Impulse problem on finite horizon with execution delay

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May 4, 2007

Motivations

Basic motivation: to study the pricing and hedging of an option on hedge fund.

- To buy or sell shares of hedge funds, the financial agents must declare their orders one or two months before they can be executed.
- The effective price is the price of the fund at the execution date.

We study the impact of this lag in a general framework. Our goal will be to estimate the cost of liquidity.

Controlled diffusion model

- In the absence of control, let X be a $\mathbb{R}^n$ valued process:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

- At any stopping time τ_i, the agent may decide to intervene on the system with an impulsion $\xi_i \in E$ based on the information availaible at τ_i.
- This impulse takes effect, with delay, at time $\tau_i + d$. This moves the system to:

$$X_{\tau_i+d} = \Gamma(X_{(\tau_i+d)^-}, \xi_i)$$

- There is a minimum lag $0 < h \leq d$ between two interventions: $\tau_{i+1} - \tau_i \geq h$.

Example

We can consider the following problem of optimal investment and/or indifference pricing:

- let S be the spot price of an hedge fund, with a Black-Scholes dynamic: $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
- let C be the amount of cash in the agent's portfolio, and N be the number of shares of the hedge fund held by the agent.
- The impulsions ξ_i represent the number of shares bought or sold by the agent at time τ_i.

Example

- The system is described by the pending orders and the vector:

$$X_t = (S_t, C_t, N_t)$$

- When the order i is executed the system moves to:

$$\begin{aligned} S_{\tau_i+d} &= S_{(\tau_i+d)^-} \\ C_{\tau_i+d} &= C_{(\tau_i+d)^-} - \xi_i S_{\tau_i+d} \\ N_{\tau_i+d} &= N_{(\tau_i+d)^-} + \xi_i \end{aligned}$$

- The optimization problem can be, for example the maximization of the expected utility of the agent:

$$\mathbb{E}\left[U(C_T + N_T S_T - g(S_T))\right]$$

Problem formulation

The goal is to solve the following problem:

$$v_0(0, x_0) = \sup_{\alpha \in \mathcal{A}} \left\{ \mathbb{E}\Big[\int_0^T f(X_t^\alpha) ds + g(X_T^\alpha) + \sum_{t < \tau_i + d \leq T} c(X^\alpha_{(\tau_i+d)^-}, \xi_i) \Big] \right\}$$

with:

$$\mathcal{A} = \{(\tau_i, \xi_i)_{i \in \mathbb{N}} \text{ s.t. } \tau_i \geq 0 \text{ is an } \mathbb{F} \text{ stopping time, } \tau_{i+1} \geq \tau_i + h \text{ and } \xi_i \text{ is } \mathcal{F}_{\tau_i}\text{-measurable} \forall i \} .$$

This an impulse control problem with delay in finite horizon.

Related literature

- Bensoussan A. and J.l. Lions (1982) : *Contrôle impulsionnel et inéquations variationnelles*, Dunod.
- Bar-Ilan A. and A. Sulem (1995) : "Explicit solution of inventory problems with delivery lags", *Math. Oper. Res.*, **20**, 709-720.
- Oksendal B. and A. Sulem (2006) : "Optimal stochastic impulse control with delayed reaction", Preprint, University of Oslo.

Markovian setting

- An important issue is that the process X_t is not Markovian by itself. Indeed, we must take the set of pending orders into account:

$$p = (t_i, \xi_i)_{i \in \{1..k\}} \in ([0, T] \times E)^k \text{ s.t. } t - d < t_i \leq t \forall i.$$

- In this framework, we have $k \leq m = \lfloor \frac{d}{h} \rfloor$, and therefore a finite dimensional system. This is why we introduced the minimum time h between two interventions.

Some notations

- The set of possible pending orders at time t is:

$$P_t(k) = \Big\{ p = (t_i, e_i)_{i\in\{1..k\}} \in ([0,T]\times E)^k \text{ s.t. } t - d < t_i \text{ and } t_{i-1} + h \le t_i \le t \forall i \le k \Big\}$$

with $k \le m$.

