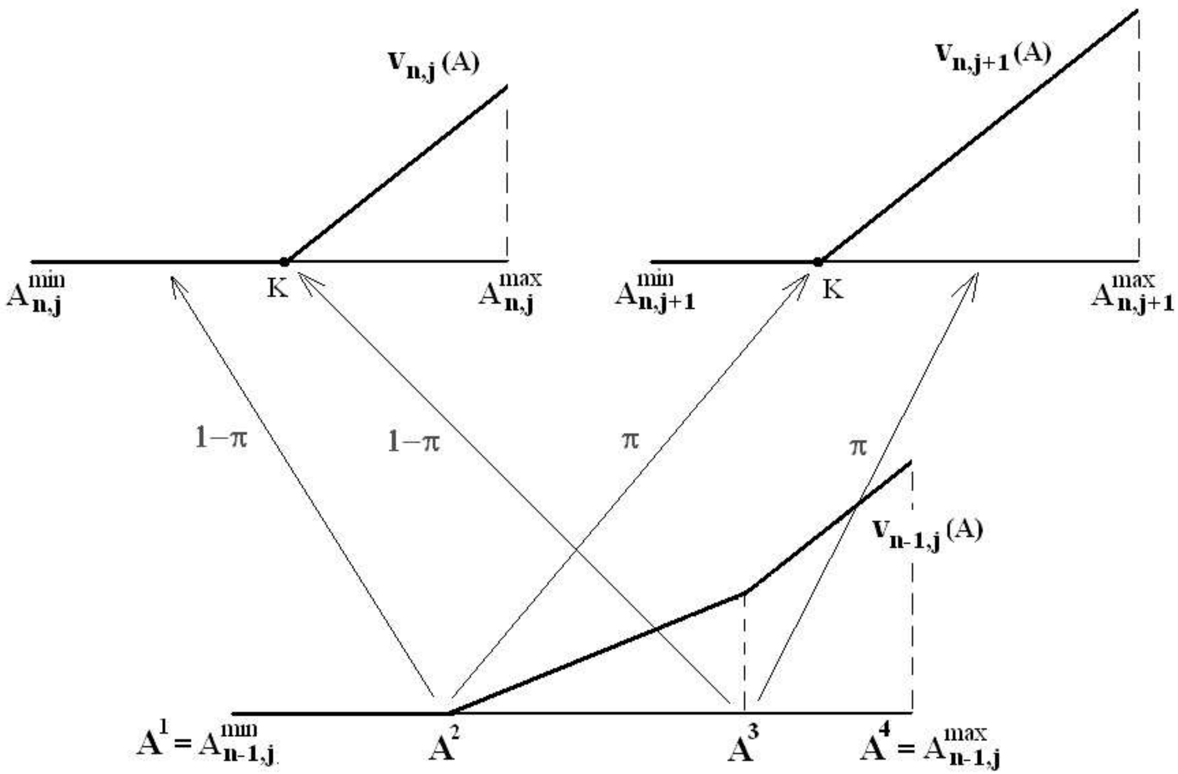

Figure 4: Singular points at i=n-1

Singular points at n-1

- Each singular average $A^l_{i+1,j}$, $l = 1, ..., L_{i+1,j}$ of the node $N_{i+1,j}$ is projected in a new average value B^l at the node $N_{i,j}$ by

$$B^l = \frac{(i+2)A^l_{i+1,j} - s_0 u^{2j-i-1}}{i+1}. \tag{6}$$

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- Let $B^l \in [A^{min}_{i,j}, A^{max}_{i,j}]$. After a down movement of the underlying, B^l transforms into $A^l_{i+1,j}$, which price is $P^l_{i+1,j}$.

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- Consider now an up movement of the underlying. In this case B^l transforms into the average: $B^l_{up} = \frac{(i+1)B_l + s_0 u^{2j-i+1}}{i+2}$. Using linear interpolation (the function is linear!) we obtain$P^l_{i+1,j+1}$.