- We denote the consistent states as:

$$\mathcal{D}_k \quad = \quad \{(t,x,p) \;:(t,x)\in[0,T]\times\mathbb{R}^d,\; p\in P_t(k)\},$$

- And the admissible controls from a given set of pending orders:

$$\mathcal{A}_{t,p} = \Big\{ \alpha = (\tau_i,\xi_i)_{i\ge1} \in \mathcal{A} \;: (\tau_i,\xi_i) = (t_i,e_i),\; i = 1,\dots,k \text{ and } \tau_{k+1} \ge t \Big\},$$

Value function

- The problem of the agent is to maximize the criterion:

$$J_k(t,x,p,\alpha) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x,p,\alpha})ds + g(X_T^{t,x,p,\alpha}) + \sum_{t<\tau_i+d\le T} c(X_{(\tau_i+d)^-}^{t,x,p,\alpha},\xi_i)\Big],$$

for $(t,x,p) \in \mathcal{D}_k$, $k = 0,\dots,m$, $\alpha = (\tau_i,\xi_i)_i \in \mathcal{A}_{t,p}$.

- We get the corresponding value functions:

$$v_k(t,x,p) = \sup_{\alpha\in\mathcal{A}_{t,p}} J_k(t,x,p,\alpha),$$

for $k = 0,\dots,m$, $(t,x,p) \in \mathcal{D}_k$,

Dynamic programming principle: no possible action

- We introduce the set:

$$\mathcal{D}_k^1 = \{(t,x,p)\ : (t,x) \in [0,T] \times \mathbb{R}^d,\ p \in P_t(k) \text{ s.t. } t_k > t - h\}$$

- If $(t,x,p) \in \mathcal{D}_k^1$ there is no immediate action possible:

$$v_k(t,x,p) = \mathbb{E}\Big[\int_t^\theta f(X_s^{t,x,0})ds + v_k(\theta, X_\theta^{t,x,0}, p)\Big].$$

- Only the diffusion of X operates, and the associated PDE is:

$$-\frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f(.) = 0, \tag{1}$$

on $\mathcal{D}_k^1$.
- Note that for v_m, no more orders can be passed until the first execution. All the consistent states of v_m lie in $\mathcal{D}_m^1$.

Dynamic programming principle: possible intervention

- We introduce the set:

$$\mathcal{D}_k^2 = \left\{(t,x,p) \ : (t,x) \in [0,T] \times \mathbb{R}^d, \ p \in P_t(k) \text{ s.t. } t_k \le t-h\right\}$$

- If $(t,x,p) \in \mathcal{D}_k^{2,m}$ the agent can take action in $[t, t+dt]$. He has the choice between letting the diffusion of X operate, and passing an order.
- If he passes an order, the value function jumps from $v_k(t,x,p)$ to $v_{k+1}(t,x,p \cup (t,\xi))$. We get that $v_k(t,x,p) \ge \sup_\xi \{v_{k+1}(t,x,p \cup (t,\xi))\}$
- This corresponds to the variational inequality on $\mathcal{D}_k^{2,m}$:

$$\min\left\{-\frac{\partial v}{\partial t} - \mathcal{L}v_k - f(.), v_k(t,x,p) - \sup_{e \in E}\{v_{k+1}(t,x,p \cup (t,e))\}\right\} = 0$$

Boundary conditions

- The terminal condition is, at time T:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p})\to(T,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = g(x) \text{ for all } (x,p) \in \mathcal{D}_k^m \qquad (3)$$

- But we must also prove the boundary condition linked to the execution of the first pending order, at time $t_1 + d < T$:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p})\to(t_1+d,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = v_{k-1}(t_1+d, \Gamma(x,e_1), p \setminus (t_1,e_1)) + c(x,e_1), \quad (4)$$

for all $(x,p) \in \mathcal{D}_k^m$, $k \geq 1$ s.t. $p = (t_1,e_1) \cup (t_i,e_i)_{i\in 2..k}$. This is more difficult due to continuity issues for v_{k-1}.