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- We can evaluate the price associated to the singular average B^l evaluating the discounted expectation value:

$$v_{i,j}(B^l) = e^{-r\Delta T}[\pi v_{i+1,j+1}(B^l_{up}) + (1-\pi)v_{i+1,j}(A^l_{i+1,j})]. \quad (7)$$

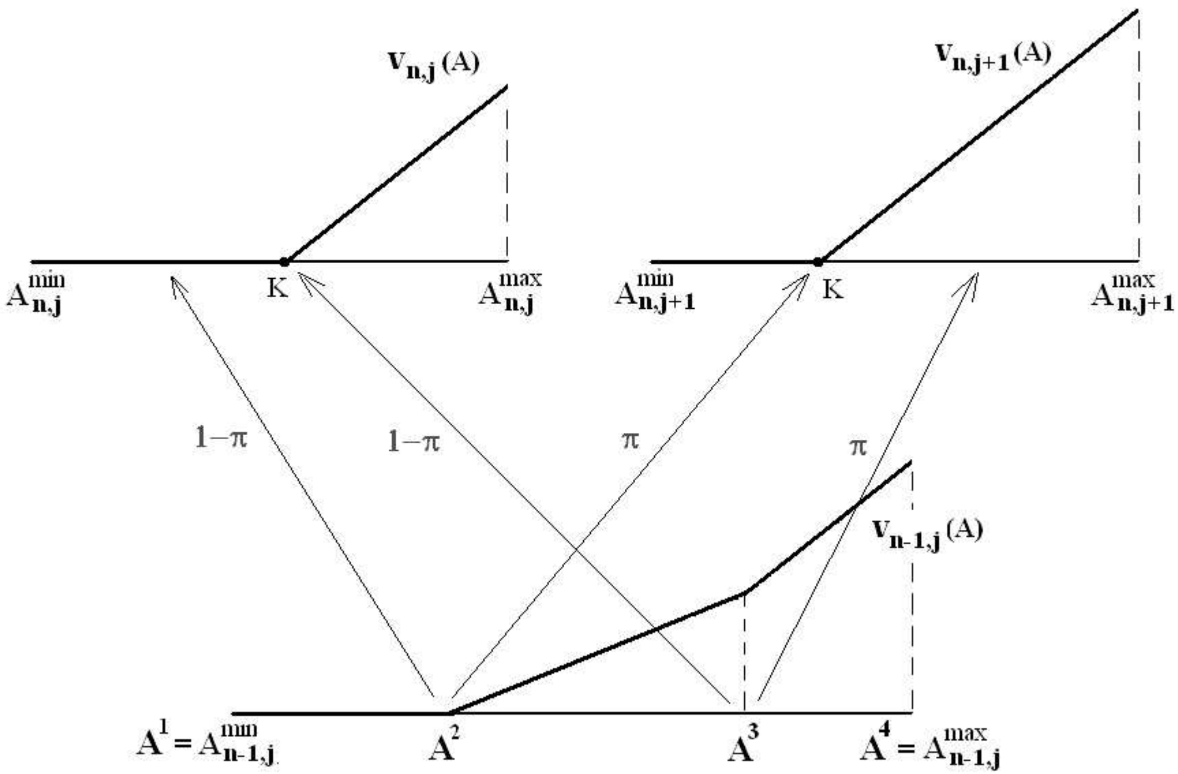

Figure 5: Singular points at i=n-1

- In a similar way each singular average $A^l_{i+1,j+1}$, $l = 1, ..., L_{i+1,j+1}$ associated to the node $N_{i+1,j+1}$ is projected in a new average C^l at the node $N_{i,j}$

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- We can evaluate the corresponding price $v_{i,j}(C^l)$ in a similar way as before.
- Finally we proceed by a sorting of the averages B^l and C^l belonging to $[A^{min}_{i,j}, A^{max}_{i,j}]$, obtaining an ordered set $\{(A^l_{i,j}, P^l_{i,j}), ..., (A^{L_{i,j}}_{i,j}, P^{L_{i,j}}_{i,j})\}$ of singular points at the node $N_{i,j}$. These are exactly all the singular points associated to this node.