The main theorem

Theorem

The family of value functions v_k, $k = 0, \ldots, m$, is the unique viscosity solution to (1) on $\mathcal{D}_k^1$ and (2) on $\mathcal{D}_k^2$, which satisfy the boundary data (3)-(4), a linear growth condition, and the condition:

$$v_k(t_k + h, x, p) \geq \sup_{e \in E} \left\{ v_{k+1}(t_k + h, x, p \cup (t_k + h, e)) \right\}.$$

Moreover, v_k is continuous on $\mathcal{D}_k$.

A nonstandard problem

To compute the value functions v_k for all k, we face an issue:

- v_k depends of v_{k+1} via the characteristic PDE on $\mathcal{D}_k^{2,m}$, because of the possibility of passing an order.
- v_{k+1} depends of v_k via the boundary condition. It corresponds to the execution of the first pending order of v_{k+1}.

This problem is solved with the following algorithm.

Initialization

The first step of the algorithm is based one the following remark:

- If an order is passed after $T - d$, it will be executed after T. Therefore it has no action on X_t for $t \leq T$.
- Hence, if p is such that $t_1 > T - d$ (i.e. all the pending orders were passed after $T - d$), the value function is such that:

$$v_k(t, x, p) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x,0}) ds + g(X_T^{t,x,0}) \Big],$$

which is easily computable.

Known calculations at step n

- We define the following set, for $k \in \{1..m\}$:

$$\mathcal{D}_k(n) = \{(t, x, p) \in \mathcal{D}_k \text{ s.t. } t_1 \geq T - nh\},$$

which represents the consistent states such that the first pending order was passed after $T - nh$.

- The hypothesis of step n is that we know v_k on $\mathcal{D}_k(n)$ for all $k \in \{1..m\}$, and v_0 for all $t \geq T - nh$.

From step n to $n+1$

One can calculate v_m on $\mathcal{D}_m(n+1) \setminus \mathcal{D}_m(n)$ for $t \in [t_m, t_1 + d]$. If $t_1 \in [T-(n+1)h, T-nh)$, then $t_2 \geq T - nh$.

- Indeed, the PDE satisfied by v_m on $\mathcal{D}_m(n+1) \setminus \mathcal{D}_k(n)$ is:

$$-\frac{\partial v_m}{\partial t} - \mathcal{L} v_m - f(.) = 0,$$

- The boundary condition of v_m is:

$$\begin{aligned} \lim_{(\tilde{t},\tilde{x},\tilde{p}) \to (t_1+d,x,p)} v_m(\tilde{t},\tilde{x},\tilde{p}) &= v_{m-1}(t_1+d, \Gamma(x,e_1), p \setminus (t_1,e_1)) \\ &+ c(x,e_1) \end{aligned}$$

with $(t_1+d, \Gamma(x,e), p \setminus (t_1,e_1)) \in \mathcal{D}_{m-1}(n)$. This has been calculated at step n.

From step n to $n+1$

For fixed n, we proceed with a backward induction on k.

- If v_{k+1} has been calculated on $\mathcal{D}_{k+1}(n+1) \setminus \mathcal{D}_{k+1}(n)$. Then v_k can be calculated on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$. Indeed, the terminal condition :

$$\begin{aligned}\lim_{(\tilde{t},\tilde{x},\tilde{p})\to(t_1+d,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) &= v_{k-1}(t_1+d,\Gamma(x,e),p\setminus(t_1,e_1)) \\ &+ c(x,e_1)\end{aligned}$$

has been calculated at step n. And the characteristic PDE involves only the knowledge of v_{k+1} on $\mathcal{D}_{k+1}(n+1) \setminus \mathcal{D}_{k+1}(n)$.

Conclusion

- We have a method to solve a large class of control problems with delay.
- The drawback is that the dimension of the problem grows quickly with m. Typically have to solve a PDE of dimension $\dim X + 2m$.
- In most cases it is numerically tractable for $m = 1$ or $m = 2$.