Extreme nodes

At the nodes $N_{i,i}$, $N_{i,0}$, there is only a singular point whose price is given by

$$P^1_{i,0} = e^{-r\Delta T}[\pi P^1_{i+1,0} + (1-\pi)P^1_{i+1,1}], \tag{8}$$

$$P^1_{i,i} = e^{-r\Delta T}[\pi P^1_{i+1,i+1} + (1-\pi)P^{L_{i+1,i}}_{i+1,i}]; \tag{9}$$

The value $P^1_{0,0}$ is exactly the binomial price relative to the tree with n steps of the fixed strike European Asian call option.

Fixed strike American call Asian options

- To taking into account the American feature

$$v_{i,j}(A) = \max\{v^c_{i,j}(A), A - K\}.$$

- $v_{i,j}(A)$, $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, is still a piecewise linear convex function.
- For this reason we can characterize it again by its singular points

Suppose that $A_{i,j}^{max} - K > v_{i,j}^{c}(A_{i,j}^{max})$ and $A_{i,j}^{min} - K < v_{i,j}^{c}(A_{i,j}^{min})$.

Then there exist an unique average $\overline{A}$ where the continuation value is equal to the early exercise.

Let j_0 be the largest index such that $A_{i,j}^{j_0} < \overline{A}$. The new set of singular points becomes:

$$\{(A_{i,j}^{1}, P_{i,j}^{1}), ..., (A_{i,j}^{j_0}, P(A_{i,j}^{j_0})), (\overline{A}, \overline{A} - K), (A_{i,j}^{max}, A_{i,j}^{max} - K)\}.$$

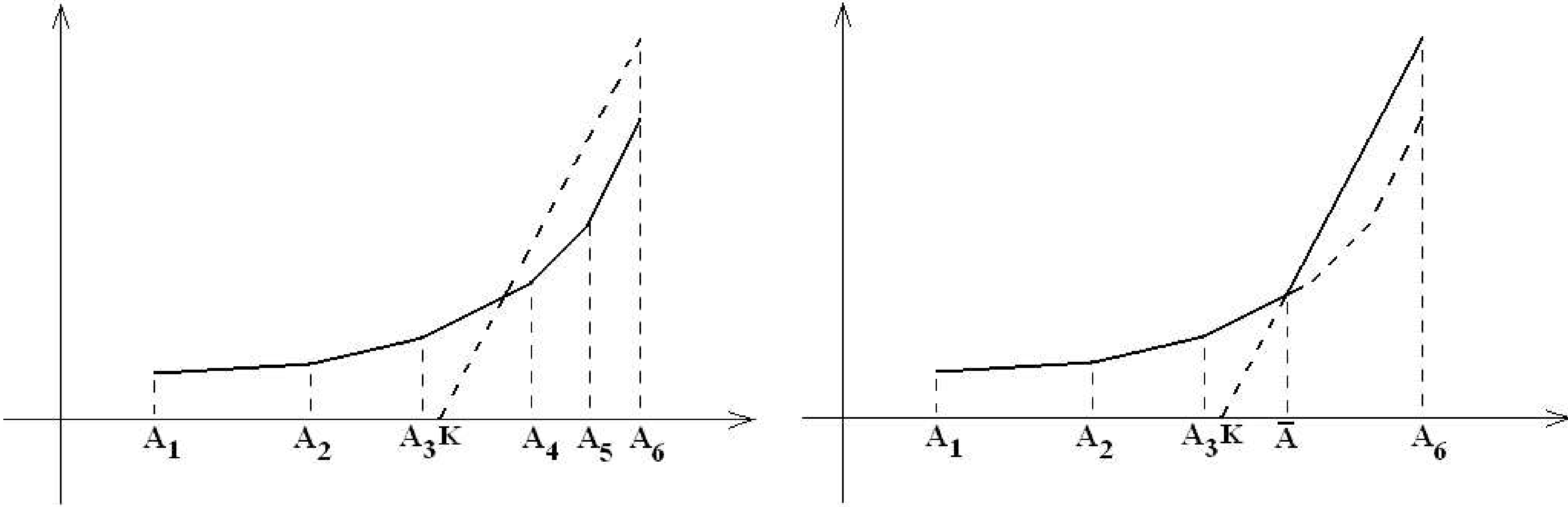

Figure 6: The point $\overline{A}$ has been inserted, A_4 and A_5 have been removed.

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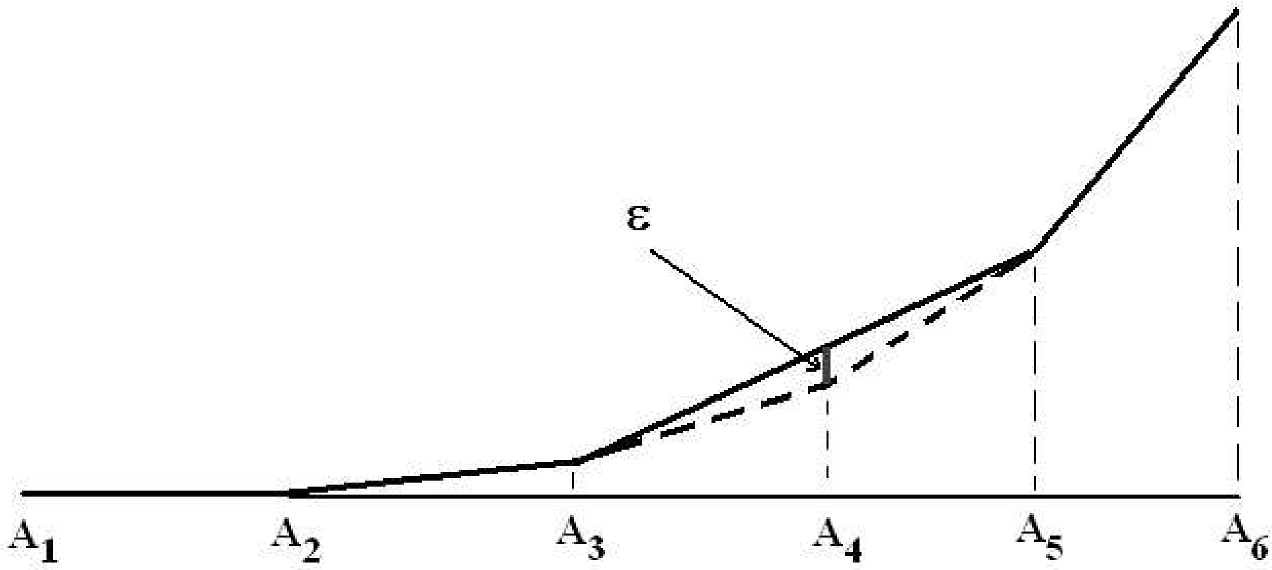
- We are able to compute an upper and a lower bound of the binomial price reducing drastically the amount of time computation and the memory requirement.
- An a-priori control of the distance of the estimates from the pure binomial price.

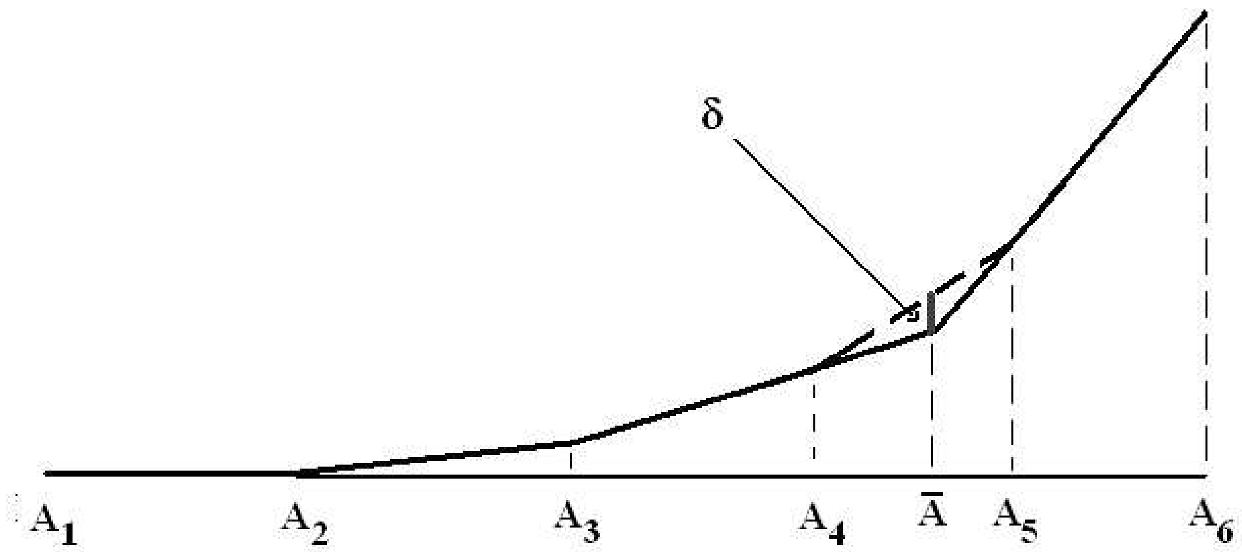
UPPER BOUND



Remove A_4 if $\epsilon \leq h$

Inductively we get that the obtained upper estimate differs from the binomial value at most for nh.

LOWER BOUND



Remove A_4 and A_5 and insert $\overline{A}$ if $\delta \le h$

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- The algoritmh in the lookback case is based on the possibility of computing the singular points in a direct way avoiding the sorting procedure.
- All the singular values of $v_{i,j}$ belonging to $(K, M_{i,j}^{max})$ are singular values of $v_{i+1,j+1}$ as well.
- In this case however the singular points are very few and their distance is much more relevant with respect to the Asian case. For this reason is not useful to compute upper and lower bounds.

Numerical Results

Fixed strike American Call Asian options

- We illustrate numerically the efficiency of singular points method.
- We compare the singular points algorithm with Hull-White,Barraquand-Pudet,Chalasani et al.
- We assume that the initial value of the stock prices are $s_0 = 100$, the maturity $T = 1$, the continuous dividend rates $q = 0.03$, while the values of the volatility $\sigma = 0.2, 0.4$, the interest rate $r = 0.1$, and the exercise price $K = 90, 100$ vary.
- We consider different time steps $n = 25, 50, 100, 200, 400, 800$

1. the pure binomial(PB) model (available only for $n = 25$),
2. the Hull-White method (HW) with $h = 0.005$,
3. the forward shooting grid method (FSG) of Barraquand-Pudet with $\rho = 0.1$,
4. the Chalasani et al. method (CJEV) that provides an upper and a lower bound, (available only for $n = 25, 50, 100$),
5. the singular points method providing an upper and a lower bound with error less than nh, for two different choices of h:
   - $h = 10^{-4}$ (SP_1);
   - $h = 10^{-5}$ (SP_2).

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σ	n	Bin	SP
0.2	100	11.06517	11.06517
		(0.004)	(0.003)
	200	11.27996	11.27996
		(0.025)	(0.016)
	400	11.43759	11.43759
		(0.184)	(0.077)
	800	11.55096	11.55096
		(1.45)	(0.38)
	1600	11.63192	11.63192
		(12.02)	(2.00)

Table 1: Fixed strike American lookback call options with $K = 110$ for binomial method and SP method